Logarithmic Geometry

Arthur Ogus

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Outline

Emphasis

- What it's for
- How it works
- What it looks like

History

Founders:

Deligne, Faltings, Fontaine–Illusie, Kazuya Kato, Chikara Nakayama, many others

Log geometry in this form was invented discovered assembled in the 80's by Fontaine and Illusie with hope of studying p-adic Galois representations associated to varieties with bad reduction. Carried out by Kato, Tsuji, Faltings, and others. (The C_{st} conjecture.)

I'll emphasize geometric analogs—currently very active—today. Related to toric and tropical geometry

Motivating problem 1: Compactification

Consider

$$S^* \xrightarrow{j} S \xleftarrow{i} Z$$

j an open immersion, i its complementary closed immersion. For example: S^* a moduli space of "smooth" objects, inside some space S of "stable" objects, Z the "degenerate" locus.

Log structure is "magic powder" which when added to S "remembers S^*."

Motivating problem 2: Degeneration

Study families, i.e., morphisms

$$\begin{tikzcd}
X^* \arrow[r] \arrow[d, "f^*"'] & X \arrow[d, "f"] & Y \arrow[l, "i"'] \arrow[d, "g"] \\
S^* \arrow[r, "j"] & S & Z \arrow[l, "i"']
\end{tikzcd}$$

Here f^* is smooth but f and g are only log smooth (magic powder).
The log structure allows f and even g to somehow "remember" f^*.

Benefits

- Log smooth maps can be understood locally, (but are still much more complicated than classically smooth maps).
- Degenerations can be studied locally on the singular locus Z.
- Log geometry has natural cohomology theories:
    - Betti
    - De Rham
    - Crystalline
    - Etale

Roots and ingredients

- Toroidal embeddings and toric geometry
- Regular singular points of ODE's, log poles and differentials
- Degenerations of Hodge structures

Remark: A key difference between local toric geometry and local log geometry:

- toric geometry based on study of cones and monoids.
- log geometry based on study of morphisms of cones and monoids.

Some applications

- Compactifying moduli spaces: K3's, abelian varieties, curves, covering spaces
- Moduli and degenerations of Hodge structures
- Crystalline and étale cohomology in the presence of bad reduction—C_{st} conjecture
- Work of Gabber and others on resolution of singularities (uniformization)
- Work of Gross and Siebert on mirror symmetry

What is Log Geometry?

What is geometry? How do we do geometry?
Locally ringed spaces: Algebra + Geometry

- Space: Topological space X (or topos): $X = (X, \{U \subseteq X\})$
- Ring: $(R, +, \cdot, 1_R)$ (usually commutative)
- Monoid: $(M, \cdot, 1_M)$ (usually commutative and cancellative)

Definition

A *locally ringed space* is a pair $(X, \mathcal{O}_X)$, where

- X is a topological space (or topos)
- $\mathcal{O}_X : \{\mathcal{O}_X(U) : U \subseteq X\}$ a *sheaf of rings on* X

such that for each $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.

Example

X a complex manifold:
For each open $U \subseteq X$, $\mathcal{O}_X(U)$ is the ring of analytic functions $U \to \mathbf{C}$.
$\mathcal{O}_{X,x}$ is the set of germs of functions at x,
$m_{X,x} := \{f : f(x) = 0\}$ is its unique maximal ideal.

Example: Compactification log structures

X scheme or analytic space, Y closed algebraic or analytic subset, $X^* = X \setminus Y$

$$X^* \xrightarrow{j} X \xleftarrow{i} Y$$

Instead of the sheaf of ideals:

$$I_Y := \{a \in \mathcal{O}_X : i^*(a) = 0\} \subseteq \mathcal{O}_X$$

consider the sheaf of multiplicative submonoids:

$$\mathcal{M}_{X^*/X} := \{a \in \mathcal{O}_X : j^*(a) \in \mathcal{O}^*_{X^*}\} \subseteq \mathcal{O}_X.$$

Log structure:

$$\alpha_{X^*/X} \colon \mathcal{M}_{X^*/X} \to \mathcal{O}_X \text{ (the inclusion mapping)}$$

Notes

- This is generally useless unless codim $(Y, X) = 1$.
- $\mathcal{M}_{X^*/X}$ is a sheaf of faces of $\mathcal{O}_X$, i.e., a sheaf $\mathcal{F}$ of submonoids such that $fg \in \mathcal{F}$ implies f and $g \in \mathcal{F}$.
- There is an exact sequence:

$$0 \to \mathcal{O}_X^* \to \mathcal{M}_{X^*/X} \to \underline{\Gamma}_Y(Div_X^-) \to 0.$$

Definition of log structures

Let $(X, \mathcal{O}_X)$ be a locally ringed space (e.g. a scheme or analytic space).
A prelog structure on X is a morphism of sheaves of (commutative) monoids

$$\alpha_X \colon \mathcal{M}_X \to \mathcal{O}_X.$$

It is a log structure if

$$\alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$$

is an isomorphism. (In this case $\mathcal{M}_X^* \cong \mathcal{O}_X^*$.)

Examples:

- $\mathcal{M}_{X/X} = \mathcal{O}_X^*$, the trivial log structure
- $\mathcal{M}_{\emptyset/X} = \mathcal{O}_X$, the empty log structure .

Logarithmic spaces

A log space is a pair (X, α_X), and a morphism of log spaces is a triple $(f, f^\sharp, f^\flat)$:

$$f\colon X \to Y, f^\sharp\colon f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X, f^\flat\colon f^{-1}(\mathcal{M}_Y) \to \mathcal{M}_X$$

Just write X for (X, α_X) when possible.
If X is a log space, let $\underline{X}$ be X with the trivial log structure.
There is a canonical map of log spaces:

$$X \to \underline{X} : (X, \mathcal{M}_X \to \mathcal{O}_X) \to (X, \mathcal{O}_X^* \to \mathcal{O}_X)$$

$$(\mathrm{id} : X \to X, \mathrm{id} : \mathcal{O}_X \to \mathcal{O}_X, inc\colon \mathcal{O}_X^* \to \mathcal{M}_X)$$

Variant: Idealized log structures

Add $\mathcal{K}_X \subseteq \mathcal{M}_X$, sheaf of ideals, such that $\alpha_X\colon (\mathcal{M}_X, \mathcal{K}_X) \to (\mathcal{O}_X, 0)$.

Example: torus embeddings and toric varieties

Example

The log line: A^1, with the compactification log structure from:

$$\begin{array}{lccccc} & \mathrm{G_m} & \xrightarrow{j} & A^1 & \xleftarrow{i} & 0 \\ \text{on points:} & \mathbf{C}^* & \longrightarrow & \mathbf{C} & \longleftarrow & 0. \end{array}$$

Generalization

$$(\mathrm{G_m})^r \subseteq A_Q$$

Here $(\mathrm{G_m})^r$ is a commutative group scheme: a (noncompact) torus,

A_Q will be a *monoid scheme*, coming from a toric monoid Q, with $Q^{gp} \cong \mathbf{Z}^r$.

Notation Let Q be a cancellative commutative monoid.

$Q^* :=$ the largest group contained in Q.

$Q^{gp} :=$ the smallest group containing Q.

$\overline{Q} := Q/Q^*$.

$\operatorname{Spec} Q$ is the set of prime ideals of Q, i.e, the complements of the faces of Q.

N.B. A *face* of Q is a submonoid F which contains a and b whenever it contains $a + b$.

Terminology: We say Q is:

- integral if Q is cancellative
- fine if Q is integral and finitely generated
- saturated if Q is integral and $nx \in Q$ implies $x \in Q$, for $x \in Q^{gp}, n \in \mathbf{N}$
- toric if Q is fine and saturated and Q^{gp} is torsion free
- sharp if $Q^* = 0$.

Generalization: toric varieties

Assume Q is toric (so $Q^{gp} \cong \mathbf{Z}^r$ for some r). Let

$\underline{\mathsf{A}}^*_{\mathsf{Q}} := \operatorname{Spec} \mathbf{C}[Q^{gp}]$, a group scheme (torus). Thus

$$\underline{\mathsf{A}}^*_{\mathsf{Q}}(\mathbf{C}) = \{Q^{gp} \to \mathbf{C}^*\} \cong (\mathbf{C}^*)^r, \quad \mathcal{O}_{\underline{\mathsf{A}}^*_{\mathsf{Q}}}(\underline{\mathsf{A}}^*_{\mathsf{Q}}) = \mathbf{C}[Q^{gp}]$$

$\underline{\mathsf{A}}_{\mathsf{Q}} := \operatorname{Spec} \mathbf{C}[Q]$, a monoid scheme. Thus

$$\underline{\mathsf{A}}_{\mathsf{Q}}(\mathbf{C}) = \{Q \to \mathbf{C}\}, \quad \mathcal{O}_{\underline{\mathsf{A}}_{\mathsf{Q}}}(\underline{\mathsf{A}}_{\mathsf{Q}}) = \mathbf{C}[Q]$$

$\mathsf{A}_{\mathsf{Q}} :=$ the log scheme given by the open immersion $j\colon \underline{\mathsf{A}}^*_{\mathsf{Q}} \to \underline{\mathsf{A}}_{\mathsf{Q}}$.

Have $\Gamma(\mathcal{M}) \cong \mathbf{C}^* \oplus Q$.

Examples

- If $Q = \mathbf{N}^r$, $\underline{\mathrm{A}}_{\mathrm{Q}}(\mathbf{C}) = \mathbf{C}^r$, $\underline{\mathrm{A}}^*_{\mathrm{Q}}(\mathbf{C}) = (\mathbf{C}^*)^r$
- If Q is the submonoid of $\mathbf{Z}^4$ spanned by $\{(1,1,0,0),(0,0,1,1),(1,0,1,0),(0,1,0,1)\}$, then

$$\underline{\mathrm{A}}_{\mathrm{Q}}(\mathbf{C}) = \{(z_1, z_2, z_3, z_4) \in \mathbf{C}^4 : z_1 z_2 = z_3 z_4\}.$$

$$\underline{\mathrm{A}}^*_{\mathrm{Q}} \cong (\mathbf{C}^*)^3.$$

Pictures

Pictures of Q:

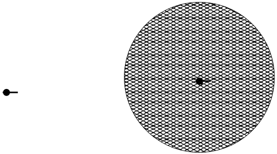
$\operatorname{Spec} Q$ is a finite topological space. Its points correspond to the orbits of the action of $\underline{\mathsf{A}}^*_{\mathsf{Q}}$ on $\underline{\mathsf{A}}_{\mathsf{Q}}$, and to the faces of the cone C_Q spanned by Q.

Pictures of a log scheme X

Embellish picture of $\underline{X}$ by attaching $\operatorname{Spec} \mathcal{M}_{X,x}$ to X at x.

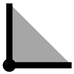
Example: The log line ($Q = \mathbf{N}$, $C_Q = \mathbf{R}_{\geq}$)

$\mathrm{Spec}(\mathbf{N})$

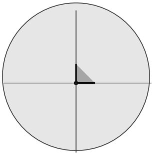


$\mathrm{Spec}(\mathbf{N} \to \mathbf{C}[\mathbf{N}])$

Example: The log plane ($Q = \mathbf{N} \oplus \mathbf{N}$, $C_Q = \mathbf{R}_{\geq} \times \mathbf{R}_{\geq}$)



$\mathrm{Spec}(\mathbf{N} \oplus \mathbf{N})$



$\mathrm{Spec}(\mathbf{N} \oplus \mathbf{N} \to \mathbf{C}[\mathbf{N} \oplus \mathbf{N}])$

Log points

The standard (hollow) log point

$t := \operatorname{Spec} \mathbf{C}$. (One point space). $\mathcal{O}_t = \mathbf{C}$ (constants)
Add log structure:

$$\alpha \colon \mathcal{M}_t := \mathbf{C}^* \oplus \mathbf{N} \to \mathbf{C} \quad (u, n) \mapsto u0^n = \begin{cases} u & if \, \mathrm{q} = 0 \\ 0 & otherwise \end{cases}$$

We usually write P for a log point.

Generalizations

- Replace $\mathbf{C}$ by any field.
- Replace $\mathbf{N}$ by any sharp monoid Q.
- Add ideal to Q.

Example: log disks

V a discrete valuation ring, e.g, $\mathbf{C}\{t\}$ (germs of holomorphic functions)
$K := frac(V)$, $m_V := max(V)$, $k_V := V/m_V$,
$\pi \in m_V$ uniformizer, $V' := V \setminus \{0\} \cong V^* \oplus \mathbf{N}$

$$T := \operatorname{Spec} V = \{\tau, t\},\ \tau := T^* := \operatorname{Spec} K,\ t := \operatorname{Spec} k.$$

Log structures on T: $\quad \Gamma(\alpha_T)\colon \Gamma(T, M_T) \to \Gamma(T, \mathcal{O}_T)$:

trivial: $\alpha_{T/T} = V^* \to V$ (inclusion): T_{triv}

standard: $\alpha_{T^*/T} = V' \to V$ (inclusion): T_{std}

hollow: $\alpha_{hol} = V' \to V$ (inclusion on V^*, 0 on m_V): T_{hol}

split$_m$ $\alpha_m = V^* \oplus \mathbf{N} \to V \quad (inc, 1 \mapsto \pi^m) : T_{spl_m}$

Note: $T_{spl_1} \cong T_{std}$ and $T_{spl_m} \to T_{hol}$ as $m \to \infty$

Inducing log structures

Pullback and pushforward

Given a map of locally ringed spaces $f : X \to Y$, we can:

Pushforward a log structure on X to Y: $f_*(\mathcal{M}_X) \to \mathcal{O}_Y$.

Pullback a log structure on Y to X: $f^*(\mathcal{M}_Y) \to \mathcal{O}_X$.

A morphism of log spaces is *strict* if $f^*(\mathcal{M}_Y) \to \mathcal{M}_X$ is an isomorphism.
A *chart* for a log space is strict map $X \to \mathrm{A}_\mathrm{Q}$ for some Q.
A log space (or structure) is *coherent* if locally on X it admits a chart.
Generalization: *relatively coherent* log structures.

Example: Log disks and log points

Let T be a log disk, t its origin. Then the log structure on T induces a log structure on t:

Log structure on T	Induced structure on t
Trivial	Trivial
Standard	Standard
Hollow	Standard
Split	Standard

Fiber products

The category of coherent log schemes has fiber products.
$\mathcal{M}_{X\times_Z Y} \to \mathcal{O}_{X\times_Z Y}$ is the log structure associated to

$$p_X^{-1}\mathcal{M}_X \oplus_{p_Z^{-1}\mathcal{M}_Z} p_Y^{-1}\mathcal{M}_Y \to \mathcal{O}_{X\times_Z Y}.$$

Danger: $\mathcal{M}_{X\times_Z Y}$ may not be integral or saturated. Fixing this can "damage" the underlying space $X \times_Z Y$.

Properties of monoid homomorphisms

A morphism $\theta\colon P \to Q$ of integral monoids is

strict if $\overline{\theta}\colon \overline{P} \to \overline{Q}$ is an isomorphism

local if $\theta^{-1}(Q^*) = P^*$

vertical if $Q/P := Im(Q \to Cok(\theta^{gp}))$ is a group.

exact if $P = (\theta^{gp})^{-1}(Q) \subseteq P^{gp}$

A morphism of log schemes $f\colon X \to Y$ has **P** if for every $x \in X$, the map $f^\flat\colon M_{Y,f(x)} \to M_{X,x}$ has **P**.

Examples of monoid homomorphisms

Examples:

- $\mathbf{N} \to \mathbf{N} \oplus \mathbf{N} : n \mapsto (n, n)$
  $\mathbf{C}^2 \to \mathbf{C} : (z_1, z_2) \mapsto z_1 z_2$
  Local, exact, and vertical.
- $\mathbf{N} \oplus \mathbf{N} \to \mathbf{N} \oplus \mathbf{N} : (m, n) \mapsto (m, m + n)$
  $\mathbf{C}^2 \to \mathbf{C}^2 : (z_1, z_2) \mapsto (z_1, z_1 z_2)$ (blowup)
  Local, not exact , vertical
- $\mathbf{N} \to Q := \langle q_1, q_2, q_3, q_4 \rangle / (q_1 + q_2 = q_3 + q_4) : n \mapsto nq_4$
  Local, exact, not vertical.

Differentials

Let $f\colon X \to Y$ be a morphism of log schemes,
Universal derivation:

$$(d, \delta) : (\mathcal{O}_X, \mathcal{M}_X) \to \Omega^1_{X/Y} \quad \text{(some write } \omega^1_{X/Y}\text{)}$$

$$d\alpha(m) = \alpha(m)\delta(m) \quad \text{so } \delta(m) = d \log m \quad \text{(sic)}$$

Geometric construction:
(gives relation to deformation theory)
Infinitesimal neighborhoods of diagonal $X \to X \times_Y X$ made strict:
$X \to \mathcal{P}^N_{X/Y}$, $\Omega^1_{X/Y} = J/J^2$.

If $\alpha_X = \alpha_{X^*/X}$ where $Z := X \setminus X^*$ is a DNC relative to Y,

$$\Omega^1_{X/Y} = \Omega^1_{\underline{X}/\underline{Y}}(\log Z)$$

In coordinates $(t_1, \ldots t_n)$, Z defined by $t_1 \cdots t_r = 0$.
$\Omega^1_{X/Y}$ has basis: $(dt_1/t_1, \ldots dt_r/t_r, dt_{r+1} \ldots dt_n)$.

Logarithmic de Rham complex

$$0 \to \mathcal{O}_X \to \Omega^1_{X/Y} \to \Omega^2_{X/Y} \cdots$$

Logarithmic connections:

$$\nabla\colon E \to \Omega^1_{X/Y} \otimes E$$

satisfying Liebnitz rule + integrability condition: $\nabla^2 = 0$.

Generalized de Rham complex

$$0 \to E \to E \otimes \Omega^1_{X/Y} \to E \otimes \Omega^2_{X/Y} \cdots$$

Smooth morphisms

The definition of smoothness of a morphism $f : X \to Y$ follows Grothendieck's geometric idea: *"formal fibration"*: Consider diagrams:

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T' \arrow[r, "h"] \arrow[ur, dashed, "g'"] & Y \end{tikzcd}$$

Here i is a strict nilpotent immersion. Then $f : X \to Y$ is

smooth if g' always exists, locally on T,
unramified if g' is always unique,
étale if g' always exists and is unique.

Examples: monoid schemes and tori

Let $\theta \colon P \to Q$ be a morphism of toric monoids. R a base ring.
Then the following are equivalent:

- $A_\theta \colon A_Q \to A_P$ is smooth
- $A^*_\theta \colon A^*_Q \to A^*_P$ is smooth
- $R \otimes \mathrm{Ker}(\theta^{gp}) = R \otimes \mathrm{Cok}(\theta^{gp})_{tors} = 0$

Similarly for étale and unramified maps.
In general, smooth (resp. unramified, étale) maps look locally like these examples.

The space X_{log} (Kato–Nakayama)

$X/\mathbf{C}$: (relatively) fine log scheme of finite type,

X_{an} : its associated log analytic space.

X_{log}: topological space, defined as follows:

Underlying set: the set of pairs (x, σ), where $x \in X_{an}$ and

$$\begin{tikzcd} \mathcal{O}^*_{X,x} \arrow[r, "x^\sharp"] \arrow[d, "\alpha_{X,x}"'] & \mathbf{C}^* \arrow[d, "\arg"] & u \arrow[d] \\ \mathcal{M}_{X,x} \arrow[r, "\sigma"] & \mathbf{S}^1 & u/|u| \end{tikzcd}$$

commutes. Hence:

$$X_{log} \xrightarrow{\tau} X_{an} \longrightarrow X$$

Each $m \in \tau^{-1}M_X$ defines a function $\arg(m)\colon X_{log} \to \mathbf{S}^1$.
X_{log} is given the weakest topology so that $\tau\colon X_{log} \to X_{an}$ and all $\arg(m)$ are continuous.
Get $\tau^{-1}\mathcal{M}_X^{gp} \xrightarrow{\arg} \underline{\mathbf{S}}^1$ extending arg on $\tau^{-1}\mathcal{O}_X^*$.

Define *sheaf of logarithms of sections of* $\tau^{-1}\mathcal{M}_X^{gp}$:

$$\begin{tikzcd}
\mathcal{L}_X \arrow[r] \arrow[d] & \tau^{-1}\mathcal{M}_X^{gp} \arrow[d] \\
\underline{\mathbf{R}}(1) \arrow[r, "\exp"] & \underline{\mathbf{S}}^1
\end{tikzcd}$$

Get "exponential" sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}(1) & \longrightarrow & \tau^{-1}\mathcal{O}_X & \longrightarrow & \tau^{-1}\mathcal{O}_X^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Z}(1) & \longrightarrow & \mathcal{L}_X & \longrightarrow & \tau^{-1}\mathcal{M}_X^{gp} & \longrightarrow & 0 \end{array}$$

Here: $\tau^{-1}\mathcal{O}_X \to \mathcal{L}_X : a \mapsto (\exp a, Im(a)) \in \tau^{-1}\mathcal{M}_X^{gp} \times \underline{\mathbf{R}}(1)$.

Construct universal sheaf of $\tau^{-1}\mathcal{O}_X$-algebras $\mathcal{O}_X^{log}$ containing $\mathcal{L}_X$

Compactification of open immersions

The map τ is an isomorphism over the set X^* where $\overline{\mathcal{M}} = 0$, so we get a diagram

$$
\begin{tikzcd}
 & X_{log} \arrow[d, "\tau"] \\
X^*_{an} \arrow[ur, "j_{log}"] \arrow[r, "j"] & X_{an}
\end{tikzcd}
$$

The map τ is proper, and for $x \in X$, $\tau^{-1}(x)$ is a torsor under $T_x := \mathrm{Hom}(\overline{\mathcal{M}}^{gp}_x, \mathbf{S}^1)$ (a finite sum of compact tori).
We think of τ as a relative compactification of j.

Example: monoid schemes

$X = A_Q := \mathrm{Spec}(Q \to \mathbf{C}[Q])$, with Q toric.

$$X_{log} = A_Q^{log} = R_Q \times T_Q \xrightarrow{\tau} X = \underline{\mathrm{A}}_{\mathrm{Q}}$$

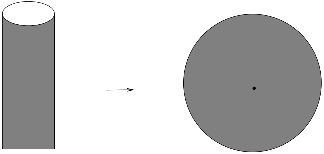
where

$\underline{A}_Q(\mathbf{C}) = \{z \colon Q \to (\mathbf{C}, \cdot)\}$ (algebraic set)
$R_Q := \{r \colon Q \to (\mathbf{R}_{\geq}, \cdot)\}$ (semialgebraic set)
$T_Q := \{\zeta \colon Q \to (\mathbf{S}^1, \cdot)\}$ (compact torus)
$\tau \colon R_Q \times T_Q \to \underline{A}_Q(\mathbf{C})$ is multiplication: $z = r\zeta$.

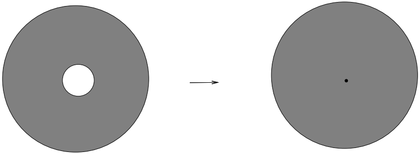
So A_Q^{log} means polar coordinates for $\underline{A}_Q$.

Example: log line, log point

If $X = \mathsf{A}_{\mathbf{N}}$, then $X_{log} = \mathbf{R}_{\geq} \times \mathbf{S}^1$.

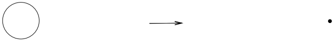


or



(Real blowup)

If $X = P = x_{\mathbf{N}}$, $X_{log} = \mathbf{S}^1$.



Example: $\mathcal{O}_P^{log}$

$$\Gamma(P_{log}, \mathcal{O}_P^{log}) = \Gamma(\mathbf{S}^1_{log}, \mathcal{O}_P^{log}) = \mathbf{C}.$$

Pull back to universal cover $\exp : \mathbf{R}(1) \to \mathbf{S}^1$

$$\Gamma(\mathbf{R}(1), \exp^* \mathcal{O}_P^{log}) = \mathbf{C}[\theta],$$

generated by θ (that is, $\log(0)$) .
Then $\pi_1(P_{log}) = Aut(\mathbf{R}(1)/\mathbf{S}^1) = \mathbf{Z}(1)$ acts, as the unique automorphism such that $\rho_\gamma(\theta) = \theta + \gamma$. In fact, if $N = d/d\theta$,

$$\rho_\gamma = e^{\gamma N}.$$

Application—Compactification

Theme: j_{log} compactifies $X^* \to X$ by adding a boundary.

Theorem

If $X/\mathbf{C}$ is (relatively) smooth, $j_{log} : X^*_{an} \to X_{log}$ is locally aspheric. In fact, $(X_{log}, X_{log} \setminus X^*_{an})$ is a manifold with boundary.

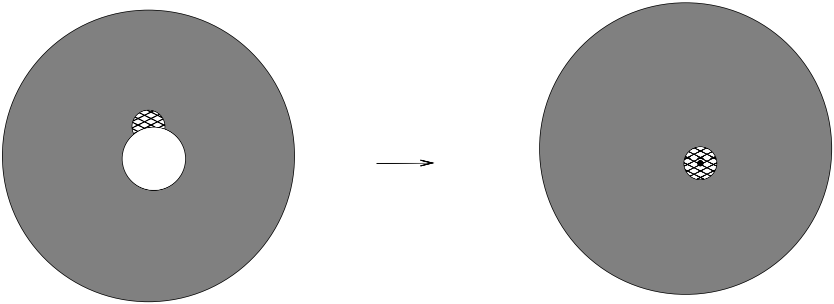
Proof.

Reduce to the case $X = \mathrm{A}_Q$. Reduce to (R_Q, R^*_Q). Use the moment map, a homeomorphism:

$$(R_Q, R^*_Q) \cong (C_Q, C^o_Q) \quad : \quad r \mapsto \sum_{a \in A} r(a)a$$

where A is a finite set of generators of Q and C_Q is the real cone spanned by Q. □

Example: The log line



Cohomology of log compactifications

Let $X/\mathbf{C}$ be (relatively) smooth, and X^* the open set where the log structure is trivial.

Theorem

$$\begin{tikzcd} & H^*(X_{log}, \mathbf{Z}) \arrow[d] \\ H^*(X^*, \mathbf{Z}) \arrow[ur, "\cong"] & H^*(X_{an}, \mathbf{Z}) \arrow[l] \end{tikzcd}$$

Log de Rham cohomology

Three de Rham complexes:

- $\Omega^{\cdot}_{X/\mathbf{C}}$ (log DR complex on X)
- $\Omega^{log\cdot}_{X/\mathbf{C}}$ (log DR compex on X_{log}
- $\Omega^{\cdot}_{X^*/\mathbf{C}}$ (ordinary DR complex on X^*

Theorem:

There is a commutative diagram of isomorphisms:

$$\begin{tikzcd} H_{DR}(X) \arrow[r] & H_{DR}(X_{log}) \arrow[r] \arrow[d] & H_{DR}(X^*) \arrow[d] \\ & H_B(X_{log}, \mathbf{C}) \arrow[r] & H_B(X^*_{an}, \mathbf{C}) \end{tikzcd}$$

X/S (relatively) smooth map of log schemes.

Theorem (Riemann-Hilbert)

Let $X/\mathbf{C}$ be (relatively) smooth. Then there is an equivalence of categories:

$$MIC_{nil}(X/\mathbf{C}) \equiv L_{un}(X_{log})$$

$$(E, \nabla) \mapsto \mathrm{Ker}(\tau^{-1}E \otimes \mathcal{O}_X^{log} \xrightarrow{\nabla} \tau^{-1}E \otimes \Omega_X^{1 log})$$

Example: $X := P$ (Standard log point)

$\Omega^1_{P/\mathbf{C}} \cong \mathbf{N} \otimes \mathbf{C} \cong \mathbf{C}$, so

$$MIC(P/\mathbf{C}) \equiv \{(E, N) : \text{vector space with endomorphism}\}$$

$P_{log} = \mathbf{S}^1$, so $L(P_{log})$ is cat of reps of $\pi_1(P_{log}) \cong \mathbf{Z}(1)$. Thus:

$$L(P_{log}) \equiv \{(V, \rho) : \text{vector space with automorphism}\}$$

Conclusion:

$$\{(E, N) : N \text{ is nilpotent}\} \equiv \{(V, \rho) : \rho \text{ is unipotent}\}$$

Use $\mathcal{O}^{log}_P = \mathbf{C}[\theta]$:

$$(V, \rho) = \operatorname{Ker}\left(\tau^* E \otimes \mathbf{C}[\theta] \to \tau^* E \otimes \mathbf{C}[\theta]\right)$$

$$N \mapsto e^{2\pi i N}$$

Application: Degenerations

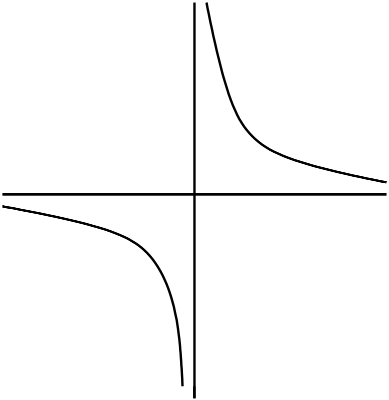
Theme: replacing f by f_{log} smooth out singularities of mappings.

Theorem (Nakayama-Ogus)

Let $f : X \to S$ be a (relatively) smooth exact morphism. Then $f_{log} : X_{log} \to S_{log}$ is a topological submersion, whose fibers are orientable topological manifolds with boundary. The boundary corresponds to the set where f_{log} is not vertical.

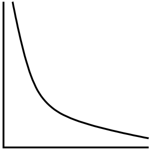
Example

Semistable reduction $\mathbf{C} \times \mathbf{C} \to \mathbf{C} : (x_1, x_2) \mapsto x_1 x_2$
This is A_θ, where $\theta\colon \mathbf{N} \to \mathbf{N} \oplus \mathbf{N} : n \mapsto (n, n)$
Topology changes: (We just draw $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$):

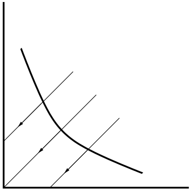


Log picture: $R_Q \times T_Q$

Just draw $R_Q \to R_{\mathbf{N}} : \mathbf{R}_{\geq} \times \mathbf{R}_{\geq} \to \mathbf{R}_{\geq} : (x_1, x_2) \mapsto x_1 x_2$



Topology unchanged, and in fact is homeomorphic to projection mapping. Proof: (Key is *exactness* of f, *integrality* of C_θ.)



Consequences

Theorem

$f : X \to S$ (relatively) smooth, proper, and exact,

1. $f_{log} : X_{log} \to S_{log}$ is a fiber bundle, and
2. $R^q f_{log*}(\mathbf{Z})$ is locally constant on S_{log}.

Monodromy

In the above situation, $R^q f_*(\mathbf{Z})$ defines a representation of $\pi_1(S_{log})$. We can study it locally, using $X_{log} \to X \times S_{log}$. (Vanishing cycles)
Restrict to $D \subseteq S$, D a log disk. Even better: to $P \subseteq D$, P a log point.

Theorem

Let $X \to P$ be (relatively) smooth, saturated, and exact.

- The action of $\pi_1(P_{log})$ on $R^q f_*(\mathbf{Z})$ is unipotent.
- Generalized Picard-Lefschetz formula for graded version of action in terms of linear data coming from: $\overline{\mathcal{M}}_P \to \overline{\mathcal{M}}_X$.

Proof uses a log construction of the Steenbrink complex

$$\Psi^{\cdot} := \mathcal{O}_P^{log} \to \mathcal{O}_P^{log} \otimes \Omega^1_{X/P} \otimes \cdots$$

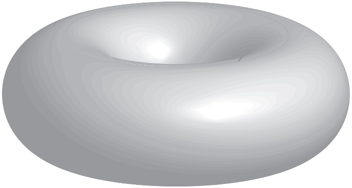
Example: Dwork families

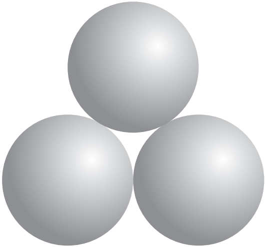
Degree 3: Family of cubic curves in $P^3 : X \to S$:

$$t(X_0^3 + X_1^3 + X_2^3) - 3X_0X_1X_2 = 0$$

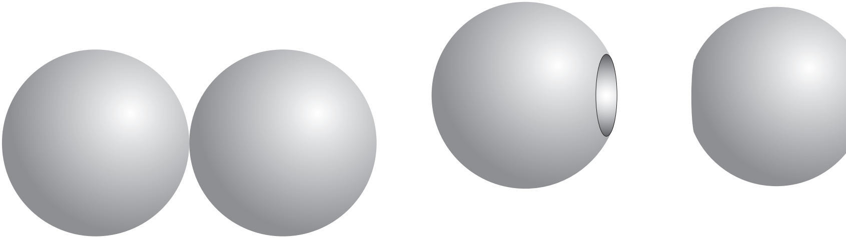
At $t = 0$, get union of three complex lines: At $t = \infty$, get smooth elliptic curve.
$X_{log} \to S_{log}$ is a fibration. How can this be?

Fibers of $X \to S$

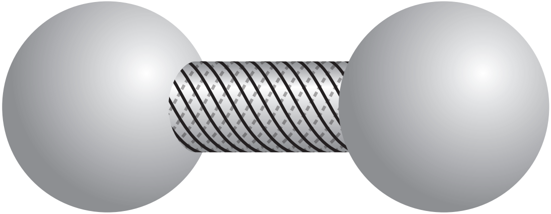




Fibers of $X_{log} \longrightarrow S_{log}$



Dehn twist



Degree 4:

$$t(X_0^4 + X_1^4 + X_2^4 + X_3^4) - 4X_0X_1X_2X_3 = 0$$

At $t - 0$, get a (complex) tetrahedron. At $t = \infty$, get a K3 surface. Need to use *relatively* coherent log structure for verticality. Still get a fibration!

Degree 5:

$$t(X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5) - 5X_0X_1X_2X_3X_4 = 0$$

Famous Calabi-Yau family from mirror symmetry.
Also used in proof of Sato-Tate

Nostalgia

$t = 5/3$ was subject of my first colloquim at Berkeley more than thirty years ago.

Conclusion

- Log geometry provides a uniform geometric perspective to treat compactification and degeneration problems in topology and in algebraic and arithmetic geometry.
- Log geometry incorporates many classical tools and techniques.
- Log geometry is not a revolution.
- Log geometry presents new problems and perspectives, both in fundamentals and in applications.

Log:

It's better than bad, it's good.