A Higgs Correspondence in Characteristic p

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Mainz, September 24, 2012

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- Joint with V. Vologodsky, 2001--2005
- Appears in Pub. I.H.E.S., 2008
- Toy Model for Simpson's and Faltings' theories
- New work by Shing, Xin, Zuo, Gros, Le Stum, Shiho......

Outline

- Quick Review
- The Cartier Transform
- Level One
- The Fundamental Extension
- The General Case

Riemann-Hilbert

$X/\mathbf{C}$ smooth projective scheme

Write $\pi_1(X)$ for $\pi_1(X_{an})$

$$Rep.(\pi_1(X)) \longrightarrow MIC.(X)$$

Simpson

$$Rep.(\pi_1(X)) \longrightarrow MIC.(X)$$

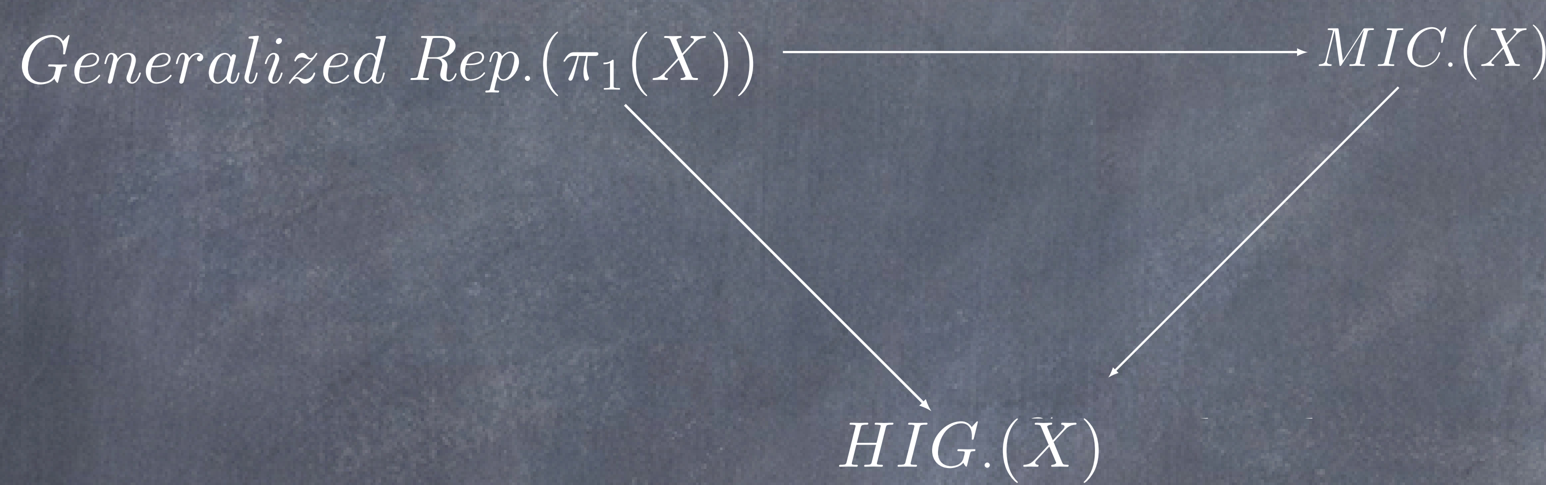
$$Rep.(\pi_1(X)) \searrow HIG.(X) \swarrow MIC.(X)$$

$$HIG.(X)$$

Somewhere: Variations of Hodge structures

Faltings

X/K smooth projective scheme, where K is a p-adic field

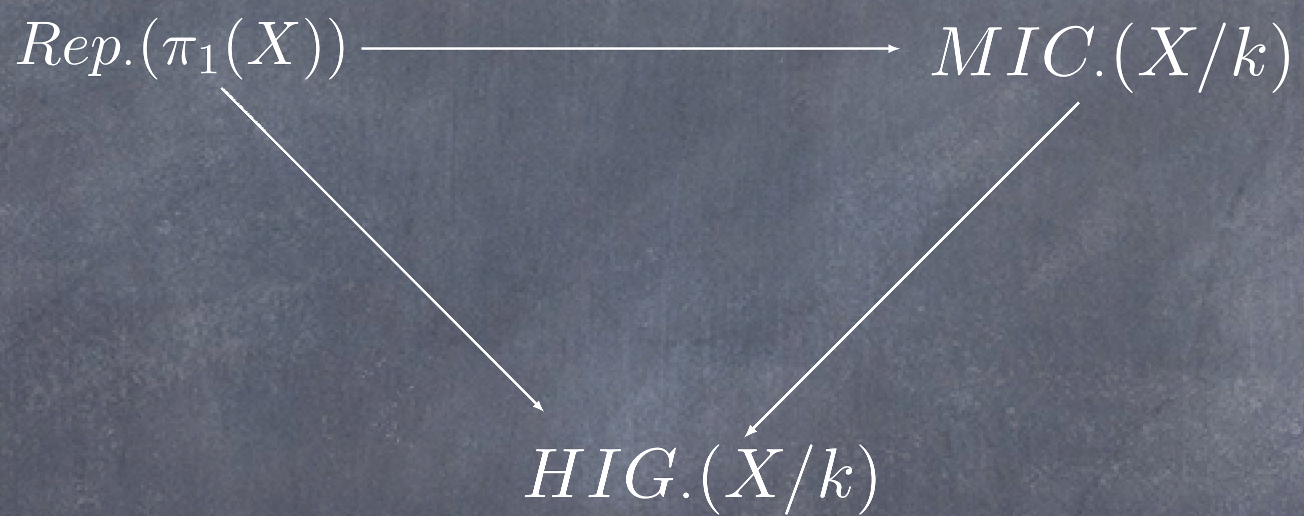


Somewhere:

Fontaine-modules on X, representations

O-Vologodsky

X/k smooth scheme, where k has characteristic p,
(with a lifting $\tilde{X}$ of X mod p^2.)



Somewhere: Fontaine modules on $\tilde{X}$

Review

- Cartier isomorphism
- De Rham decomposition (Deligne-Illusie)

Notation and setup

S scheme in characteristic p.

$\tilde{S}$ flat over $\mathbf{Z}/p^2\mathbf{Z}$, lifting S

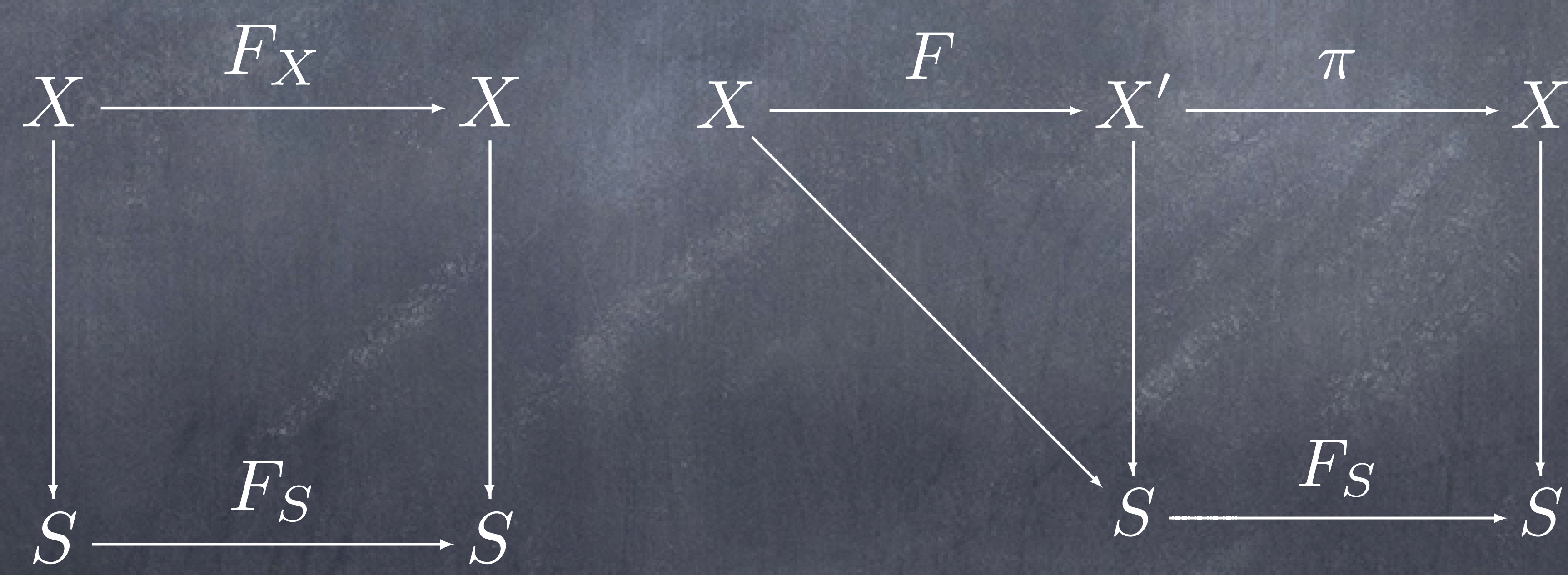
X/S smooth morphism

S scheme in characteristic p.

$\tilde{S}$ flat over $\mathbf{Z}/p^2\mathbf{Z}$, lifting S

X/S smooth morphism

$X' := X \times_{F_S} S$



Cartier Isomorphism

$$C_{X/S}^{-1}\colon \Omega^q_{X'/S} \xrightarrow{\cong} \mathcal{H}^q(F_*\Omega^{\cdot}_{X/S})$$

If $\tilde{F}\colon \tilde{X} \to \tilde{X}'$ lifts F, then $\tilde{F}^*\colon \Omega^q_{\tilde{X}'/\tilde{S}} \to \Omega^q_{\tilde{X}/\tilde{S}}$ is divisible by p^q, and we get

$$\begin{array}{ccc}
\Omega^q_{\tilde{X}'/\tilde{S}} & \xrightarrow{p^{-q}\tilde{F}^*} & F_*\mathcal{Z}^q_{X/S} \\
\downarrow & \nearrow \zeta^q_{\tilde{F}} & \downarrow \\
\Omega^q_{X'/S'} & \xrightarrow{C^{-1}_{X/S}} & F_*\mathcal{H}^q(\Omega^{\cdot}_{X/S})
\end{array}$$

Deligne-Illusie

If $\dim(X/S) < p$, a lifting $\tilde{X}'/\tilde{S}$ of X' gives an isomorphism in $D(X'/S)$:

$$C^{\cdot}_{\tilde{X}'/S}\colon (\Omega^{\cdot}_{X'/S}, 0) \sim F_*(\Omega^{\cdot}_{X/S}, d)$$

hence for every n:

$$\bigoplus_{i+j=n} H^i(X', \Omega^j_{X'/S}) \cong H^n_{DR}(X/S)$$

The Cartier Transform

Theorem: A lifting $\tilde{X}'/\tilde{S}$ of X'/S induces an equivalence of categories:

$$C_{\tilde{X}'/\tilde{S}} \colon MIC_m(X/S) \longrightarrow HIG_m(X'/S)$$

if $m < p$.

Variant: An equivalence of tensor categories:

$$C_{\tilde{X}'/\tilde{S}} \colon MIC_{\cdot}^{\gamma}(X/S) \longrightarrow HIG_{\cdot}^{\gamma}(X'/S)$$

What does " level m " mean?

$HIG_\cdot$ means *nilpotent* Higgs fields: There exists an increasing ψ-stable filtration $N_\cdot$ with $\mathrm{Gr}^N_\cdot(\psi) = 0$.

$MIC_\cdot$ means *nilpotent* connections: There exists an increasing ∇-stable filtration $N_\cdot$ such that $\mathrm{Gr}^N_\cdot(\nabla)$ is *p-integrable*.

Better: Add the filtration to the data.

"Level m" means $N_{-1}E = 0$, $N_m E = E$.

The "γ" means divided powers.

p -integrability

$$T_{X/S} \to T_{X/S} : D \mapsto D^{(p)}$$

(pth iterate of a derivation)

$$(E, \nabla) \in MIC(X/S) \quad \nabla\colon T_{X/S} \to \mathrm{End}_{\mathcal{O}_S}(E)$$

Def: ∇ is "p-integrable" if $\nabla_D^p = \nabla_{D^{(p)}}$ for all D.

Thm: iff $(F^*(E^\nabla), d \otimes \mathrm{id}) \to (E, \nabla)$ is an isomorphism.

p-curvature

$$\psi\colon T_{X/S} \to \mathrm{End}_{\mathcal{O}_S}(E) : D \mapsto \nabla_D^p - \nabla_{D^{(p)}}$$

In fact, $[\psi_{D_1}, \psi_{D_2}] = 0$ and

$$\psi\colon T_{X/S} \to F_{X*}(\mathrm{End}_{\mathcal{O}_X}(E, \nabla))$$

$$\psi\colon E \to E \otimes F^*(\Omega^1_{X'/S})$$

"F-Higgs field"

Differential Operators

$D_{X/S}$ is the sheaf of PD differential operators on X/S

(generated by $T_{X/S}$ over $\mathcal{O}_X$).

$$D \mapsto D^p - D^{(p)} : T_{X/S} \to Z_{D_{X/S}}$$

$$c\colon S^{\cdot}T_{X'/S} \cong F_*(Z_{D_{X/S}})$$

Theorem: $D_{X/S}$ is an Azumaya algebra of rank p^{2d}

Level one

$$C^{-1}_{\tilde{X}'/\tilde{S}}: HIG_1(X'/S) \to MIC_1(X/S)$$

For example

$$EXT^1_{HIG}(\mathcal{O}_{X'}, \mathcal{O}_{X'}) \xrightarrow{\cong} EXT^1_{MIC}(\mathcal{O}_X, \mathcal{O}_X)$$

$$\begin{array}{ccc}
EXT^1_{HIG}(\mathcal{O}_{X'},\mathcal{O}_{X'}) & \xrightarrow{\cong} & EXT^1_{MIC}(\mathcal{O}_X,\mathcal{O}_X) \\
\downarrow & & \downarrow \\
Ext^1_{HIG}(\mathcal{O}_{X'},\mathcal{O}_{X'}) & \xrightarrow{\cong} & Ext^1_{MIC}(\mathcal{O}_X,\mathcal{O}_X) \\
\cong\downarrow & & \downarrow\cong \\
H^0(\Omega^1_{X'/S}) \oplus H^1(\mathcal{O}_{X'}) & \xrightarrow{\cong} & H^1_{DR}(X/S)
\end{array}$$

Especially:

$$H^0(X',\Omega^1_{X'/S}) \longrightarrow EXT^1_{MIC}(\mathcal{O}_X,\mathcal{O}_X)$$

The Universal Extension

- “Universal” element of $MIC_1(X/S)$
- Similar to construction of universal extension in log geometry (Kato-Nakayama);

Theorem: Given $\tilde{X}'/\tilde{S}$, there exists a natural object of $MIC_1(X/S)$:

$$\Xi := 0 \to (\mathcal{O}_X, d) \to (\mathcal{E}_{\tilde{X}'/\tilde{S}}, \nabla) \to (F^*\Omega^1_{X'/S}, d) \to 0$$

such that

- The boundary map ∂ on cohomology

  $$H^0(X', \Omega^1_{X'/S}) = H^0_{DR}(X, F^*\Omega^1_{X'/S}) \to H^1_{DR}(X/S)$$

  is the Deligne-Illusie map (up to sign).

- The boundary map ∂ on cohomology sheaves induces $-C^{-1}_{X/S}$

  $$\mathcal{H}^0_{DR}(F^*\Omega^1_{X'/S}) \cong \Omega^1_{X'/S} \to \mathcal{H}^1_{DR}(\mathcal{O}_X).$$

$$\Xi := 0 \to (\mathcal{O}_X, d) \to (\mathcal{E}_{\tilde{X}'/\tilde{S}}, \nabla) \to (F^*\Omega^1_{X'/S}, d) \to 0$$

- The p-curvature ψ induces the identity map $F^*\Omega^1_{X'/S} \to F^*\Omega^1_{X'/S}$.

- Its class in $Ext^1(F^*\Omega^1_{X'/S}, \mathcal{O}_X) \cong H^1(F^*T_{X'/S})$ is the obstruction ξ to finding a lift $\tilde{F}$ of F.

Build it:

Choose local lifts $\tilde{F}: \tilde{U} \to \tilde{U}'$, $\zeta_{\tilde{F}}: \Omega^1_{X'/S} \to F_*(Z^1_{X/S})$

On U, let $\mathcal{E}_{\tilde{X}'/\tilde{S}} := \mathcal{O}_X \oplus F^*\Omega^1_{X'/S}$

and $\nabla(f, \omega') = (df - \zeta_{\tilde{F}}(\omega'), 0)$

Adjust gluing: On $U_1 \cap U_2$ have $\tilde{F}_2 - \tilde{F}_1 = \xi_{21} \in F^*T_{X'/S}$. Use

$$\exp \begin{pmatrix} 0 & \xi_{21} \\ 0 & 0 \end{pmatrix}$$

to glue.

OR:

Find it in Nature

Let $\mathcal{L}_{\tilde{X}'/S}$ be the sheaf of Frobenius liftings

$$U \mapsto \{(\tilde{U}, \tilde{F})\}/isom$$

Naturally an $F^*T^1_{X'/S}$-torsor, whose class $\xi \in H^1(X, F^*(T_{X'/S}))$ is the obstruction to lifting $\tilde{F}$.

Represented by a relatively affine X-scheme

$$\mathbf{L}_{\tilde{X}'/\tilde{S}} := \mathrm{Spec}_X(\mathcal{A}_{\tilde{X}'/\tilde{S}})$$

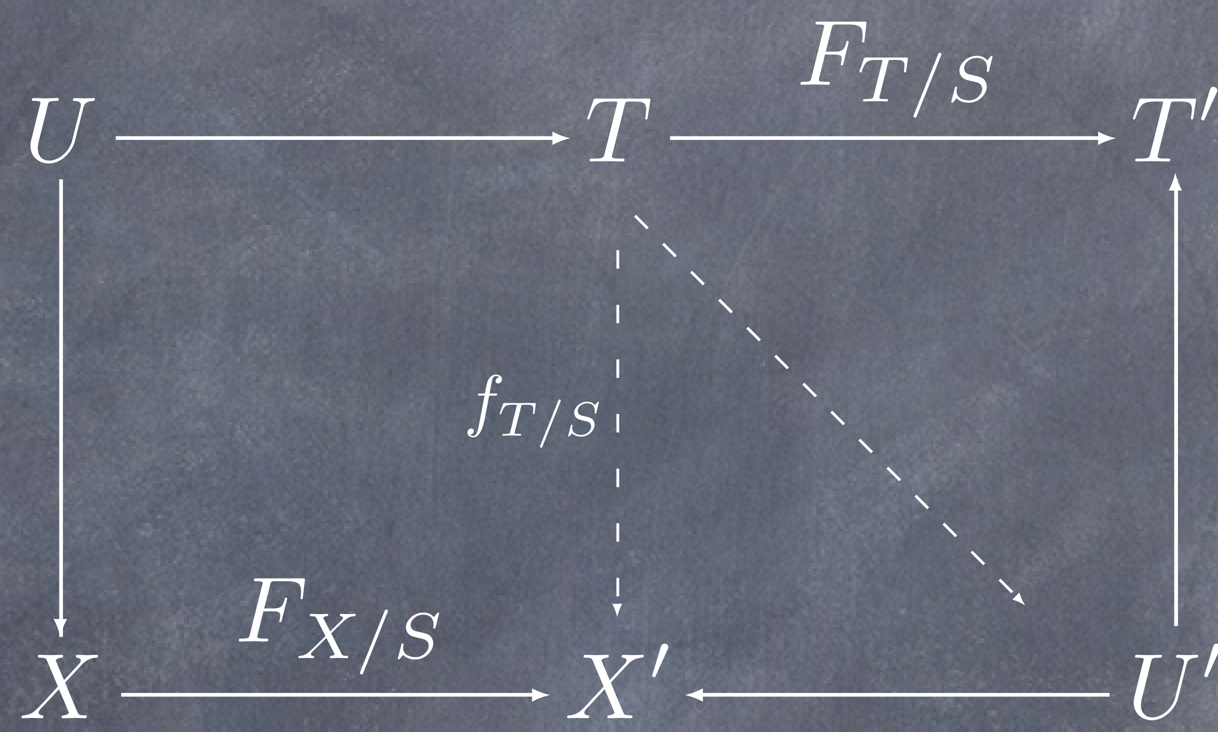
$\mathcal{E}_{\tilde{X}'/\tilde{S}} \subseteq \mathcal{A}_{\tilde{X}'/\tilde{S}}$ the affine functions

$$0 \to \mathcal{O}_X \to \mathcal{E}_{\tilde{X}'/\tilde{S}} \to F^*\Omega^1_{X'/S} \to 0$$

$$\varinjlim S^n \mathcal{E}_{\tilde{X}'/\tilde{S}} \cong \mathcal{A}_{\tilde{X}'/\tilde{S}}$$

Natural Crystal Stucture

For any $T \in Cris(X/S)$, we have



If E' is a sheaf on X', then $\{f_{T/S}^* E' : T \in Cris(X/S)\}$ is a crystal on X/S, corresponding to the Frobenius descent connection on $F^* E'$

Let $\mathcal{L}_{\tilde{X}'/S,T}$ be the sheaf of liftings of $f_{T/S}$

$$U \mapsto \{(\tilde{T}, \tilde{f}_{T/S})\}/isom$$

Sheaf, functorial in T because $dF = 0$.

Makes $\mathcal{L}_{\tilde{X}'/\tilde{S}}$, $\mathcal{A}_{\tilde{X}'/\tilde{S}}$, and $\mathcal{E}_{\tilde{X}'/\tilde{S}}$ into crystals.

Calculate ∇ and ψ of $\mathcal{L}_{\tilde{X}'/\tilde{S}}$ $\mathcal{E}_{\tilde{X}'/\tilde{S}}$, and $\mathcal{A}_{\tilde{X}'/\tilde{S}}$

Key Calculation

$$\begin{aligned}\nabla\colon \mathcal{L}_{\tilde{X}'/\tilde{S}} &\to F^*T_{X'/S}\otimes\Omega^1_{X/S}\\ &\to \mathrm{Hom}(F^*\Omega^1_{X'/S},\Omega^1_{X/S})\end{aligned}$$

Claim: $\nabla(\tilde{F}) = \zeta_{\tilde{F}} \in \mathrm{Hom}(F^*\Omega^1_{X'/S},\Omega^1_{X/S})$

$$\tilde{T} := \tilde{U}\times\tilde{U}$$

$$\begin{aligned}\nabla(\tilde{F}) &= p_2^*(\tilde{F}) - p_1^*(\tilde{F})\\ &= \tilde{F}^*\circ(p_2^*-p_1^*)\\ &= p^{-1}\tilde{F}^*\circ(p_2^*-p_1^*)\\ &= \zeta_{\tilde{F}}\circ d\end{aligned}$$

Get our desired properties!

Nicer formula for p-curvature:

$$\begin{array}{ccc}
\mathcal{A}_{\tilde{X}'/\tilde{S}} & \xrightarrow{\psi} & \mathcal{A}_{\tilde{X}'/\tilde{S}} \otimes F^*\Omega^1_{X'/S} \\
{\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle \cong} \\
\mathcal{O}_{\mathbf{L}} & \xrightarrow{d} & \Omega^1_{\mathbf{L}/X}
\end{array}$$

More:

This action of $S^{\cdot}T_{X'/S}$ on $\mathcal{A}_{\tilde{X}'/\tilde{S}}$ extends to an action of $\Gamma^{\cdot}T_{X'/S}$.

Natural filtration $N.$, with $\Gamma^i T_{X'/S} \times N_j \mathcal{A}_{\tilde{X}'/\tilde{S}} \mapsto N_{j-i}\mathcal{A}_{\tilde{X}'/\tilde{S}}$.

Object of $MIC^{\gamma}_{\cdot}(X/S)$, the category of admissibly filtered $D^{\gamma}_{X/S}$-modules.

$$D^{\gamma}_{X/S} := \Gamma^{\cdot}T_{X'/S} \otimes_{S^{\cdot}T_{X'/S}} F_* D_{X/S}.$$

The Cartier Transform

Theorem: A lifting $\tilde{X}'/\tilde{S}$ induces equivalence of tensor categories:

$$C_{\tilde{X}'/\tilde{S}} \colon MIC^{\gamma}_{\cdot}(X/S) \longrightarrow HIG^{\gamma}_{\cdot}(X'/S)$$

$$(E, \nabla, N) \mapsto (E', \psi', N)$$

$$E' := (E \otimes \mathcal{A}_{\tilde{X}'/\tilde{S}})^{(\nabla,\gamma)}$$

$$\psi' := \mathrm{id} \otimes \psi_{\mathcal{A}} = \mathrm{id} \otimes d_{\mathcal{A}/X}$$

$$C^{-1}_{\tilde{X}'/\tilde{S}} : HIG^{\gamma}_{\cdot}(X'/S) \to MIC^{\gamma}_{\cdot}(X/S)$$

$$(E', \psi', N) \mapsto (E, \nabla, N)$$

$$E := (E' \otimes \mathcal{A}_{\tilde{X}'/\tilde{S}})^{(\psi_{tot}, \gamma)}$$

$$\nabla := \mathrm{id} \otimes \nabla_{\mathcal{A}}$$

Methods of Proof

- Solve the differential equations
- Use the Azumaya property of the ring $D_{X/S}$
    - The dual of $A_{X'/S}$ is a splitting module

Cohomology

If $(E', \psi') := C_{\tilde{X}'/\tilde{S}}(E, \nabla)$, there are canonical quasi-isomorphisms:

$$\mathcal{A}^{i,j}_{\tilde{X}'/\tilde{S}} = \mathcal{A}_{\tilde{X}'/\tilde{S}} \otimes F^*\Omega^i_{X'/S} \otimes \Omega^j_{X/S}$$

$$(E \otimes \Omega^{\cdot}_{X/S}, d) \longrightarrow \mathcal{A}^{\cdot\cdot}_{\tilde{X}'/\tilde{S}}(E) \longleftarrow (E' \otimes \Omega^{\cdot}_{X'/S}, \psi')$$

if the level m of E is less than $p - d$.

$$
\begin{CD}
E' \otimes \Omega^2_{X'/S} @>>> E \otimes \mathcal{A}^{0,2}_{n-2} @>d>> E \otimes \mathcal{A}^{1,2}_{n-2} @>d>> E \otimes \mathcal{A}^{2,2}_{n-2} @>>> \cdots \\
@A{\psi'}AA @A{\mathrm{id} \otimes \psi}AA @A{\mathrm{id} \otimes \psi}AA @A{\mathrm{id} \otimes \psi}AA @. \\
E' \otimes \Omega^1_{X'/S} @>>> E \otimes \mathcal{A}^{0,1}_{n-1} @>d>> E \otimes \mathcal{A}^{1,1}_{n-1} @>d>> E \otimes \mathcal{A}^{2,1}_{n-1} @>>> \cdots \\
@A{\psi'}AA @A{\mathrm{id} \otimes \psi}AA @A{\mathrm{id} \otimes \psi}AA @A{\mathrm{id} \otimes \psi}AA @. \\
E' @>>> E \otimes \mathcal{A}^{0,0}_{n} @>d>> E \otimes \mathcal{A}^{1,0}_{n} @>d>> E \otimes \mathcal{A}^{2,0}_{n} @>>> \cdots \\
@. @AAA @AAA @AAA @. \\
@. E @>d>> E \otimes \Omega^1_{X/S} @>d>> E \otimes \Omega^2_{X/S} @>>> \cdots
\end{CD}
$$

Summary

- Lift of X′/S induces an equivalence betweeen $MIC_m(X/S)$ and $HIG_m(X'/S)$, if $m < p$.
- This equivalence is a categorification of the Deligne-Illusie decomposition
- It is compatible with cohomology