Hodge cohomology of invertible sheaves

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February 26, 2008

1 Introduction

Let k be an algebraically closed field and let X/k be a smooth projective connected k-scheme. Let L be an invertible sheaf on X, and for each integer m, let

$$H^m_{Hdg}(X/k, L) := \bigoplus_{a+b=m} H^b(X, L \otimes \Omega^a_{X/k}).$$

We wish to study how the dimensions of the k-vector spaces $H^m_{Hdg}(X/k, L)$ and $H^b(X, L \otimes \Omega^a_{X/k})$ vary with L. For example, if k has characteristic zero, Green and Lazarsefeld [4] proved that for given i, j, m, the subloci

$$\{L \in \mathrm{Pic}^0(X) : \dim\ H^i(X, \Omega^j_X \otimes L) \geq m\}$$

of $\mathrm{Pic}^0(X)$ are translates of abelian subvarieties, and Simpson [12] showed that they in fact are translates by torsions points. Both these papers use analytic methods, but Pink and Roessler [10] obtained the same results purely algebraically, using the technique of mod p reduction and the decomposition theorem of Deligne-Illusie. A key point of their proof is the fact that if that if $L^n \cong \mathcal{O}_X$ for some positive integer n, then for all natural numbers a with $(a, n) = 1$ one has

$$\dim H^m_{Hdg}(X/k, L) = \dim H^m_{Hdg}(X/k, L^a) \tag{1}$$

([10, Proposition 3.5]). They conjecture that equation 1 remains true in characteristic $p > 0$ if X/k lifts to $W_2(k)$ and has dimension $\leq p$. The purpose of this note is to discuss a few aspects of this conjecture and some variants.

Our main result (see Theorem 7) says that the conjecture is true if $n = p$ and X is ordinary in the sense of Bloch-Kato [2, Definition 7.2]. We also explain in section 2 some motivic variants of (1) and, in particular in Proposition 1, a proof (due to Pink and Roessler) of the characteristic zero case of (1), using the language of Grothendieck Chow motives. See [7, 9.3] for a discussion of a related problem using similar techniques. We should remark that there are also some log versions of these questions, which we will not make explicit.

Acknowledgements: We thank D. Roessler for explaining to us his and R. Pink's analytic proof of equation 1. We thank the referee for very useful, accurate and friendly remarks which helped us improving the exposition of this note.

2 Motivic variants

Question 1 *Let X be a smooth projective connected variety defined over an algebraically closed field k. Let L be an invertible sheaf on X and n a positive integer such that $L^n \cong \mathcal{O}_X$. Is*

$$\dim H^m_{Hdg}(X/k, L^i) = \dim H^m_{Hdg}(X/k, L)$$

for every i relatively prime to n?

Let us explain how this question can be given a motivic interpretation. We refer to [11] for the definition of Grothendieck's Chow motives over a field k. In particular, objects are triples (Y, p, n) where Y is a smooth projective variety over k, p is an element $CH^{\dim(Y)}(Y \times_k Y) \otimes \mathbb{Q}$, the rational Chow group of $\dim(Y)$-cycles, which, as a correspondence, is an idempotent, and n is a natural number.

Let $\pi : Y \to X$ be a principal bundle under a k-group scheme μ, where X and Y are smooth and projective over k. Recall that this means that there is a k-group scheme action $\mu \times_k Y \to Y$ with the property that one has an isomorphism

$$(\xi, y) \mapsto (y, \xi y) : \mu \times_k Y \cong Y \times_X Y \subseteq Y \times_k Y.$$

Thus a point $\xi \in \mu(k)$ defines a closed subset Γ_ξ of $Y \times_k Y$, the graph of the endomorphism of Y defined by ξ. The map $\xi \mapsto \Gamma_\xi$ extends uniquely to a map of $\mathbb{Q}$-vector spaces

$$\Gamma : \mathbb{Q}[\mu(k)] \to CH^{\dim(Y)}(Y \times_k Y) \otimes \mathbb{Q}.$$

Here $\mathbb{Q}[\mu(k)]$ is the $\mathbb{Q}$-group algebra, so the product structure is induced by the product of k-roots of unity. We can think of $CH^{\dim(Y)}(Y \times_k Y) \otimes \mathbb{Q}$ as a $\mathbb{Q}$-algebra of correspondences acting on $CH^*(Y) \otimes \mathbb{Q}$, where for $\beta \in CH^s(Y) \otimes \mathbb{Q}, \gamma \in CH^{\dim(Y)}(Y \times_k Y) \otimes \mathbb{Q}$, one defines as usual

$$\gamma \cdot \beta := (p_2)_*(\gamma \cup p_1^* \beta).$$

Then the map Γ is easily seen to be compatible with composition, as on closed points $y \in Y$ one has $\Gamma_\xi(y) = \xi \cdot y$. In particular if $\xi \in \mathbb{Q}[\mu]$ is idempotent in the group ring $\mathbb{Q}[\mu(k)]$, then $\Gamma_\xi \cong Y \times \xi$ is idempotent as a correspondence. In this case we let Y_ξ be the Grothendieck Chow motive $(Y, \xi, 0)$.

Let L be an n-torsion invertible sheaf on smooth irreducible projective scheme X/k. Recall that the choice of an $\mathcal{O}_X$-isomorphism $L^n \xrightarrow{\alpha} \mathcal{O}_X$ defines an $\mathcal{O}_X$-algebra structure on

$$\mathcal{A} := \bigoplus_{i=0}^{n-1} L^i \tag{2}$$

via the tensor product $L^i \times L^j \to L^i \otimes_{\mathcal{O}_X} L^j = L^{i+j}$ for $i + j < n$ and its composition with the isomorphism $L^i \times L^j \to L^i \otimes_{\mathcal{O}_X} L^j = L^{i+j} \xrightarrow{\alpha^{-1}} L^{i+j-n}$ for $0 \le i + j - n$. Then the corresponding X-scheme $\pi : Y := \operatorname{Spec}_X \mathcal{A} \to X$ is a torsor under the group scheme μ_n of nth roots of unity. Indeed, locally Zariski on X, $\mathcal{A} \cong \mathcal{O}_X[t]/(t^n - u)$ for a local unit u, the μ_n-action is defined by $\mathcal{A} \to \mathcal{A} \otimes \mathbb{Q}[\zeta]/(\zeta^n - 1),\ t \mapsto t\zeta$, and the torsor structure is given by $\mathcal{A} \otimes \mathbb{Q}[\zeta]/(\zeta^n - 1) \cong \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A},\ (t, \zeta) \mapsto (t, t\zeta)$. This construction defines an equivalence between the category of pairs (L, α) and the category of μ_n-torsors over X. Assuming now that n is invertible in k, μ_n is étale, hence π is étale and Y is smooth and projective over k. Note the character group $X_n := \operatorname{Hom}(\mu_n, \mathbf{G}_m)$ is cyclic of order n with a canonical generator (namely, the inclusion $\mu_n \to \mathbf{G}_m$). By construction, the direct sum decomposition (2) of $\mathcal{A}$ corresponds exactly to its eigenspace decomposition according to the characters of μ_n.

We can now apply the general construction of motives to this situation. Since μ_n is étale over the algebraically closed field k, it is completely determined by the finite group $\Gamma := \mu_n(k)$, which is cyclic of order n. The group algebra $\mathbb{Q}[\Gamma]$ is a finite separable algebra over $\mathbb{Q}$, hence is a product of fields:

$$\mathbb{Q}[\Gamma] = \prod E_e.$$

Here $E_e = \mathbb{Q}[T]/(\Phi_e(T)) = \mathbb{Q}(\xi_e)$, where e is a divisor of n, $\Phi_e(T)$ is the cyclotomic polynomial, and ξ_e is a primitive eth root of unity. There is an (indecomposable) idempotent e corresponding to each of these fields, and for each e we find a Chow motive Y_e.

The indecomposable idempotents of $\mathbb{Q}[\Gamma]$ can also be thought of as points of the spectrum T of $\mathbb{Q}[\Gamma]$. If K is a sufficiently large extension of $\mathbb{Q}$, then

$$T(K) = \mathrm{Hom}_{\mathrm{Alg}}(\mathbb{Q}[\Gamma], K) = \mathrm{Hom}_{\mathrm{Gr}}(\Gamma, K^*), \tag{3}$$

$$\text{and } K \otimes \mathbb{Q}[\Gamma] \cong K[\Gamma] \cong K^{T(K)}. \tag{4}$$

Thus $T(K)$ can be identified with the character group X_n of Γ, and is canonically isomorphic to $\mathbf{Z}/n\mathbf{Z}$, with canonical generator the inclusion $\Gamma \subseteq k$. Suppose that $K/\mathbb{Q}$ is Galois. Then $\mathrm{Gal}(K/\mathbb{Q})$ acts on $T(K)$, and the points of T correspond to the $\mathrm{Gal}(K/\mathbb{Q})$-orbits. By the theory of cyclotomic extensions of $\mathbb{Q}$, this action factors through a surjective map

$$\mathrm{Gal}(K/\mathbb{Q}) \to (\mathbf{Z}/n\mathbf{Z})^*$$

and the usual action of $(\mathbf{Z}/n\mathbf{Z})^*$ on $\mathbf{Z}/n\mathbf{Z}$ by multiplication. Thus the orbits correspond precisely to the divisors d of n; we shall associate to each orbit S the index d of the subgroup of $\mathbf{Z}/n\mathbf{Z}$ generated by any element of S. (Note that in fact the image of d in $\mathbf{Z}/n\mathbf{Z}$ belongs to S.) We shall thus identify the indecomposable idempotents of $\mathbb{Q}[\Gamma]$ and the divisors of n.

Let us suppose that $k = \mathbf{C}$. Then we can consider the Betti cohomologies of X and Y, and in particular the group algebra $\mathbb{Q}[\Gamma]$ operates on $H^m(Y, \mathbb{Q})$. We can thus view $H^m(Y, \mathbb{Q})$ as a $\mathbb{Q}[\Gamma]$-module, which corresponds to a coherent sheaf $\tilde{H}^m(Y, \mathbb{Q})$ on T. If e is an idempotent of $\mathbb{Q}[\Gamma]$, then $H^m(Y_e, \mathbb{Q})$ is the image of the action of e on $H^m(Y, \mathbb{Q})$, or equivalently, it is the stalk of the sheaf $\tilde{H}^m(Y, \mathbb{Q})$ at the point of T corresponding to e, or equivalently, it is $H^m(Y, \mathbb{Q}) \otimes E_e$ where the tensor product is taken over $\mathbb{Q}[\Gamma]$. If K is a sufficiently large field as above, then equation (4) induces an isomorphism of K-vectors spaces:

$$H^m(Y_e, \mathbb{Q}) \otimes_{\mathbb{Q}} K \cong \bigoplus \{H^m(Y, K)_t : t \in T^e(K)\},$$

where here $T^e(K)$ means the set of points of $T(K)$ in the Galois orbit corresponding to e, and $H^m(Y,K)_t$ means the t-eigenspace of the action of Γ on $H^m(Y,\mathbb{Q})\otimes_{\mathbb{Q}}K$. The de Rham and Hodge cohomologies of Y_e are defined in the same way: they are the images of the actions of the idempotent e acting on the k-vector spaces $H_{DR}(Y/k)$ and $H_{Hdg}(Y/k)$.

The following result is due to Pink and Roessler. Their article [10] contains a proof using reduction modulo p techniques and the results of [3]; the following analytic argument is based on oral communications with them.

Proposition 1 *The answer to question 1 is affirmative if k is a field of characteristic zero.*

Proof: As both sides of the equality in Question 1 satisfy base change with respect to field extensions, we may assume that $k = \mathbf{C}$. Let $i \to t_i$ denote the isomorphism $\mathbf{Z}/n\mathbf{Z} \cong T(\mathbf{C})$. For each divisor e of n there is a corresponding idempotent e of $\mathbb{Q}[\Gamma] \subseteq K[\Gamma]$, the sum over all i such that $t_i \in T^e(\mathbf{C})$. Consider the Hodge cohomology of the motive Y_e:

$$\begin{aligned} H^m_{Hdg}(Y_e/\mathbf{C}) := H^m_{Hdg}(Y/\mathbf{C}) \otimes_{\mathbb{Q}[\Gamma]} E_e &\cong H^m_{Hdg}(Y/\mathbf{C}) \otimes_{\mathbf{C}[\Gamma]} (\mathbf{C}\otimes E_e). \\ &\cong \bigoplus \{H^m_{Hdg}(Y/\mathbf{C})_i : i \in T^e(k)\}. \end{aligned}$$

Since $\pi\colon Y \to X$ is finite and étale,

$$\begin{aligned} H^b(Y,\Omega^a_{Y/\mathbf{C}}) \cong H^b(X,\pi_*\pi^*\Omega^a_{X/\mathbf{C}}) &\cong H^b(X,\Omega^a_{X/\mathbf{C}}\otimes \pi_*\mathcal{O}_Y) \\ &\cong \bigoplus \{H^b(X,\Omega^a_{X/\mathbf{C}}\otimes L^i) : i \in \mathbf{Z}/n\mathbf{Z}\}. \end{aligned}$$

Thus

$$H^m_{Hdg}(Y/\mathbf{C}) \cong \bigoplus \{H^m_{Hdg}(X,L^i) : i \in \mathbf{Z}/n\mathbf{Z}\},$$

and hence from the explicit description of the action of μ_n on $\mathcal{A}$ above it follows that

$$H^m_{Hdg}(Y_e/\mathbf{C}) = \bigoplus \{H^m_{Hdg}(X,L^i) : i \in T_e(\mathbf{C})\}.$$

The Hodge decomposition theorem for Y provides us with an isomorphism:

$$H^m_{Hdg}(Y/\mathbf{C}) \cong \mathbf{C}\otimes H^m(Y,\mathbb{Q}),$$

compatible with the action of $\mathbb{Q}[\Gamma]$. This gives us, for each idempotent e, an isomorphism of $\mathbf{C}\otimes E_e$-modules.

$$H^m_{Hdg}(Y_e/\mathbf{C}) \cong \mathbf{C}\otimes H^m(Y_e,\mathbb{Q}).$$

The action on $\mathbf{C} \otimes H^m(Y_e, \mathbb{Q})$ on the right just comes from the action of E_e on $H^m(Y_e, \mathbb{Q})$ by extension of scalars. Since E_e is a field, $H^m(Y_e, \mathbb{Q})$ is free as an E_e-module, and hence the $\mathbf{C} \otimes E_e$-module $H^m_{Hdg}(Y_e/\mathbf{C})$ is also free. It follows that its rank is the same at all the points $t \in T^e(\mathbf{C})$, affirming Question 1. □

Let us now formulate an analog of Question 1 for the ℓ-adic and crystalline realizations of the motive Y_e in characteristic p.

Question 2 *Suppose that k is an algebraically closed field of characteristic p and $(n, p) = 1$. Let ℓ be a prime different from p, let e be a divisor of n, and let E_e be the corresponding factor of $\mathbb{Q}[\Gamma]$. Is it true that each $H^m(Y_e, \mathbb{Q}_\ell)$ is a free $\mathbb{Q}_\ell \otimes E_e$-module? And is it true that $H^m_{cris}(Y_e/W) \otimes \mathbb{Q}$ is a free $W \otimes E_e$-module, where $W := W(k)$?*

If K is an extension of $\mathbb{Q}_\ell$ (resp. of $W(k)$) which contains a primitive nth root of unity, then as above we have a eigenspace decompositions:

$$K \otimes H^m(Y_{\acute{e}t}, \mathbb{Q}_\ell) \cong \bigoplus \{H^m(Y_{\acute{e}t}, K)_t : t \in T(K)\} \qquad (5)$$

$$K \otimes H^m(Y_{cris}/W(k)) \cong \bigoplus \{H^m(Y_{cris}, K)_t : t \in T(K)\}, \qquad (6)$$

and this question asks whether the K-dimension of the t-eigenspace is constant over the orbits $T_e(K) \subseteq T(K)$.

We show in the sequel that the question has a positive answer.

Suppose first that X/k lifts to characteristic zero, *i.e.*, that there exists a complete discrete valuation ring V with residue field k and fraction field of characteristic zero and a smooth proper $\tilde{X}/V$ whose special fiber is X/k. Let X_m be the closed subscheme of $\tilde{X}$ defined by π^{m+1}, where π is a uniformizing parameter of V. Choose a trivialization α of L^n. It follows from Theorem 18.1.2 of [6] that the étale μ_n-torsor Y on X corresponding to (L, α) lifts to X_m, uniquely up to a unique isomorphism, and hence that the same is true for (L, α). This fact can also be seen by chasing the exact sequences of cohomology corresponding to the commutative diagram of exact sequences in the étale topology

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_X & \xrightarrow{n\ \cong} & \mathcal{O}_X & & \\
 & & & & \downarrow{\scriptstyle a\mapsto 1+\pi^m a} & & \downarrow{\scriptstyle a\mapsto 1+\pi^m a} & & \\
1 & \longrightarrow & \mu_n & \longrightarrow & \mathcal{O}^\times_{X_m} & \xrightarrow{n} & \mathcal{O}^\times_{X_m} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle =} & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mu_n & \longrightarrow & \mathcal{O}^\times_{X_{m-1}} & \xrightarrow{n} & \mathcal{O}^\times_{X_{m-1}} & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
\qquad (7)
$$

By Grothendieck's fundamental theorem for proper morphisms, it follows that (L, α) and Y lift to $(\tilde{L}, \tilde{\alpha})$ and $\tilde{Y}$ on $\tilde{X}$. Then by the étale to Betti and Betti to crystalline comparison theorems, we see that under the lifting assumption, the answer to Question 2 is affirmative.

In fact, the lifting hypothesis is superfluous, but this takes a bit more work.

Claim 2 *The answer to Question 2 is affirmative.*

Proof: It is trivially true that $H^m(Y_e, \mathbb{Q}_\ell)$ is free over $\mathbb{Q}_\ell \otimes E_e$ if $\mathbb{Q}_\ell \otimes E_e$ is a field. If $(\ell, n) = 1$, this is the case if and only if $(\mathbf{Z}/e\mathbf{Z})^*$ is cyclic and generated by ℓ. More generally, assuming ℓ is relatively prime to n, there is a decomposition of $\mathbb{Q}_\ell[\Gamma]$ into a product of fields $\mathbb{Q}_\ell[\Gamma] \cong \prod E_{\ell,e}$, where now e ranges over the orbits of $\mathbf{Z}/n\mathbf{Z}$ under the action of the cyclic subgroup of $(\mathbf{Z}/n\mathbf{Z})^*$ generated by ℓ. This is indeed the unramified lift of the decomposition of $A = \mathbb{F}_\ell[\Gamma]$ into a product of finite extensions of $\mathbb{F}_\ell$, corresponding to the orbits of Frobenius on the geometric points of A. This shows at least that the dimension of $H^m(Y, K)_t$ in (5) is, as a function of t, constant over the ℓ-orbits.

For the general statement, let K be an algebraically closed field containing $\mathbb{Q}_\ell$ for all primes $\ell \neq p$, and containing $W(k)$. For $\ell \neq p$ let $V_\ell := H^m(Y_{\acute{e}t}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}\ell} K$, and let $V_p := H^m(Y_{cris}, W(k)) \otimes_{W(k)} K$. Then each V_ℓ is a finite-dimensional representation of Γ, and the isomorphisms (5) and (6) are just its decomposition as a direct sum of irreducible representations:

$$V_\ell \cong \bigoplus \{n_{\ell,i} V_i : i \in \mathbf{Z}/n\mathbf{Z}\},$$

where $V_i = K$, with $\gamma \in \Gamma$ acting by multiplication by γ^i. By [8, Theorem 2.2)] (and [1], [5] and [9] for the existence of cycle classes in crystalline cohomology) the trace of any $\gamma \in \Gamma$ acting on V_ℓ is an integer independent of ℓ, including $\ell = p$. Since Γ is a finite group, it follows from the independence of characters that for each i, $n_i := n_{\ell,i}$ is independent of ℓ. We saw above that $n_{\ell,\ell i} = n_{\ell,i}$ if $(\ell, n) = 1$ and $\ell \neq p$, so that in fact $n_{\ell i} = n_i$ for all $\ell \neq p$ with $(\ell, n) = 1$. Since the group $(\mathbf{Z}/n\mathbf{Z})^*$ is generated by all such ℓ, it follows that n_i is indeed constant over the ℓ-orbits. □

What does this tell us about Question 1? If $(p, n) = 1$ and k is algebraically closed, $W[\Gamma]$ is still semisimple, and can be written canonically as a product of copies of W, indexed by $i \in T(W) \cong \mathbf{Z}/n\mathbf{Z}$. For every $t \in T(W) \cong T(k)$, we have an injective base change map from crystalline to de Rham cohomology: $k \otimes H^m(Y/W)_t \to H^m(Y/k)_t$.

Question 3 *In the above situation, is $H^q(Y/W)$ torsion free when $(p, n) = 1$?*

If the answer is yes, then the maps $k \otimes H^m(Y/W)_t \to H^m(Y/k)_t$ are isomorphisms, and this means that we can compute the dimensions of the de Rham eigenspaces from the ℓ-adic ones. Assuming also that the Hodge to de Rham spectral sequence of Y/k degenerates at E_1, this should give an affirmative answer to Question 1. Note that if X/k lifts mod p^2, then Y/k lifts mod p^2 as well, and if the dimension is less than or equal to p, the E_1-degeneration is true by [3].

Of course, there is no reason for Question 3 to have an affirmative answer in general. Is there a reasonable hypothesis on X which guarantees it? For example, is it true if the crystalline cohomology of X/W is torsion free?

3 The p-torsion case in characteristic p

Let us assume from now on that k is a perfect field of characteristic $p > 0$. In this case we can reduce question 1 to a question about connections, using the following construction of [3]. First let us recall some standard notations.

Let X' be the pull back of X via the Frobenius of k, let $\pi\colon X' \to X$ be the projection, and let $F : X \to X'$ and $F_X\colon X \to X$ be the relative and absolute Frobenius morphisms. Then $F_X^* L = L^p = F^* L'$, where $L' := \pi^* L$. Then $L^p = F^{-1}L' \otimes_{\mathcal{O}_{X'}} \mathcal{O}_X$ is endowed with the Frobenius descent connection $1 \otimes d$, *i.e.* the unique connection spanned by its flat sections L'. In general, for a given integrable connection (E, ∇), we set

$$H^i_{DR}(X, (E, \nabla)) = \mathbb{H}^i(X/k, (\Omega^{\cdot}_{X/k} \otimes E, \nabla)),$$

and we use again the notation

$$H^i_{Hdg}(X/k, L) = \bigoplus_{a+b=i} H^b(X, \Omega^a_{X/k} \otimes L)$$

and write h^i_{DR} and h^i_{Hdg} for the respective dimensions of these spaces.

Proposition 3 *Let L be an invertible sheaf on a smooth proper scheme X over k and let ∇ be the Frobenius descent connection on L^p. Suppose that X/k lifts to $W_2(k)$ and has dimension at most p. Then for every natural number m,*

$$h^m_{DR}(X/k, (L^p, \nabla)) = h^m_{Hdg}(X/k, L).$$

Corollary 4 *Under the assumtpions of Proposition 3, if $L^p \cong \mathcal{O}_X$ and $\omega := \nabla(1)$, then for any integer a,*

$$h^m_{Hdg}(X/k, L^a) = h^m_{DR}(X/k, (\mathcal{O}_X, d + a\omega)).$$

Remark 5 If p divides a, this just means the degeneration of the Hodge to de Rham spectral sequence for $(\mathcal{O}_X, d)$.

Proof: Let $Hdg^{\cdot}_{X'/k}$ denote the Hodge complex of X'/k, *i.e.*, the direct sum $\oplus_i \Omega^i_{X'/k}[-i]$. Recall from [3] that the lifting yields an isomorphism in the bounded derived category of $\mathcal{O}_{X'}$-modules:

$$Hdg^{\cdot}_{X'/k} \cong F_*(\Omega^{\cdot}_{X/k}, d).$$

Tensoring this isomorphism with $L' := \pi^* L$ and using the projection formula for F, we find an isomorphism

$$Hdg^{\cdot}_{X'/k} \otimes L' \cong F_*(\Omega^{\cdot}_{X/k} \otimes L^p, \nabla).$$

Hence

$$H^m_{Hdg}(X/k, L) \xrightarrow{F_k^* \,\cong} H^m_{Hdg}(X'/k, L') \xleftarrow{F_* \,\cong} H^m_{DR}(X, (L^p, \nabla)).$$

This proves the proposition. If $L^p = \mathcal{O}_X$, the corresponding Frobenius descent connection ∇ on $\mathcal{O}_X$ is determined by $\omega_L := \nabla(1)$. It follows from the tensor product rule for connections that $\omega_{L^a} = a\omega_L$ for any integer a. $\square$

The corollary suggests the following question.

Question 4 *Let ω be a closed one-form on X and let c be a unit of k. Is the dimension of $H^m_{DR}(X, (\mathcal{O}_X, d + c\omega))$ independent of c?*

Remark 6 Some properness is necessary, since the p-curvature of $d_\omega := d+\omega$ can change from zero to non-zero as one multiplies by an invertible constant. If the p-curvature is non-zero, then the sheaf $\mathcal{H}^0(\Omega^{\cdot}_{X/k}, d_\omega)$ vanishes, and hence so does $H^0(X, (\Omega^{\cdot}_{X/k}, d_\omega))$. If the p-curvature vanishes, then $\mathcal{H}^0(\Omega^{\cdot}_{X/k}, d_\omega)$ is an invertible sheaf L, which can have nontrivial sections if X is allowed to shrink. However, since by definition, $L \subset \mathcal{O}_X$, it can have a global section on a proper X only if $L = \mathcal{O}_X$.

We can answer Question 4 under a strong hypothesis.

Theorem 7 *Suppose that X/k is smooth, proper, and ordinary in the sense of Bloch and Kato [2, Definition 7.2]: $H^i(X, B^j_{X/k}) = 0$ for all i, j, where*

$$B^j_{X/k} := Im\left(d\colon \Omega^{j-1}_{X/k} \to \Omega^j_{X/k}\right).$$

Then the answer to question 4 is affirmative. Hence if X/k lifts to $W_2(k)$, has dimension at most p, and if $n = p$, the answer to Question 1 is also affirmative.

We begin with the following lemmas.

Lemma 8 *Let ω be a closed one-form on X, and let*

$$d_\omega := d + \omega\wedge \quad : \quad \Omega^{\cdot}_{X/k} \to \Omega^{\cdot+1}_{X/k}.$$

Then the standard exterior derivative induces a morphism of complexes:

$$(\Omega^{\cdot}_{X/k}, d_\omega) \xrightarrow{\delta} (\Omega^{\cdot}_{X/k}, d_\omega)[1].$$

Proof: If α is a section of $\Omega^q_{X/k}$,

$$\begin{aligned} dd_\omega(\alpha) &= d(d\alpha + \omega \wedge \alpha) \\ &= dd\alpha + d\omega \wedge \alpha - \omega \wedge d\alpha \\ &= -\omega \wedge d\alpha. \end{aligned}$$

Since the sign of the differential of the complex $(\Omega^\cdot_{X/k}, d_\omega)[1]$ is the negative of the sign of the differential of $(\Omega^\cdot_{X/k}, d_\omega)$,

$$\begin{aligned} d_\omega d(\alpha) &= -(d + \omega\wedge)(d\alpha) \\ &= -\omega \wedge d\alpha \end{aligned}$$

□

Lemma 9 *Let $Z^\cdot := \ker(d) \subseteq (\Omega^\cdot_{X/k}, d_\omega)$ and $B^\cdot := \mathrm{Im}(d)[-1] \subseteq (\Omega^\cdot_{X/k}, d_\omega)$. Then for any $a \in k^*$, multiplication by a^i in degree i induces isomorphisms*

$$(Z^\cdot, d_\omega) \xrightarrow{\lambda_a} (Z^\cdot, d_{a\omega})$$

$$(B^\cdot, d_\omega) \xrightarrow{\lambda_a} (B^\cdot, d_{a\omega}).$$

Proof: It is clear that the boundary map d_ω on $Z^\cdot$ and on $B^\cdot$ is just wedge product with ω. □

Proof of Theorem 7 The morphism δ of Lemma 8 induces an exact sequence:

$$0 \to (Z^\cdot, d_\omega) \to (\Omega^\cdot_{X/k}, d_\omega) \xrightarrow{\delta} (B^\cdot, d_\omega)[1] \to 0. \tag{8}$$

As X/k is ordinary, the E_1 term of the first spectral sequence for $(B^\cdot, d_\omega)$ is $E_1^{i,j} = H^j(X, B^i) = 0$, and it follows that the hypercohomology of $(B^\cdot, d_\omega)$ vanishes, for every ω. Hence the natural map $H^q(Z^\cdot, d_\omega) \to H^q(\Omega^\cdot_{X/k}, d_\omega)$ is an isomorphism. Since the dimension of $H^q(Z^\cdot, d_\omega)$ is unchanged when ω is multiplied by a unit of k, the same is true of $H^q(\Omega^\cdot_{X/k}, d_\omega)$. This completes the proof of Theorem 7. □

Remark 10 A simple Riemann-Roch computation shows that on curves, question 1 has a positive answer with no additional assumptions. Indeed, if L is a nontrivial torsion sheaf, then its degree is zero and it has no global sections. It follows that $h^1(L) = g - 1$. Since the same is true for L^{-1}, $h^0(L \otimes \Omega^1_X) = h^1(L^{-1}) = g - 1$, and $h^1(L \otimes \Omega^1_X) = h^0(L^{-1}) = 0$.

Remark 11 In the absence of the ordinarity hypothesis, one can ask if the rank of the boundary map

$$\partial_\omega \colon H^{q+1}(B^\cdot, \omega\wedge) \to H^{q+1}(Z^\cdot, \omega\wedge)$$

of (8) changes if ω is multiplied by a unit of k. To analyze this question, let

$$c_\omega \colon (B^\cdot, \omega\wedge) \to (Z^\cdot, \omega\wedge)$$

be the morphism in the derived category $D(X', \mathcal{O}_{X'})$ defined by the exact sequence (8), so that ∂_ω can be identified with $H^{q-1}(c_\omega)$. Similarly, the exact sequence

$$0 \to (Z^\cdot, \omega\wedge) \to (\Omega^\cdot, \omega\wedge) \to (B^\cdot, \omega\wedge)[1] \to 0$$

defines a morphism

$$a_\omega \colon (B^\cdot, \omega\wedge) \to (Z^\cdot, \omega\wedge)$$

in $D(X', \mathcal{O}_{X'})$ as well. There is also an inclusion morphism:

$$b_\omega \colon (B^\cdot, \omega\wedge) \to (Z^\cdot, \omega\wedge).$$

Then it is not difficult to check that $c_\omega = a_\omega + b_\omega$. If $a \in k^*$, we have isomorphisms of complexes

$$\begin{aligned} \lambda_a \colon (Z^\cdot, \omega\wedge) &\to (Z^\cdot, a\omega\wedge) \\ \lambda_a \colon (B^\cdot, \omega\wedge) &\to (B^\cdot, a\omega\wedge) \end{aligned}$$

Using these as identifications, one can check that $c_{a\omega} = a^{-1}a_\omega + b_\omega$. This would suggest a negative answer to Question 4, but we do not have an example.

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