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Elliptic Crystals and Modular Motives

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The goal of this paper is to investigate the crystalline properties of families of elliptic curves—or, more precisely, to study the families of crystals (*elliptic crystals*) attached to families of elliptic curves.

Elliptic crystals over a point can be defined and classified very simply in terms of semilinear algebra, and this is how we begin (1.1). Our definition is cast in the logarithmic setting, so that it applies also to semistable (degenerate) elliptic curves. Roughly speaking, an elliptic crystal over a (logarithmic) perfect field k of characteristic p is a free W-module E of rank two equipped with a Frobenius-linear endomorphism Φ, a nilpotent linear endomorphism N, a two-step Hodge filtration A on $k \otimes E$, and an isomorphism tr: $\Lambda^2 E \to W$. These data are required to satisfy various compatibilities: for example, $A^1(k \otimes E)$ is the kernel of $\mathrm{id}_k \otimes \Phi$. When N is not zero, such a crystal is quite rigid, and in particular its canonical coordinates are very precisely determined (1.3). Liftings of elliptic crystals from k to its Witt ring are determined by specifying a lifting B of the Hodge filtration A of $k \otimes E$, and the classification is made explicit in (1.5).

Our first main result (2.2) concerns the relationship between deformation theory and elliptic crystals on a curve. Let X/k be the reduction modulo p of a smooth log curve Y/W over W, let (E, B) be an elliptic crystal on Y/W, and let (E, A) be the restriction of (E, B) to X/k. Following Mochizuki [8] and Gunning [6], we shall say that E is *indigenous* if its Kodaira-Spencer mapping is an isomorphism, a condition which can be checked either on Y/W or on X/k. If f: $X' \to X$ is a morphism of smooth log curves over k then a lifting g: $Y' \to Y$ of f determines a lifting $g^*(B^1 E_Y) \subseteq (f^*E)_{Y'}$ of $A^1(f^*E)_{X'/k}$. Theorem (2.2) shows that if p is odd and E is indigenous, the resulting map from the set of liftings of f to the set of liftings of the mod p Hodge filtration of f^*E_X is a bijection. As a consequence we show (reprising some of the results of Mochizuki) in Theorem (2.4) that (when the degree of $A^1 E_X$ is positive and p is odd) an indigenous bundle on X/k determines a unique lifting Y/W, characterized by the fact

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that (E, A) lifts to Y/W. By contrast, if the Kodaira–Spencer mapping vanishes, no such lifting exists, and in fact we show in Theorem (2.6) that in this case the elliptic crystal descends through the Frobenius endomorphism of X/k. As an application of these methods, we give a simple construction of the Deligne–Tate mapping and canonical coordinates for ordinary elliptic crystals; with a little care we are able to deal with the case $p = 2$ as well. Finally we give a canonical form for the universal deformation of a supersingular elliptic crystal, an explicit special case of a theorem of Faltings [4].

The last section is devoted to the study of the cohomology of the symmetric powers of families of elliptic crystals. The fascinating series of conjectures by Gouvea and Mazur and subsequent work by Coleman, Wan, and others suggests that there might be patterns to this cohomology as the weight varies. However, as in the ℓ-adic case, the cohomology groups $H^1(X/W, \mathrm{Sym}^m E)$ are rather pathological for large m—they are not auto-dual, can contain torsion, and in general fail to commute with base change. The temptation is to believe that these high weight motives have no natural well-behaved integral structure. However, Scholl has noticed [12] that a simple modification of the symmetric algebra construction, in which one takes the PD-envelope of the Hodge filtration, eliminates both of these pathologies. It turns out that the parabolic cohomology of the resulting *Scholl power* $H^1_{*!}(X/W, \mathrm{Sch}^m E)$ is torsion free, commutes with base change, and is auto-dual. For example, if Y is the modular curve of square-free level $N \geqslant 4$, then Y is log smooth over $\mathbf{Z}[1/N]$, the cohomology of the universal elliptic curve defines an indigenous elliptic crystal on $Y/\mathbf{Z}[1/N]$, and the De Rham cohomology groups of the Scholl powers are free $\mathbf{Z}[1/N]$-modules closely related to the space of cusp forms of weight $m + 2$ over $\mathbf{Z}[1/N]$. The p-adic completion of these modules can be interpreted in terms of crystalline cohomology, and in particular Frobenius acts on it.

What can one say about the action of Frobenius on these beautiful p-adic motives? The conjectures of Gouvea and Mazur [5] suggest that, for large m, the Newton polygon should lie strictly above the trivial polygon coming from duality and that there might be some sort of p-adic continuity in m for small slopes. Instead of looking at Newton polygons—which are isogeny invariants—we investigate the Hodge polygons of these F-crystals, which depend on the integral structure (and which give a lower bound for the Newton polygon).

In principle, the results of [11] give a method for calculating the Hodge polygons of crystalline cohomology with coefficients in an F-crystal. However, in the case at hand two serious difficulties arise, which we can only partially overcome. The first is that even the local Hodge filtration of $\mathrm{Sch}^m E$ is very difficult to calculate, because it is not uniform at the supersingular points. We are able to determine only its first p steps, and even this requires some rather painful calculations. The second difficulty is that the Frobenius–Hodge spectral sequence fails to degenerate, and the statements of [11] are in that case very weak. Fortunately, more refined results (which were in fact obtained to deal with the current situation) have recently become available [9] and enable us to extricate some nontrivial information. In fact, we are able to prove a remarkable p-adic continuity property of some slopes in some (but infinitely many) cases. Although the significance of this continuity is far from clear, it adds credence to the philosophy of Boyarski, Mazur, and others that important mathematical phenomena are "continuous with respect to the weight," and is evidence that the p-adically integral realization of the motives constructed by Scholl are meaningful. It is also interesting to compare our methods and results with those of Ulmer [13], who has used De Rham–Witt techniques to study the cohomology groups associated with elliptic crystals on the Igusa curve. He also found nontrivial lower bounds on the Newton polygon, but for a rather different reason: his crystals are not indigenous because the Kodaira–Spencer mapping vanishes at the supersingular points.

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1. ELLIPTIC CRYSTALS

For a general definition of the notion of an F-T-crystal, we refer to [11]. In this article, we shall deal almost exclusively with crystals on schemes and log schemes of dimension zero or one, for which we can give relatively concrete descriptions.

Let k be a perfect field of characteristic $p > 0$, let W be its Witt ring, and for $n \geqslant 0$ let $W_n := W/p^n W$; s (resp. S, resp. S_n) will denote $\operatorname{Spec} k$, (resp. $\operatorname{Spec} W$, resp. $\operatorname{Spec} W_n$). The inclusion $W' \to W$ of the multiplicative monoid of nonzero elements of W into W defines a log structure on S, and we denote by $S^\times$ the corresponding log scheme $\operatorname{Spec}(W' \to W)$. Note that the monoid W' fits in an exact sequence

$$1 \to W^* \to W' \to \mathbf{N} \to 0,$$

which is split, in some sense naturally, by the choice of $p \in W'$. For each $n \geqslant 0$, let $S_n^\times \subseteq S^\times$ denote the exact closed log subscheme of $S^\times$ defined by the ideal p^n of W'. Thus the underlying scheme $\underline{S}_n^\times$ of $S_n^\times$ is just S_n, the monoid $M_{S_n^\times}$ is the quotient of W' by the set of units congruent to 1 modulo p^n, and the map $M_{S_n^\times} \to W/p^n W$ is the obvious one. It is also useful to consider the *idealized log scheme* S_n^+, which is just the log scheme $S_n^\times$ endowed with the ideal of M_S generated by p^n as structural ideal (which as required maps to zero in $W/p^n W$). There are natural maps $S_n^+ \subseteq S_n^\times \to S$ in the category of idealized log schemes.

In fact, the notions of idealized log geometry will not be used in any deep way, but they underly the following concrete definitions. These will require one more piece of terminology: if A is a ring (or sheaf of rings in a topos) and M is a locally free A-module of rank n, then by a *line in M* we mean a submodule L such that M/L is locally free of rank $n-1$.

DEFINITION 1.1. 1. An *elliptic crystal on s/W* is a free finitely generated W-module E of rank 2, equipped with an isomorphism $tr\colon \Lambda^2 E \to W$. An *elliptic crystal on s^+/W* is an elliptic crystal (E, tr) on s/W, together with a W-linear endomorphism N, such that $tr(Nx \wedge y) + tr(x \wedge Ny) = 0$ for x and y in E, and all of whose eigenvalues modulo p lie in $\mathbf{F}_p$. An elliptic crystal on S_n/W (resp. on S_n^+/W) is just the same as such an object on s/W (resp. s^+/W).

2. If $n \in \mathbf{N}$, an *elliptic T-crystal on S_n/W* (resp., on S_n^+) is an elliptic crystal E on s/W (resp., on s^+) together with a line B^1E_n in $E_n := E \otimes W_n$, and an elliptic T-crystal on S/W is a compatible family of lines B^1E_n for all n or, equivalently, a line B^1E in E.

3. An *elliptic F-crystal on S_n/W* (resp. S_n^+/W) is an elliptic crystal E (resp. (E, N)) together with a linear map $\Phi\colon F_W^* E \to E$ such that $tr \circ \Lambda^2\Phi = pF^*\, tr$ (resp. and $N\Phi = p\Phi F_W^*(N)$). An *elliptic F-T-crystal on S_n/W* (resp. S_n^+/W) consists of data (E, tr, Φ, B), (resp. (E, tr, Φ, N, B)) such that (E, tr, Φ) is an elliptic F-crystal on s/W (resp. ...) and (E, tr, B) is an elliptic T-crystal on s/W such that $F^*B^1E_1$ is the kernel of the map $\Phi_1\colon F^*E_1 \to E_1$.

Remark 1.2. To give the isomorphism tr in (1.1.1) is the same as giving a perfect and alternating bilinear form $\langle\ ,\ \rangle$ on E, which we call a *polarization* of E. The compatibility condition in (1.1.2) then says that

$$\langle \Phi(x), \Phi(y) \rangle = pF_W^* \langle x, y \rangle$$

for all x and y in E. This relation implies that the rank of the reduction of Φ modulo p is one. Furthermore, if V is the transpose of Φ with respect to the form $\langle\ ,\ \rangle$, then $V \circ \Phi = \Phi \circ V = p$, and the kernel of the reduction modulo p of Φ coincides with the image of the reduction modulo p of V.

Since k is perfect, the natural map $E_1 \to F_* F^* E_1$ is a bijection, and so this line in F^*E_1 defines a line A^1E_1 in E_1, hence a T-crystal on k/W. Thus, A^1E_1 is the unique line in E_1 such that (E, Φ, A, tr) is an elliptic F-T-crystal on k/W. We shall often abuse notation by also writing Φ for the map $E \to F_{W*}E$ corresponding to $\Phi: F_W^* E \to E$ by adjunction. To express the Tate twist in the compatibility between tr and Φ, one often writes $tr: \Lambda^2 E \to W(1)$.

Let us begin by reviewing the classification of elliptic F-T-crystals on s^+/W and on $S^\times/W$. If E is an elliptic crystal on s^+/W, we write E_0 or E_1 or E_s for the reduction of E modulo p. By definition, A^1E_s is the kernel of the map

$$E_s \to F_{S*}F_S^*E_s \xrightarrow{\Phi} E_s;$$

let N^1E_s denote its image.

Theorem 1.3. *Suppose k is algebraically closed and (E, Φ, tr) is an elliptic F-crystal on s^+/W.*

1. *The following are equivalent.*

   (a) *The subspaces N^0E_1 and A^1E_1 are distinct.*

   (b) *There exists a basis (x, y) for E such that $\Phi x = px$, $\Phi y = y$, $\langle x, y\rangle = 1$, $Ny = 0$ and $Nx = \eta y$, with $\eta \in \mathbf{Z}_p$. Furthermore, (x, y) can be chosen so that η is the smallest nonnegative integer in its class in $\mathbf{Z}_p/\mathbf{Z}_p^{*2}$ and if η is not zero, this choice determines (x, y) up to a sign.*

   (c) *The automorphism group of (E, Φ, tr) is abelian.*

2. *The following are equivalent.*

   (a) *The subspaces N^0E_1 and A^1E_1 coincide.*

   (b) *There exists a basis (x, y) for E such that $\Phi x = py$, $\Phi y = -x$, and $\langle x, y\rangle = 1$. Furthermore, $N = 0$ in this case.*

   (c) *The automorphism group of (E, Φ, tr) is isomorphic to the group $\mathbf{U}$ of elements of norm one in the nonsplit quaternion algebra of rank two over $\mathbf{Z}_p$.*

If the conditions of (1) hold, the class of η in $\mathbf{Z}_p/\mathbf{Z}_p^{*2}$ determines the isomorphism class of (E, tr, Φ, N) uniquely, and its automorphism group is $\mathbf{Z}_p^*$ if N is zero and is μ_2 otherwise.

Proof. We shall not review the first case (called the *ordinary* case), which is well-known, except perhaps for the statements about η, which present no real difficulty. If E is not ordinary, it is called *supersingular*, and in this case Φ^2 is divisible by p and $-p^{-1}\Phi^2$ defines an F_W^{*2}-linear bijection

$\Psi: E \to E$. If $q := p^2$, the set $L(E)$ of all elements e of E such that $\Psi(e) = e$ is a free $W(\mathbf{F}_{p^2})$-module of rank 2, and $W \otimes_{W(\mathbf{F}_{p^2})} L(E) \to E$ is an isomorphism; furthermore it is still true that $\mathrm{id} \otimes \Phi: \mathbf{F}_{p^2} \otimes L(E) \to \mathbf{F}_{p^2} \otimes L(E)$ has rank one, and $A^1 E_1$ is defined over $\mathbf{F}_{p^2}$. Let us call a line in $L(E)$ *quasi-canonical* if its image in $k \otimes E$ is $A^1 E_1$. Given such a line, the set of all its elements x such that $\langle x, \Phi(1 \otimes x) \rangle = p$ is a torsor under the kernel of the norm $W(\mathbf{F}_{p^2}) \to W(\mathbf{F}_p)$, and we call such elements *quasi-canonical vectors*. If x is a quasi-canonical vector and if $y := p^{-1}\Phi(1 \otimes x)$, then (x, y) is a basis for E, and $\Phi(1 \otimes x) = py$, $\Phi(1 \otimes y) = -x$, and $\langle x, y \rangle = 1$. This shows that there is just one isomorphism class of supersingular elliptic F-crystals on s/W. If α is a W-linear endomorphism of E and $\alpha(x) = ax + by$, $\alpha(y) = cx + dy$, then α is compatible with Φ if and only if a and c lie in $W(\mathbf{F}_{p^2})$ and $b = -pF^*(c)$ and $d = F^*(a)$; α is compatible with tr if and only if $ad - bc = 1$. Thus the group of automorphisms of (E, Φ) can be identified with the set of pairs (a, c) in $W(\mathbf{F}_{p^2})$ such that $aF^*(a) + pcF^*(c) = 1$, i.e., with $\mathbf{U}$. If E is supersingular, it is easy to verify that N must be zero. To do this without using the notion of slopes, observe that if (x, y) is a basis as above, then the compatibility of Φ and N implies that $Ny = \Phi Nx$ and $Nx = -p\Phi Ny$. This shows that Nx and Ny are infinitely divisible by p, hence zero. Thus any supersingular elliptic F-crystal on s^+/W descends to s/W. ▮

Remark 1.4. We call a basis satisfying the conditions in the statement of the theorem (including the minimality in the choice of η) a *quasi-canonical basis* for E, and the filtration B on E defined by setting $B^1 E$ to be the span of x a *quasi-canonical filtration*. In the ordinary case, the quasi-canonical filtration is in fact unique, as is the splitting provided by the basis $\{x, y\}$.

It is straightforward to carry out the classification of elliptic F-T-crystals on $S^\times/W$.

THEOREM 1.5. *Let (E, B, Φ, N) be an elliptic F-T-crystal on $S^\times/W$, with k algebraically closed, and let (E, Φ, N) be its restriction to s^+/W. Let (x, y) be a quasi-canonical basis for E. Then $B^1 E$ admits a unique basis z such that $z = x + \lambda y$, with $\lambda \in pW$. Furthermore,*

1. *If E is ordinary and $N = 0$, then the class of λ modulo the action of $\mathbf{Z}_p^{*2}$ is independent of the choice of (x, y). This gives a bijection between the set isomorphism classes of elliptic F-T-crystals on S/W and the orbit space $pW/\mathbf{Z}_p^{*2}$.*

2. *If the endomorphism N of E is not zero, then λ is independent of the choice of quasi-canonical basis and determines the class of E. Thus there is a bijection between the set of isomorphism classes of elliptic F-T-crystals with*

nonzero N on $S^{\times}/W$ and the set of pairs (n, λ), where $n \in \mathbf{Z}^+/\mathbf{Z}^+ \cap \mathbf{Z}_p^{*2}$ and $\lambda \in pW$.

3. *If E is supersingular, the class of λ modulo the action of $\mathbf{U}$ given by the formula $(a, c)\,\lambda = (pF_W^*(c) + a\lambda)/(F_W^*(a) - c\lambda))$ is well-defined, and the isomorphism class of E is determined by the orbit of λ.*

2. ELLIPTIC CRYSTALS ON CURVES AND THEIR DEFORMATIONS

Let X/k be a smooth fs log curve, and let $\underline{X}$ be the underlying curve with trivial log structure. Then $\underline{X}/k$ is smooth, and there is a dense open set U such that M_X is the sheaf of all sections of $\mathcal{O}_X$ which are invertible on U. The complement of U in X is a reduced divisor of X which we call the *cusps of X* and denote by ∞. The log scheme X/k admits liftings to W, and this fact simplifies the study of crystals on X/W. If we fix a (p-adic) formal lifting Y/W, then a crystal on X/W is just a sheaf E_Y on Y/W equipped with a (log) connection

$$\nabla\colon E_Y \to \Omega^1_{Y/W} \otimes E$$

whose reduction modulo p has nilpotent p-curvature [11]. An *elliptic crystal on X/W* is a crystal of locally free $\mathcal{O}_{X/W}$-modules of rank two, equipped with a horizontal isomorphism tr: $\Lambda^2 E \to \mathcal{O}_{X/W}$, and an *elliptic T-crystal on X/W* (*resp.* Y/W) is an elliptic crystal (E, tr) on X/W together with a line A^1E_X in E_X (resp. a line B^1E_Y in E_Y). We often write ω for the invertible sheaf A^1E_X or B^1E_Y and we note that the polarization then induces an isomorphism $\omega^{-1} \to \mathrm{Gr}^0_A E_X = E_X/A^1E_X$. By abuse of language we call sections of ω^k *modular forms of weight k*. Sometimes we write A^1E_Y to mean the inverse image of A^1E_X via the natural map $E_Y \to E_X$. Then if (E, B) is an elliptic T-crystal on Y/W, its restriction to X/W is given by (E, A), where $A^1E_Y = B^1E_Y + pE_Y$. An *elliptic F-crystal on X/W* is an elliptic crystal E together with a morphism Φ: $F_X^*E \to E$ compatible with tr as above. Locally on Y there exist liftings F_Y of the Frobenius endomorphism of X, and $(F_X^*E)_Y \cong F_Y^*(E_Y)$. To give Φ amounts to giving, for each local lift (Y, F_Y), a horizontal map $\Phi_Y\colon F_Y^*(E_Y) \to E_Y$. Finally, an *elliptic F-T-crystal on X/W* (resp. Y/W) is data (E, tr, Φ, A), where (E, tr, Φ) is an elliptic F-crystal on X/W, (E, tr, A) is an elliptic T-crystal on X/W, and $F_X^*(A^1E_X)$ is the kernel of $\Phi_X\colon F_X^*E_X \to E_X$ (resp. and also A^1E_X is the image of B^1E_Y).

Remark 2.1. An elliptic F-crystal is automatically uniform of level one; that is, there exists a morphism $V\colon E \to F^*E$ such that $\Phi V = V\Phi = \cdot p$, and the kernel and cokernel of $\Phi_X\colon F_X^*E_X \to E_X$ are locally free. Indeed, for V we just take the transpose of Φ with respect to the form $\langle\ ,\ \rangle$. To see that the cokernel of Φ_X is locally free, suppose that $x \in F_X^*E_X$, $x' \in E_X$, $f \in \mathcal{O}_X$, and $fx' = \Phi(x)$, with f not zero. Let (Y, F_Y) be a local lifting of (X, F_X) and let y, y', and g be local liftings of x, x', and f. Then $\Phi(y) = gy' + py''$ for some $y'' \in E_Y$. Hence $py = gVy' + pVy''$, and since (g, p) is a regular sequence for E_Y, it follows that $Vy' \in pE_Y$, say $Vy' = pz$. Then $Vy' = pz = V\Phi z$, so $y' = \Phi z$ and so x' is in the image of Φ_X, and the cokernel of Φ_X is torsion free. It follows by Cartier descent cf. [1.3.6] of [11]) that there is a unique T-crystal structure on E_X compatible with Φ. Thus an elliptic F-crystal on X/W is the same as an elliptic F-T-crystal on X/W. The map $\Phi_X\colon F_X^*E_X \to E_X$ factors through an injection $F_X^*\operatorname{Gr}_A^0 E_X \to E_X$; its image N^0E_X is a horizontal subspace, and its cokernel $\operatorname{Gr}_N^{-1} E_X$ is locally free of rank one. The composite $F_X^*\operatorname{Gr}_A^0 E_X \to E_X \to \operatorname{Gr}_A^0 E_X$ is called the *Hasse–Witt map* of E. Thus the Hasse–Witt map is a linear map $h\colon F_X^*(\omega^{-1}) \to \omega^{-1}$.

We begin with the deformation theory of elliptic T-crystals. In this part of the theory, the Frobenius structure plays no role.

The map $\kappa\colon \operatorname{Gr}_A^1 E_X \to \operatorname{Gr}_A^0 E_X \otimes \Omega^1_{X/k}$ induced by the connection ∇ is called the *Kodaira–Spencer mapping* of (E, A), and we view it as defining a complex of length 2, called the *Kodaira–Spencer complex* of (E, A). We can also view κ as a map $\omega \to \omega^{-1} \otimes \Omega^1_{X/k}$, or as a map $\omega^2 \to \Omega^1_{X/k}$, or $T_{X/k} \to \omega^{-2}$. If κ is not zero, there is a unique effective divisor R such that κ defines an isomorphism

$$\omega \cong \omega^{-1} \otimes I_R \Omega^1_{X/S}, \tag{2.1.1}$$

which we can also regard as an isomorphism $\omega^2(R) \cong \Omega^1_{X/k}$. Following Mochizuki [8], we say that (E, A) is *indigenous* if R is empty, i.e., if κ is an isomorphism, and we say that (E, A) is *degenerate* if κ is zero.

When X is proper,

$$\deg \omega = g - 1 + 1/2(\deg \infty - \deg R) \tag{2.1.2}$$

where g is the genus of the curve $\underline{X}$. In particular, $\deg \omega$ is "usually" positive. If this is the case, observe that $\omega \subseteq E_X$ is the unique line bundle contained in E_X whose degree is greater than or equal to $\deg \omega$. Indeed, if $L \to E_X$ is any nonzero map from a line bundle of positive degree to E_X, then the map

$$L \to E_X \to E_X/\omega \cong \omega^{-1}$$

vanishes and hence factors through a map $L \to \omega$, which must be an isomorphism if $\deg L \geqslant \deg \omega$.

Recall that the sheaf $T_{X/k}$ of logarithmic vector fields on X is the sheaf of derivations of $\mathcal{O}_X$ which preserve the ideal I_∞ of the inclusion $i_\infty\colon \infty \to X$. It is a (noncommutative) restricted Lie algebra, and the p-curvature of a connection [11, 1.2] ∇ on a sheaf E of $\mathcal{O}_X$-modules is the map $F_X^* T_{X/k} \to \operatorname{End}_{\mathcal{O}_X}(E_X)$ sending $F_X^*\partial$ to $(\nabla_\partial)^p - \nabla_{\partial^{(p)}}$. There is a canonical residue map $\rho\colon \Omega^1_{X/k} \to \mathcal{O}_\infty$ which can be interpreted as the natural map

$$\Omega^1_{X/k} \to \Omega^1_{\infty/k} \cong \mathcal{O}_\infty,$$

where ∞ is the viewed as a log scheme with the induced structure from X. Then ∇ induces a linear endomorphism N of $i_\infty^*(E)$, which can also be viewed as the connection of the corresponding crystal on ∞/k. If t is a local coordinate at a point of ∞ and $\partial := t\partial/\partial t$ is the corresponding logarithmic derivation, then by the formula of [11, 1.2.2],

$$i_\infty^*(\psi_{F_X^*\partial}) = N^p - N. \tag{2.1.3}$$

Furthermore, if (E, ∇) is the Frobenius pullback of a coherent sheaf on X/k, with the induced connection, then both ψ and N vanish. Conversely, if ψ and N vanish and E is locally free in a neighborhood of ∞, then E descends by Frobenius [11, 1.3]. We say that ∇ is *nilpotent* if ψ is, and that it is *residually nilpotent* if N is.

Composing the p-curvature of (E_X, ∇) with the map

$$\operatorname{End}(E_X) \to \operatorname{Hom}(\operatorname{Gr}^1_A E_X, \operatorname{Gr}^0_A E_X),$$

we obtain a map

$$\bar{\psi}\colon F_X^* T_{X/k} \to \operatorname{Hom}(\operatorname{Gr}^1_A E_X, \operatorname{Gr}^0_A E_X) \cong \omega^{-2} \tag{2.1.4}$$

We can also view the Kodaira–Spencer mapping as a map

$$\kappa\colon T_{X/k} \to \omega^{-2},$$

and when E is indigenous, we can combine the inverse of this map with $\bar{\psi}$ to obtain a map $F_X^* T_{X/k} \to T_{X/k}$. This observation is the *starting point* of the work of Mochizuki.

Let Y/W be a smooth lift of X/k, let (E, B) be an elliptic T-crystal on Y/W, and let $f\colon X' \to X$ be a map of smooth log curves over k. Then if $h\colon Y' \to Y$ lifts f, there is a natural isomorphism

$$\theta_h\colon h^*(E_Y) \to (f^*E)_{Y'},$$

and the image of $h^*(B)$ under this map is a line in $(f^*E)_{Y'}$ which lifts the line f^*A. This fact underlies the following result, which illustrates the tight connection between deformation theory and T-crystals.

THEOREM 2.2. *Let Y/W_n and Y'/W_n be log smooth curves as above, let $m \leqslant n$ be a positive integer, and let g: $Y'_m \to Y_m$ be a map of their reductions modulo p^m. Suppose that (E, B) is an indigenous (resp. nondegenerate) elliptic T-crystal on Y/W. Then if p is odd or $m > 1$ or $g^*\bar{\psi} = 0$, the natural map from the set of liftings h: $Y' \to Y$ of g to the set of liftings $B' \subseteq (g^*E)_{Y'}$ of g^*B is bijective (resp. injective).*

Proof. An induction argument on $n - m$ reduces to the case in which $m = n - 1$. Let $Z' := Y'_m$, and let X' be the reduction of Z' modulo p. The ideal $J = p^{n-1}\mathcal{O}_{Y'}$ of Z' in Y' satisfies $J^2 = pJ = 0$, and a divided power structure on such an ideal amounts to a linear map γ_p: $F^*_{X'}J \to J$ [1]. We shall (as we must) use the standard divided power structure γ coming from the divided power structure on (p); then γ_p is the zero map if p is odd or $n > 2$, but if $p = n = 2$, the class of 2 defines a basis for J which identifies γ_p with the standard Frobenius endomorphism of the structure sheaf. Using this γ, we construct the following diagram,

$$\begin{tikzcd} F^*_{X'}(f^*T_{X/k} \otimes J) \cong f^*(F^*_X T_{X/k}) \otimes F^*_{X'}(J) \arrow[dr, "\psi_{f,\gamma}"'] \arrow[d, "f^*(\bar{\psi}) \otimes \gamma"] \\ f^*(\omega^{-2}) \otimes J \end{tikzcd}$$

where $\bar{\psi}$ is the graded version of the p-curvature (2.1.4) and f is g mod p.

The set $L(g)$ of all lifts of g to Y'/W_n is naturally a torsor under the sheaf of groups $f^*T_{X/k} \otimes J$, and the set $L(B)$ of lifts of $g^*B^1E_{Y_{n-1}} \subseteq g^*(E_{Z'}) \cong (f^*E)_{Z'}$ is naturally a torsor under the sheaf of groups $f^*\mathrm{Hom}(\mathrm{Gr}^1_A E_X, \mathrm{Gr}^0_A E_X) \otimes J \cong f^*\omega^{-2} \otimes J$.

LEMMA 2.3. *Let δ: $L(g) \to L(B)$ be the above map from the $f^*T_{X/k}$-torsor of liftings of g to the $f^*\omega^{-2}$-torsor of liftings of $f^*A^1E_X$ defined by pullback and θ_h. Then δ is a morphism of torsors over the homomorphism*

$$f^*\kappa + \psi_{f,\gamma}F^*_{X'}: f^*T_{X/k} \otimes J \to f^*\omega^{-2} \otimes J.$$

That is, for each $\tau \in f^*T_{X/k} \otimes J$ and each lifting h,

$$\delta(\tau h) = (f^*\kappa + \psi_{f,\gamma}F^*_{X'})(\tau)\,\delta(h).$$

Proof. The main point is to make explicit the formula[1] for the isomorphism θ_h.

[1] A similar formula appears in a letter from Deligne to Shafarevich about lifting K3 surfaces, from sometime in the 1970's.

We work locally, with the aid of logarithmic coordinates as described in [11, p. 12]. Thus, we choose a section m of M_Y with the property that $\operatorname{dlog} m$ is a basis for $\Omega^1_{Y/W}$. In our context, we can think of this concretely as follows. If the log structure of Y is trivial, m is a unit, and $u := \alpha(m)$ is a unit of $\mathcal{O}_Y$; if the log structure of Y is defined by an effective divisor with local equation t, then $\alpha(m) = t$. Let ∂_1 denote the dual basis for $T_{Y/W}$, i.e., $u\partial/\partial u$ or $t\partial/\partial t$. Then a basis for the algebra of log differential operators is given by the operators:

$$\partial_N := \prod_{i=0}^{N-1} (\partial_1 - i) \qquad \text{for} \quad N > 0, \quad \partial_0 = 1.$$

Modulo p,

$$\partial_p \equiv \partial_1^p - \partial_1 \qquad \text{and} \qquad \partial_{p^n} \equiv (\partial_p)^{p^{n-1}}.$$

Furthermore, with the restricted Lie algebra structure of $T_{X/k}$, $\partial_1^{(p)} = \partial_1$, and hence $\partial_p \equiv \partial_1^p - \partial_1^{(p)}$. Thus

$$\nabla(\partial_{p^n}) = (\nabla(\partial_p))^{p^{n-1}} = ((\nabla\partial_1)^p - \nabla(\partial_1^{(p)}))^{p^{n-1}} = (\psi_{\partial_1})^{p^{n-1}},$$

where ψ is the p-curvature of the connection ∇. The map taking a section D of $T_{X/k}$ to $(\psi_D)^{p^{n-1}}$ is a linear map

$$\psi^{(n)}\colon (F_X^n)^* \, (T_{X/k}) \to \operatorname{End}(E).$$

Now if g_1 and g_2 are two morphisms $Y' \to Y$ which agree on the exact closed subscheme defined by a PD ideal I, then $g_2^* m = (1+\eta)\, g_1^* m$ for some $\eta \in I$, and the isomorphism ε: $g_2^* E \to g_1^* E$ is given by:

$$\varepsilon g_2^*(e) = \sum_m \gamma_m(\eta)\, g_1^*(\nabla(\partial_m)(e)).$$

If also $I^2 = 0$, then

$$\tau := f^*(\partial_1 \otimes \eta) \in f^*(T_{X/k} \otimes I)$$

is the element such that $\tau g_1 = g_2$. Applying this in our case with J in place of I, $I^2 = pI = 0$, and it follows that $\gamma_m(\eta) = 0$ unless $m = 0$ or $m = p^n$ for some n; furthermore, $\gamma_{p^n} = (\gamma_p)^n$. Thus our formula reduces to

$$\varepsilon g_2^*(e) = g_1^*(e) + \eta f^*(\nabla(\partial_1)) + \sum_{n=1}^{\infty} \gamma_{p^n}(\eta)\, f^*(\nabla(\partial_{p^n})\, e)$$

Let

$$\psi_{f,\gamma}^{(n)}\colon (F_{X'}^n)^*(f^*(T_{X/k})\otimes J)\to f^*(\operatorname{End} E)\otimes J$$

be the map obtained by combining the pullback of $\psi^{(n)}$ with γ_{p^n}. Then we get:

$$\varepsilon g_2^*(e)=g_1^*(e)+\eta f^*(\nabla(\partial_1)\,e)+\sum_{n>0}\psi_{f,\gamma}^{(n)}(\tau)(f^*e)$$

Now in our context, $\psi^2=0$, so our formula reduces to a sum of three terms:

$$\varepsilon g_2^*(e)=g_1^*(e)+\eta f^*(\nabla(\partial_1)\,e)+\psi_{\gamma_p}(\tau)(f^*e).$$

If $e\in B^1E$, then modulo B^1E, $g_2^*(e)$ is $f^*(\kappa)(\tau)(e)+\psi_{f,\gamma}(\tau)(e)$, as claimed in Lemma (2.3). ∎

Now to prove the theorem, note that when p or m is greater than 2, $\gamma_p=0$, and hence $f^*\bar{\psi}_\gamma=0$, and the same is true if $f^*\bar{\psi}=0$. Thus in these cases the morphism of groups in Lemma (2.3) can be identified with κ. ∎

THEOREM 2.4. *Let Y/W_n be a smooth fs log curve and let (E, B) be an elliptic T-crystal on Y/W. Suppose that the* Kodaira–Spencer *mapping κ of the restriction (E, A) of (E, B) to X/k is not zero and that either p is odd or $n>1$. Then there exist a formally smooth $\tilde{Y}/W$ lifting Y/W_n and a lifting of $(E, \tilde{B})$ of (E, B) to $\tilde{Y}/W$. If (E, A) is indigenous, then Y/W is determined up to unique isomorphism by (E, B). If X/k is proper and* $\deg\omega>0$, *then $(E, \tilde{B})$ is uniquely determined by Y/W and (E, B).*

Proof. It is clear that liftings exist when Y is affine, and since in this case any two liftings are isomorphic, the statement is clear from (2.2) in the affine case. In the complete case, it will suffice to prove that we can lift from W_n to W_{n+1}. Let J be the ideal p^nW_{n+1}. Then the set of isomorphism classes of lifts of Y to W_{n+1} is a torsor under $H^1(X, T_{X/k}\otimes J)$, and given a lifting Z/W_{n+1}, the obstruction ξ to lifting (E, B) to Z/W_{n+1} lies in $H^1(X, \omega^{-2}\otimes J)$. If we change the lifting Z/W_n by some element τ of $H^1(X, T_{X/k}\otimes J)$, then it follows from (2.3) that the new obstruction is $\xi'=\xi+\kappa(\tau)$. Thus it will suffice to prove that the group homomorphism

$$\kappa\colon T_{X/k}\otimes J\to\omega^{-2}\otimes J \tag{2.4.5}$$

induces a surjection on H^1. But since κ is injective, its cokernel has finite support, and consequently $H^1(\kappa)$ is surjective. Furthermore, if (E, A) is indigenous, the homomorphism (2.4.5) is an isomorphism, and it follows that Z/W is unique. Finally, the set of lifts of B is a pseudo-torsor under $\mathrm{Hom}(\mathrm{Gr}^1_A, \mathrm{Gr}^0_A \otimes J) \cong H^0(\omega^{-2} \otimes J)$, which vanishes if ω has positive degree.

Remark 2.5. For an example where $(X/k, (E, A))$ cannot be lifted, start with an indigenous (E, A) on X/k, let (E, B) be its lift to Y/W, and let (E', A) be its pullback by the absolute Frobenius endomorphism F_X of X. Then the Kodaira–Spencer mapping of (E', A) vanishes, and it cannot be lifted. Indeed, if Y'/W is any lifting of X'/k, then according to (2.2), liftings (E', B) of (E', A) to Y'/W correspond to liftings of F_X, and if the degree of $\Omega^1_{X/k}$ is positive, no such liftings exist. The next result gives a partial converse.

THEOREM 2.6. *Suppose that* (E, Φ, tr) *is an elliptic F-T-crystal on* X/W. *Then the* Kodaira–Spencer *mapping of* E *vanishes if and only if there is an elliptic F-T-crystal* (E', Φ', tr') *on* X/W *such that* $(E, \Phi, tr) \cong F^*_X(E', \Phi', tr')$.

Proof. If (E, A) is a T-crystal arising from an F-crystal, then there is a profound relationship between the Kodaira–Spencer mapping κ and the p-curvature ψ, first discovered by Katz in the geometric setting. It is proved in the abstract case (without log structures) in [10]—and we should point out that it is true for F-spans, not just F-crystals. Recall that the F-crystal structure defines a filtration N on E_X by horizontal subsheaves such that the p-curvature ψ of $\mathrm{Gr}_N E_X$ vanishes, so that ψ induces maps

$$\psi_i \colon \mathrm{Gr}^i_N E_X \to F^*_X \Omega^1_{X/k} \otimes \mathrm{Gr}^{i+1}_N E_X.$$

Moreover, Φ induces isomorphisms:

$$\Phi_i \colon \mathrm{Gr}^i_A E_X \to F_{X*}(\mathrm{Gr}^{-i}_N E_X)^{\nabla}.$$

LEMMA 2.7 (Katz's formula). *Suppose* (E, Φ, A) *is an F-T-crystal on* X/k. *Then there is a commutative diagram*:

$$\begin{array}{ccc}
F^*_X \mathrm{Gr}^i_A E_X & \xrightarrow{F^*_X(\kappa_i)} & F^*_X(\Omega^1_{X/k}) \otimes (\mathrm{Gr}^{i-1}_A E_X) \\
\downarrow{\scriptstyle \Phi_i} & & \downarrow{\scriptstyle \mathrm{id} \otimes \Phi_{i-1}} \\
(\mathrm{Gr}^{-i}_N E_X) & \xrightarrow{-\psi_{-i}} & F^*_X(\Omega^1_{X/k}) \otimes (\mathrm{Gr}^{1-i}_N E_X)
\end{array}$$

It follows from this and (2.1.3) that:

COROLLARY 2.8. *If* (E, Φ, A) *is an F-T-crystal on* X/k, *and* ∞ *is the divisor of cusps, the diagram below is commutative.*

$$\begin{tikzcd} F_k^* i_\infty^*(\mathrm{Gr}_A^i E_X) \arrow[r, "F_k^* i_\infty^*(\kappa)"] \arrow[d, "\Phi_i"'] & F_k^* i_\infty^*(\Omega^1_{X/k} \otimes \mathrm{Gr}_A^{i-1} E_X) \arrow[d, "\rho \otimes \Phi_{i-1}"] \\ i_\infty^*(\mathrm{Gr}_N^{-i} E_X) \arrow[r, "N"] & i_\infty(\mathrm{Gr}_N^{1-i} E_X) \end{tikzcd}$$

Now suppose that the Kodaira–Spencer mapping of (E, A) vanishes. Then ψ_0 vanishes, and since the conjugate filtration N on E_X has only two steps, ψ itself vanishes. Furthermore, the corollary shows that the graded version of the logarithmic monodromy map N also vanishes. However, at each cusp, E is ordinary, and the explicit description (1.3) of E shows that the endomorphism N of $i_\infty^*(E)$ is nilpotent. Since it vanishes on N^0E, it is determined by its graded version ψ_{-1} and hence it vanishes. Now the descent theorem of [11, 13.10] implies that E, Φ, and tr, all descend.

As an immediate consequence of Katz's formula, we have:

COROLLARY 2.9. *If (E, Φ, A) is an elliptic F-T-crystal on X/k, then there is a commutative diagram,*

$$\begin{tikzcd} F_X^*(\omega) \arrow[r, "F^*(\kappa)"] & F_X^*(\omega^{-1} \otimes \Omega^1_{X/k}) \cong F_X^*(\omega^{-1}) \otimes F_X^*(\Omega^1_{X/k}) \arrow[dl, "h \otimes \mathrm{id}"] \\ \omega \arrow[u, "h^{-1}"] \arrow[r, "-\bar{\psi}"] & \omega^{-1} \otimes F_X^* \Omega^1_{X/k} \end{tikzcd}$$

where h: $F_X^*(\omega^{-1}) \to (\omega)^{-1}$ is the Hasse–Witt morphism. In particular, if E is indigenous and if we use κ to identify $T_{X/k}$ with ω^{-2}, then Mochizuki's map $\bar{\psi}$: $F_X^* T_{X/k} \to T_{X/k}$ is identified with $-h^2$.

We say that an elliptic F-crystal on X/k is *ordinary* if it is so at every point of X, or, equivalently, if the natural map $A^1E_X \oplus N^0E_X \to E_X$ is an isomorphism. Our classification theorem (2.2) for lifts of morphisms gives a very quick way of describing the *canonical lifting of Frobenius* (sometimes called the Deligne–Tate mapping) and the theory of canonical coordinates (q) in the ordinary case.

PROPOSITION 2.10. *Suppose that (E, Φ, B) is an ordinary indigenous F–T-crystal on Y/W and that p is odd. Then there is a unique lift F_Y of F_X such that Φ: $F_Y^*E \to E$ maps B^1E_Y to B^1E_Y.*

Proof. Since E is ordinary, the image of V: $A^1E_X \to F_X^*E_X$ is $F_X^*A^1E_X$, and hence the image of V: $B^1E_Y \to F_X^*E_Y$ is a lifting of $F_X^*A^1E_X$. Thus, the existence and uniqueness of F_Y follow from (2.4). ∎

Remark 2.11. Let us look more closely at the case of characteristic 2, when we are trying to lift a morphism or T-crystal from k to $W_2(k)$. In this case, we are working with a morphism of torsors over the morphism of groups $\kappa + \bar{\psi} \circ F_X^*$. If $\bar{\psi}$ is not zero, this morphism is not linear, and in fact if we choose bases for $T_{X/k}$ and ω, it looks locally like a map of the form $t \mapsto at + bt^2$. Here a is nonzero if κ is not zero, and b vanishes at the points where $\bar{\psi}$ is zero. For example, if (E, A) is indigenous, we see that in Theorem (2.2), the map from liftings of f to liftings of A is neither injective nor surjective, in general. More specifically, it follows that, after a finite étale cover of degree two, we can find a lifting F_Y as described in (2.10), and the set of such liftings will be a torsor under the additive group of all sections τ of ω^{-2} such that $\tau + F_X^*(h\tau) = 0$, which is isomorphic to $\mathbf{Z}/2\mathbf{Z}$. Similarly, in (2.4), one sees that, when X/k is proper and k is algebraically closed, there is a finite set of liftings of X/k to which (E, A) extends.

We can now make explicit the local deformation theory of indigenous elliptic F–T-crystals. A k-valued point σ of Y necessarily has as its image a point y of Y at which the log structure is trivial, so σ induces a morphism: $\sigma_k^\flat \colon M_{Y,y} = \mathcal{O}_{Y_y}^* \to k^*$. To deal with the points where the log structure is not trivial, we consider instead the morphisms of log schemes $\sigma \colon s^+ \to Y$. Such a σ again induces a map $\sigma^\flat \colon M_Y \to M_s$, and because of our choice of the generator p of the monoid W', we obtain a map $\sigma_k^\flat \colon M_Y \to k^*$.

THEOREM 2.12. *Let (E, Φ, B) be an indigenous elliptic F–T-crystal on Y/W and let y be a closed point of Y at which E is ordinary, where now k is algebraically closed, and let T be the formal completion of Y at y. If σ is a map $s^+ \to Y$ at y, let (x_0, y_0) be a quasi-canonical basis* (1.4) *for σ^*E, let F_T be the restriction of the Deligne–Tate mapping* (2.10)[2] *to T, and let $\tau \in T(W)$ denote the Teichmuller lifting of σ with respect to F_T. Then there exists a unique basis (x, y) for E which is compatible with B and such that $\tau^*(x, y) = (x_0, y_0)$ and such that*

$$\Phi(F_T^*(x)) = px \qquad \text{and} \qquad \Phi(F_T^*(y)) = y.$$

Furthermore, there exists a unique $q \in M_T$ such that $\nabla y = 0$, $\nabla x = dq/q \otimes y$, $F_T^*(q) = q^p$, and $\sigma_k^\flat(q) = 1$.

Proof. Except perhaps for the possibility of log structures, this result is well-known; *cf.* [2], for example. For the sake of completeness let us indicate the steps in the proof which reduce it to a straightforward consequence of our methods, using the language of log structures when convenient.

[2] In characteristic 2, there are in fact two such mappings.

LEMMA 2.13. *Let Y/W_n be a fine saturated log smooth curve over W_n, let X/k be its reduction modulo p, and let $F\colon Y \to Y$ be a lift of the absolute Frobenius endomorphism F_X of X. If $\omega \in \Gamma(Y, \Omega^1_{Y/W})$ is a differential form such that $F^*(\omega) \equiv p\omega$, then locally on Y there is a $q \in M_Y$ such that* $\operatorname{dlog} q \equiv \omega \bmod p^{n-1}$ *and $F^*(q) = q^p$.*

Proof. The proof uses the Cartier operator $C_{X/k}\colon F_{X*}\Omega^1_{X/k} \to \Omega^1_{X/k}$ and the theory of Cartier descent for log schemes. Note that X/k is of Cartier type and that k is perfect, so that the Frobenius morphism is exact. The key point is Tsuji's generalization to the logarithmic case of the well-known statement that a differential on X/k which is fixed by $C_{X/k}$ can locally be written as $\operatorname{dlog} m$ for some section m of M_X [7, 3.5].

The lemma is trivial for $n=1$: just take $q=1$. We proceed by induction on n. Supposing that $n \geqslant 1$ and that the lemma is true for n, we prove that it is also true for $n+1$. If Y/W_{n+1} is smooth and $F^*(\omega) = p\omega$ on Y, then by the induction hypothesis there exist q and θ such that $\omega = \operatorname{dlog} q + p^{n-1}\theta$ and $F^*(q) \equiv q^p \bmod p^n$. The latter condition means that $F^*(q) = q^p u$, where u is an element of $\mathcal{O}^*_Y$ which is congruent to 1 modulo p^n. Then $\log u := (u-1) - (u-1)^2/2 + (u-1)^3/3 + \cdots = p^n g$ for some $g \in \mathcal{O}_Y$, and $\operatorname{dlog} u = p^n\, dg$. Now $p^{n-1}\theta = \omega - \operatorname{dlog} q$, so

$$\begin{aligned} p^n\theta - p^{n-1}F^*(\theta) &= p\omega - p\operatorname{dlog} q - F^*(\omega) + F^*(\operatorname{dlog} q) \\ &= -\operatorname{dlog}(q^p) + \operatorname{dlog} F^*(q) \\ &= -\operatorname{dlog}(q^p) + \operatorname{dlog}(q^p) + \operatorname{dlog} u \\ &= p^n dg \end{aligned}$$

Since Y is flat over W_{n+1}, we can divide both sides by p^n, and writing a subscript 0 to indicate reduction modulo p, we find that $\theta_0 - (p^{-1}F^*(\theta))_0 = dg_0$ on X/k. By the formula [11, 1.2.7] for the Cartier operator and the fact that it kills exact forms, it follows that $\theta_0 = C_{X/k}(\theta_0)$. Hence by Cartier descent we can write $\theta = \operatorname{dlog} m + p\tilde{\theta}$, where m is a section of M_Y. Since F lifts Frobenius, $F^*(m) = m^p(1+pf)$ for some $f \in \mathcal{O}_Y$. Let $w := m^{p^{n-1}}$, and note that

$$F^*(w) = F^*(m^{p^{n-1}}) = (F^*(m))^{p^{n-1}} = (m^p(1+pf))^{p^{n-1}} = m^{p^n}(1+pf)^{p^{n-1}}.$$

Since $m^{p^n} = w^p$ and since $(1+pf)^{p^{n-1}} \equiv 1 \bmod p^n$, we conclude that $F^*(w) \equiv w^p \bmod p^n$. Let $\tilde{q} := qw$, so that again $F^*(\tilde{q}) \equiv \tilde{q}^p \bmod p^n$. Then

$$\begin{aligned} \omega &= \operatorname{dlog} q + p^{n-1}\theta \\ &= \operatorname{dlog} q + p^{n-1}(\operatorname{dlog} m + p\tilde{\theta}) \\ &= \operatorname{dlog} q + \operatorname{dlog} w + p^n\tilde{\theta} \\ &= \operatorname{dlog} \tilde{q} + p^n\tilde{\theta} \end{aligned}$$

Thus we have shown that by changing our choices of q and θ, we may arrange to have $\omega = \operatorname{dlog} q + p^n\theta$, with $F^*(q) = q^p u$, and $u \equiv 1 \bmod p^n$. Write $u = 1 + p^n f$. Arguing as before, we find that

$$p^{n+1}\theta - p^n F^*(\theta) = \operatorname{dlog} u = p^n u^{-1}\, df,$$

so df is zero mod p. Then there exists an h such that so that $f \equiv -F^*(h)$ mod p. Let $\tilde{q} := q(1 + p^n h)$; then $F^*(\tilde{q}) \equiv \tilde{q}^p \bmod p^{n+1}$. Moreover it is still the case that $\omega \equiv \operatorname{dlog} q \bmod p^n$. ∎

It is now an easy matter to prove (2.12); we give only a sketch. Because (E, Φ) is ordinary, there is a unique subcrystal U of rank one such that the restriction of Φ to U is an isomorphism (*c.f.* [10, 3.13], for example). Then $U \subseteq E_Y$ is a direct summand, and $U \oplus B^1 E_Y \cong E_Y$. Let F_Y denote the lifting of Frobenius described in (2.10). After restricting to the formal completion T, one can find a basis y for U_T such that $\Phi(F_T^*(y)) = y$, and if x is the dual basis for $B^1 E_T$, one finds that $\Phi(F_T^*(x)) = px$. Now one checks easily that $\nabla(y) = 0$ and that $\nabla(x) = \omega \otimes y$, where $\omega \in \Omega^1_{T/W}$ satisfies $F_T^*(\omega) = p\omega$. Furthermore, the residue of ω is the element η appearing in the description of $\sigma^*(E)$ in (1.3) and hence lies in $\mathbf{N}$. Now we can apply the previous lemma to find a section q of M_T such that $F_T^*(q) \equiv q^p \bmod p^n$ and $\omega \equiv \operatorname{dlog} q \bmod p^{n-1}$. Since in the construction the q at level n is compatible with the given q at level $n-1$, we may pass to the limit and obtain a q on the formal scheme. Dividing q by the Teichmuller lifting to W of $\sigma^*(q)$, we obtain $\sigma_k^*(q) = 1$. The uniqueness of q in the limit is clear, since the conditions that $\operatorname{dlog} q = \omega$ and that it map to η determine it up to a multiplicative constant $\lambda \in W^*$, and the condition that $F_T^*(q) = q^p$ means that λ is the Teichmuller lifting of its image in k, and hence is determined by the fact that $\sigma^*(q) = 1$. ∎

Remark 2.14. When p is two, the choices above are not unique, and in fact if q is a parameter as above, then it is easy to see that the "other" choice of a Deligne–Tate mapping is given by sending q to $-q^2$. Indeed, if $\tilde{F}$ is any other lifting of Frobenius, then

$$\varepsilon\tilde{F}^*(x) = F^*(x) + (\tilde{F}^*(\log q) - F^*(\log q))\, y.$$

Thus if $\tilde{F}^*(q) = -F^*(q)$, $\varepsilon\tilde{F}^*(x) = F^*(x)$, since the 2-adic logarithm of -1 is zero. The parameter $\tilde{q}$ corresponding to $\tilde{F}$ is $-q$.

The supersingular case is less well known, although Adrian Vasiu (personal communication) explained to me how the existence follows from a result of Faltings [4] and Johan De Jong (personal communication) pointed out that it can be deduced from the well-known prorepresentability for deformations of p-divisible groups.

Proposition 2.15. *Suppose that k is algebraically closed, and let T be the formal completion of Y at a k-valued point σ of X at which (E, B) is supersingular and indigenous. Choose a quasi-canonical basis (x_0, y_0) for σ^*E. Then there exists a unique formal parameter s of T such that, with respect to the lifting of Frobenius sending s to s^p, there is a basis (x, y) for E_T (also unique) which is compatible with B such that $\sigma^*(x, y) = (x_0, y_0)$, $\langle x, y\rangle = 1$, $\Phi F_T^*(x) = py$, and $\Phi F_T^*(y) = sy - x$.*

Proof. First of all, observe that by (2.2), there is a unique $\tau \in Y(W)$ lifting σ such that $\tau^* B^1 E_Y$ is the span of x_0. (This even holds in characteristic two because σ^*E is supersingular, so that $\bar{\psi} = 0$.) If $t \in \mathcal{O}_T$ is any formal parameter such that $\tau^*(t) = 0$, the lifting F of Frobenius sending t to t^p makes τ the Teichmuller lifting of σ with respect to F. Hence the F-crystal (σ^*E, Φ) can be computed by evaluating the map $\Phi: F^*E_T \to E_T$ at τ. Choose any element x of B^1E_T which lifts $x_0 \in \tau^*E$, and let $y =: p^{-1}\Phi F^*(x)$. Then (x, y) is a basis of E_T and $\tau^*(x, y) = (x_0, y_0)$. Thus we can write $\Phi F^*(y) = \alpha x + \beta y$, where $\alpha \equiv 1 \bmod t$ and $\beta \equiv 0 \bmod t$. If we multiply x by a suitable $\lambda \in W[[t]]^*$, then we can arrange to have $\alpha = -1$; furthermore, if in fact the original α is congruent to $-1 \bmod t^i$, then $\lambda \equiv 1 \bmod t^i$, and β is unchanged modulo t^i. Moreover, λ is uniquely determined if we insist (as we may) that it be congruent to 1 modulo t.

The above argument shows that, once t is chosen, there is a unique $x \in B^1E_T$ lifting x_0 such that $\Phi F^*(y) = -x + sy$, for some $s \in W[[t]]$. Igusa's principle, asserting that the Hasse invariant of a indigenous family of elliptic curves has simple zeroes, suggests that in fact s is a uniformizer of T. To see that this is indeed the case it will suffice to prove that the class of ds is not zero in $\Omega^1_{S/k}(0)$. Since $-x + sy$ is in the image of ΦF^* it is horizontal modulo p, and hence $-\nabla x + ds \otimes y + s\nabla y \equiv 0 \bmod p$. Since s belongs to the maximal ideal, we see that $\nabla x = ds \otimes y$ in $\Omega^1_{S/k} \otimes E(0)$, and since (E, A) is indigenous, this is not zero.[3] Since $s \in (t)$, we conclude that $s = ut$ where u is a unit of $W[[t]]$, say $u \equiv 1 \bmod t^i$, with $i \geqslant 0$.

Now we can define a new lifting of Frobenius $\tilde{F}$ so that $\tilde{F}^*(s) = s^p$. It is still true that $\tau^* s = 0$, so that τ is again the Teichmuller lifting of σ with respect to $\tilde{F}$. Then

$$\begin{aligned}\tilde{F}^*(s) - F^*(s) &= s^p - F^*(u)\, u^{-p} s^p \\ &= s^p p f,\end{aligned}$$

where $f =: p^{-1}(1 - F^*(u)\, u^{-p})$. Since $u \equiv 1 \bmod s^i$, $f \equiv 0 \bmod (s^i)$. (This last assertion is of course trivial if $i = 0$.)

[3] This argument is originally due to Deligne and is also proved for indigenous bundles by Mochizuki.

Recall that the connection ∇ on E induces an isomorphism $\varepsilon\colon \tilde{F}^*E \to F^*E$ given by

$$\begin{aligned}\varepsilon\tilde{F}^*(z) &= F^*(z) + \sum_{n>0} (\tilde{F}^*(s) - F^*(s))^{[n]}\, F^*(\nabla(\partial/\partial s)^n\, z)\\ &= F^*(z) + \sum_{n>0} p^{[n]} s^{pn} f^n F^*(\nabla(\partial/\partial s)^n\, z)\\ &= F^*(z) + ps^p f\,\delta(z),\end{aligned}$$

where $\delta(z)$ is some element of $F^*(E)$. Since Φ is horizontal, $\Phi\circ\varepsilon = \Phi$, and thus if $y' =: p^{-1}\Phi\tilde{F}^*(x)$, we have $y' = y + s^p f\,\delta'$ for some $\delta' \in E$. Then

$$\begin{aligned}\varepsilon\tilde{F}^*(y') &= \varepsilon\tilde{F}^*(y) + s^{p^2}\tilde{F}^*(f)\,\varepsilon(\tilde{F}^*(\delta'))\\ \varepsilon\tilde{F}^*(y') &= F^*(y) + ps^p f\,\delta(y) + s^{p^2}\tilde{F}^*(f)\,\varepsilon(\tilde{F}^*(\delta'))\\ \Phi(\varepsilon\tilde{F}^*(y')) &= -x + sy + ps^p f\Phi\,\delta(y) + s^{p^2}\tilde{F}^*(f)\,\Phi F^*(\delta)\end{aligned}$$

Since $f \in (s^i)$ and, consequently, $\tilde{F}^*(f) \in (s^{pi})$, we can write $\Phi(\tilde{F}^*(y')) = \alpha x + s\beta y'$, where $\alpha \equiv -1 \bmod s^{p+i}$ and $\beta \equiv 1 \bmod s^{p+i-1}$. Then as we saw above, we can find a unit λ congruent to 1 modulo s^{p+i} so that if $\tilde{x} := \lambda x$ and $\tilde{y}$: $p^{-1}\Phi\tilde{F}^*(\tilde{x})$, $\Phi\tilde{F}^*(\tilde{y}) = \tilde{x} + s\lambda'\beta y$, where λ' is another unit congruent to 1 modulo s^{i+p}. But then $\tilde{u} := \lambda'\beta$ is a unit congruent to 1 modulo p^{i+p-1}, and since $p > 1$, we have moved closer to our goal. We can continue in this way until we obtain in the limit the desired parameter s.

To prove that the parameter s above is unique, let us suppose we are given another such parameter $\tilde{s}$ and another basis $(\tilde{x}, \tilde{y})$ with the same properties. Let $\tilde{\tau}$ be the element of $T(W)$ defined by sending $\tilde{s}$ to zero. Since $\tilde{x}$ lies in B^1E_T and $\tilde{\tau}^*\tilde{x} = \tau^*x$, it follows that $\tau^*B^1E_T = \tilde{\tau}^*B^1E_T$, and hence, as we observed above, $\tilde{\tau} = \tau$. In other words, $\tau^*\tilde{s} = 0$, and hence we can write $\tilde{s} = us$, where u is a unit which we may assume to be congruent to 1 modulo s^i, where $i \geqslant 0$. It will suffice to show that such a congruence implies that in fact $u \equiv 1 \bmod s^{i+1}$. As we saw above, the isomorphism $\varepsilon\colon \tilde{F}^*E \to F^*E$ has the property that $\varepsilon\tilde{F}^*(z) \equiv F^*z \bmod ps^{p+i}$. Write $\tilde{x} = \gamma x$, where $\gamma \equiv 1 \bmod s$. Then $p\tilde{y} = \Phi\tilde{F}^*\tilde{x} \equiv \tilde{F}^*(\gamma)\,py \bmod ps^{p+i}$, and hence $\tilde{y} \equiv \tilde{F}^*(\gamma)\,y \bmod s^{p+i}$. Arguing in the same way and working modulo s^{p+i}, we see that $\Phi\tilde{F}^*(\tilde{y}) \equiv F^{*2}(\gamma)(-x+sy)$, and hence that $-\tilde{x} + \tilde{s}\tilde{y} \equiv F^{*2}(\gamma)(-x+sy)$. Then we find that $\gamma \equiv \tilde{F}^{*2}(\gamma)$ and that $us\tilde{F}^*(\gamma) \equiv \tilde{F}^{*2}(\gamma)\,s$. Since $\gamma \equiv 1 \bmod \tilde{s}$, the first of these implies that $\gamma \equiv 1 \bmod s^{p+i}$, and the second therefore implies that $u \equiv 1 \bmod s^{p+i-1}$. ∎

Unfortunately the connection matrix in the coordinates of (2.15) does not seem attractive. We shall need a formula for its reduction modulo p:

LEMMA 2.16. *Let $b(s)$ be the power series in $k[[s]]$ such that $b(s) = s^{p-1} + b^p s^{3p-1}$ and let $\beta := b(s)\, ds$. Then modulo p, the connection for ∇ in the basis (x, y) above is given by*

$$\nabla(x) = -s\beta \otimes x + (ds + s^2\beta) \otimes y$$
$$\nabla(y) = -\beta \otimes x + s\beta \otimes y$$

Proof. If $e \in E$, write $1 \otimes e$ for its image in F^*E. The differential of Frobenius is divisible by p and can be written as $p\zeta$, where ζ is a map $\Omega^1_{Y/W} \to F_* \Omega^1_{Y/W}$. Then for any $e \in E$,

$$p(\zeta \otimes \Phi)\, \nabla e = \nabla \Phi(1 \otimes e). \tag{2.16.6}$$

Let $z := sy - x$. Then (y, z) is a basis for E, and

$$\Phi(1 \otimes y) = z$$
$$\Phi(1 \otimes z) = s^p z - py$$
$$p(\zeta \otimes \Phi)\, \nabla(y) = \nabla(z)$$
$$p(\zeta \otimes \Phi)\, \nabla(z) = \nabla(s^p z - py)$$

Substituting and dividing by p, we get

$$(\zeta \otimes \Phi)\, p(\zeta \otimes \Phi)\, \nabla(y) = s^{p-1}\, ds \otimes z + s^p (\zeta \otimes \Phi)\, \nabla(y) - \nabla(y).$$

Thus, modulo p,

$$\nabla(z) \equiv 0$$
$$\nabla(y) \equiv s^{p-1}\, ds \otimes z + s^p (\zeta \otimes \Phi)\, \nabla(y).$$

Write $\nabla(y) = \alpha \otimes y + \beta \otimes z$. Then from the formula for Φ we see that, modulo p,

$$(\zeta \otimes \Phi)(\nabla(y)) = (\zeta \otimes \Phi)(\alpha \otimes y + \beta \otimes z) = \zeta(\alpha) \otimes z + s^p(\zeta(\beta) \otimes z).$$

Then

$$\alpha \otimes y + \beta \otimes z = s^{p-1} ds \otimes z + s^p(\zeta(\alpha) + s^p \zeta(\beta)) \otimes z.$$

It follows that $\alpha = 0$ and that

$$\beta = s^{p-1} ds + s^{2p} \zeta(\beta).$$

Then β is divisible by s, and if we write $\beta = b(s)\, ds$, this equation says

$$b(s) = s^{p-1} + b(s^p)\, s^{3p-1}. \quad \blacksquare$$

3. HODGE NUMBERS OF MODULAR MOTIVES

The Galois representations attached to modular forms are constructed from the étale cohomology of a modular curve with coefficients in the symmetric powers of the Tate-module of the universal elliptic curve. It seems natural to try to study the analogous crystalline objects, which in our context means looking at $H^1(X/W, \mathrm{Sym}^m E)$. However, if we work p-adically and $m \geqslant p$, this construction is plagued with several (related) problems: $\mathrm{Sym}^m E$ is not autodual, the Kodaira–Spencer maps are not isomorphisms, and the cohomology can have torsion and fail to commute with base change. In an attempt to avoid some of these pathologies, we shall investigate instead Scholl's modification [12] of the symmetric powers.

Let (E, B) be an elliptic F–T-crystal on W/W, let $\mathrm{Sym}\, E$ denote the symmetric algebra of E, and let (x, y) be a basis for E compatible with B. Then (x, y) determines an identification of $\mathrm{Sym}\, E$ with the polynomial algebra $W[x, y]$, mapping the ideal I_B of $\mathrm{Sym}\, E$ generated by B isomorphically to the ideal (x) of $W[x, y]$. Denote by $\mathrm{Sch}^m(E, B)$ the homogeneous component of degree m of the divided power envelope $(D_{I_B}(\mathrm{Sym}\, E), \bar{I}_B)$ of I_B in $\mathrm{Sym}\, E$. Thus $\mathrm{Sch}^m(E, B)$ has a basis $\{x^{[i]}y^j\colon i+j=m\}$, where $x^{[i]} := x^i/i!$. Let B also denote the divided power filtration of $\mathrm{Sch}(E, B)$, so that $B^i\mathrm{Sch}^m(E, B) := \bar{I}_B^{[i]} \cap \mathrm{Sch}^m(E, B)$. The divided power structure on the ideal $\bar{I}_B$ is automatically compatible with the divided power structure of the ideal (p) of W, and hence $\mathrm{Sch}^m(E, B)$ depends only on the image of B mod p, *i.e.* on the corresponding F–T-crystal (E, A) on k/W.

A similar construction can be carried out for crystals on curves. As before, let Y/W be a formally smooth log curve and let X/k be its reduction modulo p. Let (E, B) be an elliptic F–T-crystal on Y/W and let (E, A) be its restriction to X/W. For each PD-thickening (T, J_T, γ) of an open subset of the restriction of Y to some W_n (such that the divided powers on J_T are compatible with the standard divided powers on (p)), let $\mathrm{Sch}(E, B)_T$ denote the divided power envelope of $\bar{I}_B$ compatible with $(J_T + p, \gamma)$. Then $\mathrm{Sch}(E, B)_T$ depends only on $B + J_T E$, hence only on the restriction (E, A) of (E, B) to X/W. Moreover, by [1, 6.2], if $g\colon (T', J_{T'}) \to (T, J_T)$, is a PD-morphism, the natural map $g^*\,\mathrm{Sch}(E, B)_T \to \mathrm{Sch}(E, B)_{T'}$ is an isomorphism. It follows that if (E, A) is an F–T-crystal on X/k, then $\mathrm{Sch}^m(E, A)$ forms a crystal on X/W. If ∇ is the connection on E given by its crystal structure, then ∇ acts on $\mathrm{Sch}^m(E, A)$ by the rule:

$$\nabla x^{[i]}y^j = x^{[i-1]}y^j\,\nabla(x) + j^{[i]}y^{j-1}\,\nabla y. \tag{3.0.7}$$

Thus the filtration B of $\mathrm{Sch}^m(E, A)$ satisfies Griffiths transversality and so defines an F–T-crystal in the sense of [11].

In contrast to the case of symmetric powers, a principal polarization of E defines a principal polarization of $\mathrm{Sch}^m(E, A)$ for all m. Indeed, the usual pairing on $S^m(E)$ induces a pairing

$$\mathrm{Sch}^m(E, A) \times \mathrm{Sch}^m(E, A) \to \mathbf{Q} \otimes \mathcal{O}_{X/W},$$

and we can check that this pairing is integral and perfect locally with the aid of any basis for $\mathrm{Sch}^m(E, A)$. Recall first that if $(e_1, ..., e_m)$ and $(f_1, ..., f_m)$ are m-tuples of elements of E and $e = \prod e_i$, $f = \prod f_i \in S^m E$ are their respective products, then

$$\langle e, f \rangle := \sum_{\sigma} \prod_{i} \langle e_{\sigma(i)}, f_i \rangle,$$

where the sum is taken over the symmetric group. In particular, if (x, y) is a basis for E compatible with B normalized so that $\langle x, y \rangle = 1$, we have:

$$\langle x^{[i]} y^{m-i}, x^{[i']} y^{m-i'} \rangle = (-1)^{(m-i)} \frac{i!(m-i)!\,\delta_{i, m-i'}}{i!\, i'!}.$$

This is in fact $(-1)^{m-i}$ if $i = m - i'$ and is zero otherwise.

The following result shows that the cohomology of the Scholl powers is much more nicely behaved than that of the usual symmetric powers.

THEOREM 3.1. *Let (E, B) be an elliptic F–T-crystal on Y/W and let $\mathcal{E} :=$ $\mathrm{Sch}^m E$ for some positive integer m, with its filtration B. Suppose that (E, B) is indigenous and that $d := \deg \omega > 0$. Then $H^q(Y/W, \mathcal{E})$ vanishes for $q \neq 1$. When $q = 1$ it is free of finite rank over W, and there is an exact sequence*

$$0 \to \Gamma(Y, \omega^{m+2}) \to H^1(Y/W, \mathcal{E}) \to H^1(Y, \omega^{-m}) \to 0.$$

Furthermore, if B is the filtration on H^1 induced by the filtration B on $\mathcal{E}$, the image of $\Gamma(Y, \omega^{m+2})$ in $H^1(Y/W, \mathcal{E})$ is $B^{m+1} H^1 = B^1 H^1$.

Proof. Locally on Y we can choose a basis (x, y) for E_Y such that x is a basis for $\omega := B^1 E_Y$. Then $B^i\, \mathrm{Sch}^m(E_Y, B)$ has as a basis the set of elements of the form $x^{[i']} y^j$ such that $i' + j = m$ and $i' \geqslant i$, and

$$\mathrm{Gr}^i_B \mathcal{E}_Y \cong (\mathrm{Gr}^1_B \mathcal{E})^{\otimes i} \otimes (\mathrm{Gr}^0_B \mathcal{E})^{\otimes m-i} \cong \omega^{2i-m},$$

with local basis the class of $x^{[i]} \otimes y^{m-i}$. Let $C^{\cdot}_{Y/W}$ denote the (logarithmic) De Rham complex of $\mathcal{E}$:

$$C^{\cdot}_{Y/W} := \mathcal{E}_Y \xrightarrow{\nabla} \mathcal{E}_Y \otimes \Omega^1_{Y/W},$$

with its filtration B. Then $H^n(Y/W, \mathscr{E})$ is calculated by the the hypercohomology $H^n(Y, C^{\cdot}_{Y/W})$, and $\mathrm{Gr}^i_B C^{\cdot}_{Y/W}$ is the complex:

$$\begin{array}{ccc}
\mathrm{Gr}^i_B \mathscr{E}_Y & \xrightarrow{\xi_i} & \mathrm{Gr}^{i-1}_B \mathscr{E}_Y \otimes \Omega^1_{Y/S} \\
\cong \downarrow & & \downarrow \cong \\
\omega^{2i-m} & \longrightarrow & \omega^{2i-m-2} \otimes \Omega^1_{Y/S}
\end{array}$$

It follows from the formula (3.0.7) for the Gauss–Manin connection that for $0<i<m+1$ the bottom arrow can be identified with the map

$$\omega^{2i-m} \cong \omega^{2i-m-2} \otimes \omega^2 \xrightarrow{\mathrm{id}\otimes\xi} \omega^{2i-m-2} \otimes \Omega^1_{Y/W},$$

where ξ is the Kodaira–Spencer isomorphism. Thus $\mathrm{Gr}^i_B C^{\cdot}_{Y/W}$ is acyclic if i is not equal to 0 or $m+1$, and there are isomorphisms

$$\mathrm{Gr}^0_B C^{\cdot}_{Y/W} \cong \omega^{-m}$$

$$\mathrm{Gr}^{m+1}_B C^{\cdot}_{Y/W} \cong \omega^m \otimes \Omega^1_{Y/S}[-1] \cong \omega^{m+2}[-1]$$

Since $H^1(Y, \omega^{m+2})$ vanishes if $m>0$, it follows that $H^q(Y, C^{\cdot}_{Y/W})$ vanishes unless $q=1$, that $H^1(Y, C^{\cdot}_{Y/W})$ is free over W, and that its filtration is as described in the theorem. ∎

Remark 3.2. It is a little nicer to look at parabolic cohomology. In the ℓ-adic case, this is done by considering the image of cohomology with compact supports in ordinary cohomology, or, better, by computing the cohomology of X with coefficients in the sheaf $j_* \mathscr{E}$, where j is the inclusion of the open set X^* (the complement of the cusps) in X. The analogous log crystalline construction has been described by Faltings in [3]. The subject deserves a more thorough exposition, which we cannot give here. For our purposes, it suffices to say that, if $C^{\cdot}_{Y/W}$ is the De Rham complex of a crystal $\mathscr{E}$ on a smooth log curve Y/W, then the parabolic cohomology $H^1_{!*}(Y/W, \mathscr{E})$ is computed by the hypercohomology of the subcomplex

$$K^{\cdot}_{Y/W} := \mathscr{E}'_Y \to \mathscr{E}_Y \otimes \underline{\Omega}^1_{Y/W}.$$

Here $\underline{\Omega}^1_{Y/W}$ is the usual sheaf of (nonlogarithmic) differentials, and $\mathscr{E}'_Y$ is its inverse image in $\mathscr{E}$ under ∇. If $i\colon \infty \to Y$ is the inclusion of the cusps, the residue isomorphism

$$\rho\colon \mathscr{E}_Y \otimes \Omega^1_{Y/W}/\underline{\Omega}^1_{Y/W} \to i^*\mathscr{E}$$

and ∇ induce a W-linear endomorphism N of $i^*\mathscr{E}$. It follows that there is a short exact sequence of complexes,

$$0 \to K^{\cdot}_{Y/W} \to C^{\cdot}_{Y/W} \to C^{\cdot}_{\infty} \to 0,$$

where $C^{\cdot}_{\infty}$ is the complex,

$$\mathrm{Im}(N) \to i^*\mathscr{E},$$

and which is quasi-isomorphic to the complex consisting of the single term $\mathrm{Cok}(N)$ in degree one. If E is indigenous, then E is necessarily ordinary at the cusps, and we can use the coordinates of (2.12) to calculate. Indeed, i^*E has a basis (x, y) such that $Nx = y$ and $Ny = 0$; then $N(x^{[i]}y^j) = x^{[i-1]}y^{j+1}$, and the class of $x^{[m]}$ is a basis for the cokernel of N acting on $\mathscr{E}$. It follows that

$$H^1(B^{m+1}C^{\cdot}_{\infty}) \cong H^1(C^{\cdot}_{\infty}) \cong \Gamma(\infty, i^*\omega^m).$$

Thus $H^1_{!*}(Y/W, \mathscr{E})$ is a direct summand of $H^1(Y/W, \mathscr{E})$ and fits in an exact ladder:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(Y, I_\infty \omega^{m+2}) \arrow[r] \arrow[d] & H^1_{!*}(Y/W, \mathscr{E}) \arrow[r] \arrow[d] & H^1(Y, \omega^{-m}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(Y, \omega^{m+2}) \arrow[r] & H^1(Y/W, \mathscr{E}) \arrow[r] & H^1(Y, \omega^{-m}) \arrow[r] & 0
\end{tikzcd}$$

Here $\Gamma(Y, I_\infty \otimes \omega^{m+2})$ is the space S_{m+2} of cusp forms of weight $m+2$, and by Serre duality $H^1(Y, \omega^{-m})$ is dual to

$$H^0(Y, \Omega^1_{Y/W} \otimes \omega^m) \cong H^0(Y, I_\infty \otimes \omega^2 \otimes \omega^m) \cong \Gamma(Y, I_\infty \otimes \omega^{m+2})$$

and $H^1_{!*}(Y/W, \mathscr{E})$ is autodual.

Now suppose that (E, Φ, A) is an elliptic F–T-crystal on X/S. Then Φ induces a map $F^*_X \operatorname{Sch}(E, A) \to \operatorname{Sch}(E, A)$. Let us check this locally, with the aid of liftings Y, F_Y of some open subset U of X and its Frobenius endomorphism F_U. We have

$$F^*_Y(\operatorname{Sch}(E_Y, A)) \cong \operatorname{Sch} F^*_Y(E_Y, A) \cong \operatorname{Sch}(F^*_Y E, M^1),$$

and since Φ sends $M^1 F^*_Y E_Y$ to $pE_Y \subseteq A^1 E_Y$, it follows that it induces a map $\operatorname{Sch}(F^*_Y E, M^1) \to \operatorname{Sch}(E_Y, A^1 E_Y)$. Then the cohomology of $\operatorname{Sch}^m E$ becomes an F-crystal, and it is natural to ask for information about its Hodge and Newton polygons. We shall see that for large m, the Hodge polygon (and hence also the Newton polygon) lies strictly above the trivial Hodge polygon associated with the Hodge filtration B (which has just two slopes, 0 and $m+1$.)

The key to analyzing the F-crystal structure of the cohomology of $\mathrm{Sch}^m E$ will be the results of [11], and in particular the behavior of the Hodge and conjugate spectral sequences. Consider the relative Frobenius diagram

$$\begin{tikzcd} X \arrow[r, "F_{X/W}"] \arrow[dr] & X' \arrow[r, "\pi_{X/W}"] \arrow[d] & X \arrow[d] \\ & \operatorname{Spec} W \arrow[r, "F_W"] & \operatorname{Spec} W, \end{tikzcd}$$

and write $\mathscr{E}'$ for $\pi^*_{X/W}\mathscr{E}$ on X'. As explained in [11], the F-crystal structure on $\mathscr{E}$ induces an abstract Hodge filtration A_Φ on $\mathscr{E}'$, endowing it with the structure of an F–T-crystal on X'/W. Since F_W and $\pi_{X/W}$ are isomorphisms, A_Φ can also be viewed as a filtration of $\mathscr{E}$. To apply the results of [11], one must begin by computing A_Φ. It turns out that this filtration is not uniform [11, 2.4] and is difficult to compute or even express, and we have to content ourselves with a partial and cumbersome result. To state it, we need the following notation.

DEFINITION 3.3. If i and k are natural numbers, let

$$v_i(k) := \begin{cases} \inf\{ j\colon \operatorname{ord}_p(j!) + j \geqslant \operatorname{ord}_p(i!) - i + k\} & \text{if } \ i \geqslant k \\ \infty & \text{if } \ i < k. \end{cases}$$

Note that $\operatorname{ord}_p(j!) + j = \operatorname{ord}(pj!)$ and is a strictly increasing function of j. Consequently

$$pv_i(k) := \begin{cases} \inf\{ j\colon \operatorname{ord}_p(j!) \geqslant \operatorname{ord}_p(i!) - i + k\} & \text{if } \ i \geqslant k, \\ \infty & \text{if } \ i < k. \end{cases} \tag{3.3.8}$$

Furthermore,

$$v_m(m) = \left[\frac{m}{p} \right]. \tag{3.3.9}$$

We shall give a more explicit description of v_i later (3.17).

THEOREM 3.4. *Let* $\mathscr{E} := \mathrm{Sch}^m(E, A)$, *let* A^k_Φ *denote its abstract Hodge filtration, and let* $I_\Sigma \subseteq \mathcal{O}_X$ *denote the ideal of the supersingular locus of* (E, Φ).

1. *For any* k, $A^k_\Phi \mathscr{E}'_X \subseteq A^k \mathscr{E}'_X$ *with equality on the ordinary locus of* X (*and everywhere if* $k < p$).
2. *If* $m' := [\frac{m}{p}]$, $A^m_\Phi \mathscr{E}'_{X'} = I^{m'}_\Sigma A^m \mathscr{E}'_{X'}$.
3. *More generally, for any* i *and* k, $A^k_\Phi \operatorname{Gr}^i_A \mathscr{E}'_{X'} = I^{v_i(k)}_\Sigma \operatorname{Gr}^i_A \mathscr{E}'_{X'}$ *provided that* $m - k < p$.

The proof will involve several intermediate results. We begin by recalling the construction of the filtration A_Φ. We work locally on X, in a logarithmically parallelizable [11, 1.2.6] lifted situation $\mathscr{T} := (Y, F_Y)$. Then $M^k F^* \mathscr{E}_Y$ is defined to be $\Phi^{-1}(p^k \mathscr{E}_Y)$, $A^k_{\mathscr{T}} \mathscr{E}$ is the inverse image of $M^k \mathscr{E}$ under the natural map

$$\mathscr{E} \to F_{Y*} F^*_Y \mathscr{E},$$

and

$$A^k_\Phi \mathscr{E}_Y := \sum_{i=0}^{\infty} A^{k-i}_{\mathscr{T}} p^{[i]} \mathscr{E}_Y.$$

The general results of [11] show that in fact $A_{\mathscr{T}}$ descends the filtration M, and this can be checked directly in the current case.

It suffices to prove (3.4) after passing to the formal completion of Y at every point, and in particular we can work with the canonical coordinates of Section 1, since the liftings of Frobenius are parallelizable. In either case, the coordinates identify the complete local ring of X at the point in question with the ring $R := W[[t]]$ and $\mathrm{Sch}\, E$ with the PD-envelope S of the ideal (x) in the polynomial ring $R[x, y]$, which has a basis $\{x^{[i]} y^j\}$. In the ordinary case, $\Phi F^*_Y(x^{[i]} y^j) = p^i x^{[i]} y^j$ so that $M^k F^*_Y \mathrm{Sch}(E, A)$ is spanned by $\{p^{i'} F^*_Y x^{[i]} y^j : i + i' \geqslant k\}$, and A_Φ is just the filtration A. This proves the first statement of Theorem 3.4.

To prove the remaining statements, we must look at the formal neighborhood of a supersingular point. Thus any element of S can be written uniquely as a sum $f := \sum_{ij} a_{ij} x^{[i]} y^j$, with $a_{ij} \in R$, and Φ corresponds to the divided power homomorphism Ψ: $F^* S \to S$ sending $F^*(x)$ to py and $F^*(y)$ to $ty - x$. Notice that $\Phi F^*(x^{[p]}) = p^{[p]} y^p$ is divisible by $p^{[p-1]}$ but not by p^p, so that A^p_Φ will not be the same as A^p. To get an idea of what is behind (3.4), set $m_0 := m - pm'$ and

$$\eta := tF^*(x) - pF^*(y), \tag{3.4.10}$$

Then $\Phi(\eta)=px$, so $\Phi(\eta^{[m]})=(px)^{[m]}=p^m x^{[m]}$ and $\eta^{[m]}\in M^m F^* E_{\mathscr{Y}}$. It follows that $F^*(x^{[m_0]})\,\eta^{[pm']}\in M^m F^*\mathscr{E}_{\mathscr{Y}}$, which modulo p reduces to a unit times $t^{pm'}F^*(x^{[m]})=F^*(t^{m'}x^{[m]})$. This shows that $I_{\Sigma}^{m'}A^m\mathscr{E}_X\subseteq A_{\Phi}^m\mathscr{E}_X$.

To investigate A_Φ more deeply, let us simplify the notation by writing $\tilde{S}$ for F^*S, $\tilde{x}$ for $F^*(x)$, and $\tilde{y}$ for $F^*(y)$. Then an element of $\tilde{S}$ can be thought of as a divided power polynomial $f(\tilde{x},\tilde{y})$ in $\tilde{x}$ and $\tilde{y}$, and $\Psi(f)=f(py,ty-x)$.

Proposition 3.5. *If $f(\tilde{x},\tilde{y})\in\tilde{S}^m$ and $k\in\mathbf{N}$, the following are equivalent:*

1. $f(py,ty-x)\in p^kS$.
2. $(\partial_{\tilde{y}}^i f)(p,t)\in p^kR$ *for all* i.
3. $(\partial_{\tilde{x}}^i f)(p,t)\in p^{k-i}R$ *for all* i.

Proof. A homogeneous $g\in S$ is divisible by p^k if and only if $g(x,1)\in R\langle x\rangle$ is divisible by p^k. In particular, if $f\in\tilde{S}^m$, then $f(py,ty-x)\in p^kS$ if and only if $f(p,t-x)\in p^kR\langle x\rangle$. By Taylor's theorem for PD-polynomials,

$$f(p,t-x)=f(p,t)-x\partial_{\tilde{y}}f(p,t)+x^{[2]}\partial_{\tilde{y}}^2 f(p,t)+\cdots,$$

and so (1) is equivalent to (2).

To prove the equivalence of (2) and (3), observe that for any $n\geqslant 1$,

$$\tilde{x}^n\partial_{\tilde{x}}^n=\prod_{i=0}^{n-1}(\tilde{x}\partial_{\tilde{x}}-i).$$

It follows that the additive subgroup G_N of the ring of endomorphisms of $\tilde{\mathscr{E}}$ generated by $\{\tilde{x}^i\partial_{\tilde{x}}^i : 0\leqslant i\leqslant N\}$ is the same as that generated by $\{(\tilde{x}\partial_{\tilde{x}})^i : 0\leqslant i\leqslant N\}$. Euler's identity asserts that $\tilde{x}\partial_{\tilde{x}}+\tilde{y}\partial_{\tilde{y}}=m$ as endomorphisms of $\tilde{\mathscr{E}}$, and it follows that G_N is also the group generated by $\{\tilde{y}^i\partial_{\tilde{y}}^i : 0\leqslant i\leqslant N\}$. Since (p,t) is a regular sequence, an element $f\in\tilde{E}$ satisfies (2) if and only if $\tilde{y}^i\partial_{\tilde{y}}^i f(p,t)\in p^kR$ for all i, and this is true if and only if $\tilde{x}^i\partial_{\tilde{x}}^i f(p,t)\in p^kR$ for all i, i.e., if and only if (3) holds. ∎

We can simplify a little by dehomogenizing. If $f:=\sum_{i+j=m}a_{ij}\tilde{x}^{[i]}\tilde{y}^j\in\tilde{S}^m$ set

$$\Theta(f):=f(z,t):=\sum_i a_{i,m-i}t^{m-i}z^{[i]}\in R\langle z\rangle. \tag{3.5.11}$$

Then $\Theta: f\mapsto g$ induces a bijection between $\tilde{S}^m$ and the set I_m of all $g\in R\langle z\rangle$ of degree m such that $g^{(i)}(0)\in t^{m-i}R$ for all i.

COROLLARY 3.6. *Let*

$$J_k := \{g \in R\langle z\rangle : g^{(i)}(p) \in p^{k-i}R \text{ for all } i\}.$$

Then an element $f \in \tilde{S}^m$ belongs to $M^k\tilde{S}^m$ if and only if $\Theta(f) \in J_k$. Thus Θ induces a bijection between $M^k\tilde{S}^m$ and $I_m \cap J_k$.

We can now prove that $A^k_\Phi \mathscr{E}_X \subseteq A^k E_X$. It suffices to check that $M^k F^* \mathscr{E}_X \subseteq A^k F^* E_X$. Suppose that $f := \sum_{i+j=m} a_{ij} \tilde{x}^{[i]} \tilde{y}^j \in M^k F^* \mathscr{E}$ and let $g := f(z, t)$. Taylor's theorem applied to $g^{(i)}$ implies that

$$g^{(i)}(0) \equiv g^{(i)}(p) \qquad (\mathrm{mod}\ p),$$

which is divisible by p if $i < k$. Since $t^j a_{ij} = g^{(i)}(0)$ and (p, t) is a regular sequence in R, it follows that $a_{ij} \in pR$ if $i < k$, as required.

The proof of the remaining statements is more complicated. We use the fact that any divided power polynomial $g \in R\langle z\rangle$ can be written uniquely

$$g(z) = \sum_{i=0}^{p-1} g_i(z^{[p]})\, z^{[i]},$$

where each g_i is a PD-polynomial, say in $R\langle w\rangle$.

LEMMA 3.7. 1. *If $k > 0$, J_k is a sub PD-ideal of $(p, z) \subseteq R\langle z\rangle$.*

2. *For $k, k' \in \mathbf{N}$, $J_k J_{k'} \subseteq J_{k+k'}$.*

3. *If $g(z) = \sum_{i=0}^{p-1} g_i(z^{[p]})\, z^{[i]}$, then $g \in J_k$ if and only if $g_i^{(j)}(p^{[p]}) \in p^{k-i-pj}R$ for all i and all j. In particular, $g \in J_k$ if and only if each $g_i(z^{[p]}) \in J_k$.*

Proof. If $f \in J_k$ and $g \in J_{k'}$ then $\mathrm{ord}_p(fg)(p) \geqslant k + k'$. Furthermore $(fg)' = f'g + fg'$, and $f' \in J_{k-1}$ and $g' \in J_{k'-1}$. By an induction on $k + k'$, $f'g + fg' \in J_{k+k'-1}$ and so $fg \in J_{k+k'}$. If $g \in J_k$ and $n > 0$, $g(p)^{[n]} \in (p^k)^{[n]} \subseteq (p^k)$. The derivative of $g^{[n]}$ is $g^{[n-1]}g'$ and by induction on n, $g^{[n-1]} \in J_k$. Since $g' \in J_{k-1}$, the product lies in $J_{2k-1} \subseteq J_{k-1}$. This completes the proof of (1) and (2); (3) is more involved. Let $\pi := p^{[p]}$ and

$$\tilde{J}_k := \{h \in R\langle w\rangle : h^{(j)}(\pi) \in p^{k-pj}R \text{ for all } j\}.$$

We first show that if $h \in \tilde{J}_k$, then $g(z) := h(z^{[p]})$ lies in J_k. Of course $g(p) = h(\pi)$ which by assumption is divisible by p^k. Furthermore $g'(z) = z^{[p-1]} h'(z^{[p]})$, and an induction assumption applied to h' tells us that $h'(z^{[p]}) \in J_{k-p}$. Since $z^{[p-1]} \in J_{p-1}$, it follows that $g'(z) \in J_{k-1}$ and hence that $g(z) \in J_k$. Then if $g(z) = \sum_{i=0}^{p-1} g_i(z^{[p]})\, z^{[i]}$ and each $g_i \in \tilde{J}_{k-i}$, then $g \in J_k$.

For the converse we first verify the following claim.

Claim 3.8. If $g \in R\langle w\rangle$ and $g + pcwg' \in \tilde{J}_k$ for some $c \in R$, then $g \in \tilde{J}_k$.

Again we argue by induction on k, and the case of $k=0$ is trivial. If $h = g + pcwg' \in \tilde{J}_k$, then

$$h' = g' + pcg' + pcwg'' \in \tilde{J}_{k-p}.$$

Let $c' := c(1+pc)^{-1}$, so that $g' + pc'wg'' = (1+pc)^{-1}h' \in \tilde{J}_{k-p}$. Hence by the induction assumption $g' \in \tilde{J}_{k-p}$. Finally $g(\pi) = h(\pi) - pc\pi g'(\pi)$, and since $\mathrm{ord}_p(g'(\pi)) \geqslant k-p$ and $\mathrm{ord}_p(h(\pi)) \geqslant k$, $\mathrm{ord}_p g(\pi) \geqslant k$. Thus $g \in \tilde{J}_k$ and the claim is proved.

We now compute that if $g(z) = \sum_{i=0}^{p-1} g_i(z^{[p]})\, z^{[i]}$, then

$$\begin{aligned} g'(z) &= \sum_{i=0}^{p-1} g_i'(z^{[p]})\, z^{[p-1]} z^{[i]} + \sum_{i=1}^{p-1} g_i(z^{[p]})\, z^{[i-1]} \\ &= g_0'(z^{[p]})\, z^{[p-1]} + \sum_{i=1}^{p-1} g_i'(z^{[p]})\, z^{[p]} i^{-1} p z^{[i-1]} + \sum_{i=1}^{p-1} g_i(z^{[p]})\, z^{[i-1]}. \end{aligned}$$

That is, the ith component $(g')_i(w)$ of $g'(z)$ is $g_0'(w)$ if $i = p-1$ and is $i^{-1}pwg_{i+1}'(w) + g_{i+1}(w)$ if $i < p-1$. Since $g' \in J_{k-1}$, an induction assumption tells us that $(g')_i \in \tilde{J}_{k-1}$ for all i, and then the claim above implies that $g_{i+1} \in \tilde{J}_{k-i-1}$ for $i < p-1$. Thus $g_i \in \tilde{J}_{k-i}$ if $i \neq 0$, and $g_0' \in \tilde{J}_{k-p}$. It remains only to check that $g_0(\pi) \in p^k R$. But we know now that for $i > 0$, $g_i(z^{[p]})\, z^{[i]} \in J_k$, and so

$$g_0(\pi) = g(p) - \sum_{i=1}^{p-1} g_i(\pi)\, p^{[i]} \in p^k R. \quad \blacksquare$$

The following lemma is the crucial step in our upper bound for A_Φ.

Lemma 3.9. *Let g be an element of $J_k \cap I_m$ and let $r := pv_m(k)$, where $v_m(k)$ is as defined in* (3.3). *Then $g^{(m)}(0)$ belongs to* (p, t^r).

Proof. Taylor's theorem for the PD-polynomial $g^{(i)}$ implies that

$$g^{(i)}(0) = \sum_{j=i}^{m} (-p)^{[j-i]}\, g^{(j)}(p).$$

Multiplying by $(-p)^{[i]}$ we find that

$$(-p)^{[i]}\, g^{(i)}(0) = \sum_{j=i}^{m} \binom{j}{i} (-p)^{[j]}\, g^{(j)}(p).$$

Now since $g \in J_k$,

$$\operatorname{ord}_p((-p)^{[j]}\, g^{(j)}(p)) \geqslant j - \operatorname{ord}_p(j!) + k - j = k - \operatorname{ord}_p(j!).$$

It follows from (3.3.8) that for any $j < r$, $\operatorname{ord}_p(j!) < \operatorname{ord}_p(m!) - m + k$, so that

$$\operatorname{ord}_p(-p)^{[j]}\, g^{(j)}(p) \geqslant k - \operatorname{ord}_p(j!) > k - (\operatorname{ord}_p(m!) - m + k) = \operatorname{ord}_p p^{[m]}.$$

Thus for each i,

$$(-p^{[i]})\, g^{(i)}(0) \equiv \sum_{j=r}^{m} \binom{j}{i} (-p)^{[j-i]}\, g^{(j)}(p) \qquad (\operatorname{mod} pp^{[m]})$$

Consider these equations for $i = 0, ..., m-r$. Lemma (3.10) below implies that the matrix of coefficients $\binom{j}{i}$ is invertible. Thus, the sequence $(-p)^{[j-i]}\, g^{(j)}(p)$, for $j = r, ..., m$, is a linear combination of the sequence $(-p)^{[i]}\, g^{(i)}(0)$ for $i = 0, ..., m-r$. In particular, $p^{[m]} g^{[m]}(p)$ can be written as a linear combination of $g(0), ..., g^{(m-r)}(0) \operatorname{mod} pp^{[m]}$. Since $g \in I_m$, it follows that $p^{[m]} g^{(m)}(p) \in (pp^{[m]}, t^r)\, R$, and since $(p^{[m]}, t^r)$ is an R-regular sequence, $g^{(m)}(p) \in (p, t^r)\, R$. But g has degree m, so $g^{(m)}(p) = g^{(m)}(0)$. ∎

LEMMA 3.10. *Let r and e be natural numbers and let $A(r, e)$ denote the $(e+1)$ by $(e+1)$ matrix whose ijth entry is $\binom{r+j}{i}$, where i and j range between* 0 *and e. Then* $\det A(r, e) = 1$.

Proof. We prove this by induction on e, noting that $A(r, 0)$ is the 1×1 identity matrix. The determinant of $A(r, e)$ is unchanged if we subtract the $(j-1)$st column from the jth. We do this successively, starting on the right. The new matrix A' that we obtain has the same determinant as A, and if $i, j > 0$, its ijth component is

$$\binom{r+j}{i} - \binom{r+j-1}{i} = \binom{r+j-1}{i-1}.$$

In fact A' looks like

$$\begin{pmatrix} 1 & 0 & 0 \\ * & A(r, e-1) & \\ * & \cdots & \end{pmatrix}.$$

Hence $A(r, e)$ and $A(r, e-1)$ have the same determinant, and so the induction hypothesis tells us that the determinant is one. ∎

The next lemma gives a lower bound for A_Φ.

LEMMA 3.11. *If $k \geqslant m-p$, there exists $g_{m,k} \in A^k_\Phi \operatorname{Sch}^m \mathscr{E}$ and with leading term $t^{\nu_m(k)} x^{[m]}$.*

Proof. Let $n := [\frac{m}{p}]$, $s := \nu_m(k)$, $\pi := p^{[p]}$, $r := ps$, and

$$h(w) := \binom{n}{s}^{-1} (w-\pi)^{[s]}\, w^{[n-s]} \in \mathbf{Q}[w].$$

We claim that in fact $h \in W\langle w \rangle$. To check this we expand in powers of $(w-\pi)$:

$$\begin{aligned}
h(w) &= \binom{n}{s}^{-1} (w-\pi)^{[s]}(w-\pi+\pi)^{[n-s]}\\
&= \binom{n}{s}^{-1} (w-\pi)^{[s]} \sum_{j=0}^{n-s} (w-\pi)^{[j]}\, \pi^{[n-j-s]}\\
&= \binom{n}{s}^{-1} \sum_{j=0}^{n-s} \binom{s+j}{s} (w-\pi)^{[j+s]}\, \pi^{[n-j-s]}\\
&= \binom{n}{s}^{-1} \sum_{i=s}^{n} \binom{i}{s} (w-\pi)^{[i]}\, \pi^{[n-i]}\\
&= \sum_{i=s}^{n} \binom{n}{s}^{-1} \binom{i}{s}\binom{n}{i} (w-\pi)^{[i]}\, \pi^{[n]}\pi^{-[i]}\\
&= \sum_{i=s}^{n} \binom{n-s}{i-s} (w-\pi)^{[i]}\, \pi^{[n]}\pi^{-[i]}
\end{aligned}$$

Thus it suffices to show that $\operatorname{ord}_p(\pi^{[n]}) \geqslant \operatorname{ord}_p(\pi^{[i]})$ for $s \leqslant i \leqslant n$. If $i = n$ this is clear. Recall that for any j, $\operatorname{ord}_p(pj!) = \operatorname{ord}_p(j!) + j$, and that by definition $\operatorname{ord}_p(r!) \geqslant \operatorname{ord}_p(m!) + k - m$. We compute

$$\begin{aligned}
\operatorname{ord}_p(\pi^{[n]}) - \operatorname{ord}_p(\pi^{[i]}) &= n(p-1) - \operatorname{ord}_p(n!) - i(p-1) + \operatorname{ord}_p(i!)\\
&= np - ip - \operatorname{ord}_p(pn!) + \operatorname{ord}_p(pi!)\\
&\geqslant np - ip - \operatorname{ord}_p(pn!) + \operatorname{ord}_p(ps!)\\
&\geqslant p(n-i) - \operatorname{ord}_p(m!) + \operatorname{ord}_p(r!)\\
&\geqslant p(n-i) - \operatorname{ord}_p(m!) + \operatorname{ord}_p(m!) + k - m\\
&\geqslant p(n-i) + k - m.
\end{aligned}$$

If $i < n$, $p(n-i) \geqslant p$, and since $k - m \geqslant -p$, $p(n-i) + k - m \geqslant 0$.

This verifies the claim that $h \in W\langle w\rangle$. Furthermore, $h^{(i)}(\pi)$ is the coefficient of $(w-\pi)^{[i]}$ in the above expansion, which is zero if $i<s$, and, as we have seen, has p-adic ordinal at least $p(n-i)+k-m=k-m_0-pi$ for all i. Hence $h \in \tilde{J}_{k-m_0}$, and so by (3.8), $g(z) := z^{m_0}h(z^{[p]}) \in J_k$, with leading term a unit times $z^{[m]}$. Evidently $g^{(i)}(0)=0$ if $i<m-r$, so that $(t^r g)^{(i)}(0)$ is divisible by t^{m-i} for all i. Thus $t^r g \in I_m \cap J_k$ and so corresponds to an element $\tilde{f}$ of $M^k\tilde{\mathscr{E}}$. Since the coefficients of g are constants and r is divisible by p, there is an element f of $A^k_\Phi\mathscr{E}$ such that $F^*f=\tilde{f}$, and f has leading term a unit times $t^s x^{[m]}$. ∎

Proposition 3.12. *Suppose that $m-p<k\leqslant m$. Then an element $f(x, y) := \sum_i a_i x^{[i]} y^{m-i}$ of $\mathscr{E}_X$ lies in $A^k_\Phi\mathscr{E}_X$ if and only for all $i\geqslant k$, $\operatorname{ord}_t(a_i)\geqslant v_i(k)$ and for all $i<k$, $a_i=0$.*

Proof. By theorem (3.4), this statement can be verified locally on X, in a neighborhood the supersingular points. Suppose first that $f\in A^k_\Phi\mathscr{E}_X$. By definition, it is then the reduction of an element $g\in A^k_\Phi\mathscr{E}_Y$, and $\tilde{g} := F^*_Y g \in M^k\tilde{\mathscr{E}}_X$. For each $r\in \mathbf{Z}/p\mathbf{Z}$, let

$$\tilde{g}_r := \sum \{\tilde{a}_i \tilde{x}^{[i]}\tilde{y}^j : i\in r\},$$

where $\tilde{a}_i = F^*_Y(a_i)$. It follows from (3.7) that each $\tilde{g}_r \in M^k\tilde{S}^m$. If $i<k$, $\tilde{a}_i$ is zero mod p, by (3.4.1), and hence the same is true of a_i. If $i\geqslant k$, i is the unique element of $r\cap[k, m]$, so $\tilde{a}_i\tilde{y}^{m-i}\tilde{x}^{[i]}$ is the leading term (in $\tilde{x}$) of $\tilde{g}_r$. In fact we can write $\tilde{g}_r = \tilde{y}^{m-i}h$ with $h\in M^kF^*\operatorname{Sch}^i_Y$. By (3.9) the leading coefficient of $\tilde{g}_r$ belongs to $(p, t^{pv_i(k)})$, and hence the reduction modulo p of a_i belongs to $t^{v_i(k)}$.

For the converse, suppose that the coefficients of f satisfy the conditions of the proposition. If the degree n of f in x is less than k, then f is zero, and there is nothing to prove; we proceed by induction on this degree. Since $\operatorname{ord}_t(a_n)\geqslant v_n(k)$, Lemma (3.9) implies that there exists an element g of $A^k_\Phi\operatorname{Sch}^n E$ with leading term a_n. Then $y^{m-n}g\in A^k_\Phi\mathscr{E}_Y$, and $f-y^{m-n}g$ still satisfies the conditions of the proposition and has smaller degree in x. The induction hypothesis implies that it lies in $A^k_\Phi\mathscr{E}_X$, and since $y^{m-n}g\in A^k_\Phi\mathscr{E}_X$, the same is true of f.

This concludes the proof of Proposition (3.11), and Theorem (3.6) follows.

Remark 3.13. Here are some examples to show the limitations of the above results. We take $p=3$. Proposition (3.11) shows that if $m-k<p$, then the result of (3.9) is sharp; here is an example showing that this is not the case in general. Take $m:=81$ and $k:=42$. Recall that $\operatorname{ord}_p(n!) = (n-s(n))/(p-1)$, where $s(n)$ is the sum of the p-adic digits of n. Thus

$\operatorname{ord}_3(81!) = 40$. Then $\min\{j: j + \operatorname{ord}_p(j!) \geqslant \operatorname{ord}_p(m!) - m + k\} = 1$, so (3.9) says that if $g \in I_{81} \cap J_{42}$, then the coefficient of x^{81} belongs to (p, t^3). Thanks to (3.8), we may assume that all the divided powers are divisible by 3, so that we can write $g = ax^{[81]} + bt^3x^{[78]} + t^6h$ for some h. Then $g(3) = a3^{[81]} + bt^3 3^{[78]} + t^6h(3)$, and $\operatorname{ord}_3(g(3)) = 42 = \operatorname{ord}_3(3^{[78]})$, while $\operatorname{ord}_3(3^{[81]}) = 41$. Thus $h(3)$ is divisible by 3^{41} and a belongs to the ideal $3^{-41}(g(3), t^6h(3), bt^3e^{[78]}) \subseteq (3, t^6)$. Let us also observe how (3.11) can fail if $m - k \geqslant p$. Take $m = 27$ and $k = 15$. Then $\min\{j: j + \operatorname{ord}_p(j!) \geqslant \operatorname{ord}_p(m!) - m + k\} = 1$ again. But in fact $t^3x^{[27]}$ does not belong to the reduction modulo p of $M^{15}\tilde{S}^{27}$, as a similar calculation shows. On the other hand, $t^3x^{[27]} - y^3x^{[24]}b$, where $b := (26 \cdot 25)^{-1}$, does belong to $M^{15}\tilde{S}^{27}$. This shows that we cannot expect a simple term-by-term description of M^k, even mod p.

We now study the cohomological consequences of these calculations. The next two results depend only on the first two statements of (3.4) (the "easy") part.

PROPOSITION 3.14. *Let (E, Φ) be an indigenous F–T-crystal on X/W such that* $\deg(\omega) > 0$ *and let* $\mathscr{E} := \operatorname{Sch}^m(E, A)$ *with* $m > 0$. *Write* $\operatorname{Gr}^i_\Phi \mathscr{E}_{X/k}$ *for the sheaf* $A^i_\Phi \mathscr{E}_{X/k}/A^{i+1}_\Phi \mathscr{E}_{X/k}$ *on Cris*(X/k).

1. $H^2(X/k, \operatorname{Gr}^i_\Phi \mathscr{E}_{X/k}) = H^2_{!*}(X/k, \operatorname{Gr}^i_\Phi \mathscr{E}_{X/k}) = 0$ *for all i.*
2. *If* $m := m'p + m_0$ *with* $0 \leqslant m_0 < p$,

$$
\begin{aligned}
H^1_{!*}(X/k, \operatorname{Gr}^0_\Phi \mathscr{E}_{X/k}) &\cong H^1(X/k, \operatorname{Gr}^0_\Phi \mathscr{E}_{X/k}) \cong H^1(X, \omega^{-m}) \\
H^1(X/k, \operatorname{Gr}^{m+1}_\Phi \mathscr{E}_{X/k}) &\cong H^0(X, I^{m'}_\Sigma \otimes \omega^{m+2}) \\
&\cong H^0(X, \omega^{m'+m_0+2}) \\
H^1_{!*}(X/k, \operatorname{Gr}^{m+1}_\Phi \mathscr{E}_{X/k}) &\cong H^0(X, I^{m'}_\Sigma \otimes I_\infty \otimes \omega^{m+2}) \\
&\cong H^0(X, I_\infty \otimes \omega^{m'+m_0+2}) \qquad (3.11)
\end{aligned}
$$

Proof. Let $\mathscr{E}^{\cdot}_{X/k}$ denote the De Rham complex of $\mathscr{E}$ on X, with its filtration A_Φ. The complex $\operatorname{Gr}_\Phi \mathscr{E}^{\cdot}_{X/k}$. calculates the cohomology of $\operatorname{Gr}_\Phi \mathscr{E}_{X/k}$, and its differentials are $\mathscr{O}_X$-linear. Because A and A_Φ agree away from the supersingular locus Σ, it follows from Theorem (3.1) that $\operatorname{Gr}^i_\Phi \mathscr{E}^{\cdot}_{X/k}$ has finite support unless $i = 0$ or $m + 1$, and since it is concentrated in degrees zero and one, $H^2(X, \operatorname{Gr}^i_\Phi \mathscr{E}^{\cdot}_{X/k})$ vanishes unless $i = 0$ or $m + 1$. When $i = 0$,

$$
\operatorname{Gr}^0_\Phi \mathscr{E}^{\cdot}_{X/k} = \operatorname{Gr}^0_A \mathscr{E}^{\cdot}_{X/k},
$$

which as we have seen is concentrated in degree zero and hence has no H^2. When $i=m+1$,

$$\begin{aligned}\mathrm{Gr}_{\Phi}^{m+1}\mathscr{E}^{\cdot}_{X/k} &\cong A^m_{\Phi}\mathscr{E}_X\otimes\Omega^1_{X/S}[-1]\\ &\cong I^{m'}_{\Sigma}\,\omega^m\otimes\Omega^1_{X/S}[-1]\cong\omega^{m'+m_0}\otimes\omega^2[-1]\end{aligned}$$

since $I_\Sigma\cong\omega^{1-p}$ and $\omega^2\cong\Omega^1_{X/S}$.

Thus

$$H^2(X/k,\mathrm{Gr}_{\Phi}^{m+1})\cong H^1(X,\omega^{m'+m_0}\otimes\Omega^1_{X/S}).$$

Since $\omega^{m'+m_0}$ is an invertible sheaf of degree greater than the degree of $\Omega^1_{X/S}$, it is nonspecial, and it follows that the cohomology vanishes. A similar calculation works with parabolic cohomology, using (3.2). ∎

We are now in position to apply the results of [9] to analyze the Frobenius Hodge numbers of $H^1(X/W,\mathscr{E})$. Let N_Φ denote the conjugate filtration of $\mathscr{E}$, let $\bar{F}_\Phi$ denote the décalé of the filtered complex $Ru_{X/W*}(\mathscr{E},N_\Phi)$, and let F_Φ be the conjugate of $\bar{F}_\Phi$, as defined in [9]. Modulo p, the filtration F_Φ can be identified with the filtration A_Φ. Then the previous result shows that the hypotheses of Theorem (4.7) of [9] are satisfied, so that $(\mathscr{E},\bar{F}_\Phi)$ and $(\mathscr{E},F_\Phi)$ are *cohomologically concentrated in degree 1*. That is:

THEOREM 3.15. *For all i and q, $H^q(X/W,F^i_\Phi\mathscr{E})$ is torsion free, and the maps $H^1(X/W,F^i_\Phi\mathscr{E})\to H^q(X/W,\mathscr{E})$ are injective. Furthermore, $H^q(X/W,F^i_\Phi\mathscr{E})=0$ if $q\neq 1$. The same is true with $\bar{F}_\Phi$ in place of F_Φ and with parabolic cohomology in place of cohomology.*

COROLLARY 3.16. *With the above notation, let (H,F) denote $H^1(X/W,\mathscr{E})$ with the filtration induced by F_Φ. Then the mod p Frobenius Hodge and conjugate spectral sequences of X/k with coefficients in $\mathscr{E}$ coincide with the Hodge and conjugate spectral sequences* [9, 1.7] *of (H,F). Furthermore, the Hodge polygon of (H,Φ) lies between the Hodge and conjugate polygons of the filtered object (H,F)*

It is somewhat tedious to obtain explicit consequences of these results. In what follows we have tried to strike a balance by going far enough to show what is possible (and to reveal what we feel is an interesting phenomenon), without taxing the reader's patience unduly.

Fix an integer $m\geqslant p$. Then $H^1(X/W,\mathscr{E})$ has "weight" $m+1$, and we are interested in its Hodge and Frobenius Hodge numbers h^k for $k\in\{0,1,\ldots,m+1\}$. It seems to be hopeless to analyze the spectral sequence of the filtered object $(\mathscr{E}_{X/k},A_\Phi)$ directly. Instead we shall fix an integer k and look just at the map $H^1(X/k, A^k_\Phi, \mathscr{E}_{X/k}) \to H^1(X/k, \mathscr{E}_{X/k})$, which we attack using the filtration A of source and target. Recall from Theorem (3.1) that

$$H^1(X/k, A^{m+1}\mathscr{E}_{X/k}) \cong H^1(X/k, A^1\mathscr{E}_{X/k}) \cong M_{m+2} := \Gamma(X, \omega^{m+2}).$$

LEMMA 3.17. *Suppose that i and k are natural numbers with $i-k \leqslant p$. As above, let $(i_0, i_1, \ldots)$ denote the digits in the p-adic expansion of i, and let $i' := [\frac{i}{p}]$ and $i'' := [\frac{i'}{p}]$. Then if v_i is as in* (3.3),

$$v_i(k) = \begin{cases} \infty & \text{if } \; i-k<0 \\ i'+k-i & \text{if } \; 0 \leqslant i-k \leqslant i_1 \\ i'-i_1 & \text{if } \; i_1 \leqslant i-k \leqslant i_1+\operatorname{ord}_p(i'')+1 \\ i'+k-i+\operatorname{ord}_p(i'')+1 & \text{if } \; i_1+\operatorname{ord}_p(i'')+1 \leqslant i-k \\ & \qquad \leqslant i'+\operatorname{ord}_p(i'')+1 \\ 0 & \text{if } \; i-k \geqslant i'+\operatorname{ord}_p(i'')+1. \end{cases}$$

Proof. The proof is straightforward but tedious; we check case by case. If $k>i$, then $v_i(k)=\infty$ by definition.

Suppose $0 \leqslant (i-k) \leqslant i_1$, and let $j := i'-i+k$. Then $\operatorname{ord}_p(j!) = \operatorname{ord}_p(i'!)$, and so

$$j+\operatorname{ord}_p(j!) = i'-i+k+\operatorname{ord}_p(i'!) = \operatorname{ord}_p(i!)-i+k.$$

Since $j+\operatorname{ord}_p(j!)$ is a strictly increasing function of j, it follows that $v_i(k)=j$, as claimed.

Suppose that $i_1 \leqslant i-k \leqslant i_1+\operatorname{ord}_p(i'')+1$. As before, we see that

$$i'-i_1+\operatorname{ord}_p((i'-i_1)!) = i'-i_1+\operatorname{ord}_p(i'!) \geqslant \operatorname{ord}_p(i!)-i+k.$$

Thus $v_i(k) \leqslant i'-i_1$. On the other hand, if $j<i'-i_1$, then $\operatorname{ord}_p(j!) < \operatorname{ord}_p(i'!)-\operatorname{ord}_p(i'')$, so

$$\begin{aligned} j+\operatorname{ord}_p(j!) &< j+\operatorname{ord}_p(i'!)-\operatorname{ord}_p(i'') \\ &< i'-i_1-1+\operatorname{ord}_p(i'!)-\operatorname{ord}_p(i'') \\ &< \operatorname{ord}_p(i!)-i_1-1-\operatorname{ord}_p(i'') \\ &< \operatorname{ord}_p(i!)-i+k. \end{aligned}$$

Thus such a j cannot work, and $v_i(k)=i'-i_1$.

If $i-k=i_1+\operatorname{ord}_p(i'')+1$, the two formulas for $v_i(k)$ agree.

Suppose $i_1+\operatorname{ord}_p(i'')+1<i-k \leqslant i'+\operatorname{ord}_p(i'')+1$, and let $j := i'+k-i+1+\operatorname{ord}_p(i'')$. From the assumption on $i-k$ and the fact that

$i-k \leqslant p$ it follows that $i'-i_1 > j > i'-p$. Hence $\operatorname{ord}_p(j!) = \operatorname{ord}_p(i'!) - \operatorname{ord}_p(i'') - 1$, and so

$$\begin{aligned} j + \operatorname{ord}_p(j!) &= j + \operatorname{ord}_p(i'!) - \operatorname{ord}_p(i'') - 1 \\ &= i' + k - i + 1 + \operatorname{ord}_p(i'!) + \operatorname{ord}_p(i'') - \operatorname{ord}_p(i'') - 1 \\ &= \operatorname{ord}_p(i!) + k - i. \end{aligned}$$

Hence $v_i(k) = j$.

Finally, if $i-k \geqslant i' + \operatorname{ord}_p(i'') + 1$, $\operatorname{ord}_p(i!) - i + k \leqslant 0$, so $j = 0$ will do. ▮

COROLLARY 3.18. *If i is not divisible by p and $i-k \leqslant p$, then $v_k(i) = v_{k-1}(i-1)$.*

Proof. If i is not divisible by p, then all the digits of the p-adic expansions of i and $i-1$ after the unit digit agree. Thus $i_1 = (i-1)_1$, and $(i-1)' = i'$ and $(i-1)'' = i''$. Since the formula for $v_i(i)$ depends only on $i-k$ and on these digits, $v_i(k) = v_{i-1}(k-1)$. ▮

We shall find it convenient to introduce the following notation:

$$\begin{aligned} \bar{k} &:= m + 1 - k \\ \mu(\bar{k}) &:= v_{pm'}(k) \\ \mu'(\bar{k}) &:= v_{pm'-1}(k-1) \\ \mu''(\bar{k}) &:= v_m(k-1). \end{aligned}$$

Explicitly:

COROLLARY 3.19. *Let $m_0 + pm_1 + p^2 m_2 = \cdots$ be the p-adic expansion of m and suppose that $0 \leqslant \bar{k} \leqslant p$. Then*

$$\mu(\bar{k}) = \begin{cases} \infty & \text{if } \bar{k} \leqslant m_0 \\ m_0 + m' - \bar{k} + 1 & \text{if } m_0 + 1 \leqslant \bar{k} \leqslant m_0 + m_1 + 1 \\ m' - m_1 & \text{if } m_0 + m_1 + 1 \leqslant \bar{k} \\ & \qquad \leqslant m_0 + m_1 + \operatorname{ord}_p(m'') + 2 \\ m_0 + m' + \operatorname{ord}_p(m'') - \bar{k} + 2 & \text{if } m_0 + m_1 + \operatorname{ord}_p(m'') + 2 \leqslant \\ & \qquad \bar{k} \leqslant m_0 + m' + \operatorname{ord}_p(m'') + 2 \\ 0 & \text{if } \bar{k} \geqslant m_0 + m' + \operatorname{ord}_p(m'') + 2. \end{cases}$$

If $m_1 \neq 0$,

$$\mu'(\bar{k}) = \begin{cases} \infty & \text{if } \bar{k} \leqslant m_0 \\ m_0 + m' - \bar{k} & \text{if } m_0 + 1 \leqslant \bar{k} \leqslant m_0 + m_1 \\ m' - m_1 & \text{if } m_0 + m_1 \leqslant \bar{k} \\ & \qquad \leqslant m_0 + m_1 + \operatorname{ord}_p(m'') + 1 \\ m_0 + m' + \operatorname{ord}_p(m'') - \bar{k} + 1 & \text{if } m_0 + m_1 + \operatorname{ord}_p(m'') \leqslant \bar{k} \\ & \qquad \leqslant m' + m_0 + \operatorname{ord}_p(m'') + 1 \\ 0 & \text{if } m' + m_0 + \operatorname{ord}_p(m'') + 1. \end{cases}$$

On the other hand, if $m_1 = 0$, *then*

$$\mu'(\bar{k}) = \begin{cases} \infty & \text{if } \bar{k} \leqslant m_0 \\ m_0 + m' - \bar{k} & \text{if } m_0 + 1 \leqslant \bar{k} \leqslant m' + m_0 \\ 0 & \text{if } m' + m_0 \leqslant \bar{k}. \end{cases}$$

In any case, $\mu'(\bar{k}) \geqslant m' + m_0 - \bar{k}$.

Finally,

$$\mu''(\bar{k}) = \begin{cases} \infty & \text{if } \bar{k} < 0 \\ m' - \bar{k} & \text{if } 0 \leqslant \bar{k} \leqslant m_1 \\ m' - m_1 & \text{if } m_1 \leqslant \bar{k} \leqslant m_1 + \operatorname{ord}_p(m'') + 1 \\ m' - \bar{k} + \operatorname{ord}_p(m'') + 1 & \text{if } m_1 + \operatorname{ord}_p(m'') + 1 \leqslant \bar{k} \\ & \qquad \leqslant m' + \operatorname{ord}_p(m'') + 1 \\ 0 & \text{if } m' + \operatorname{ord}_p(m'') + 1 \leqslant \bar{k}. \end{cases}$$

In any case, $m' - \bar{k} \leqslant \mu''(\bar{k}) \leqslant m'$.

Proof. Let $i := pm' - 1$. Then $i' = m' - 1$, and i_1 is $m_1 - 1$ if $m_1 \neq 0$ and is $p - 1$ if $m_1 = 0$. Furthermore, if $m_1 \neq 0$, then $i'' = m''$. The formula for μ' then simply follows by substituting $k - 1$ for k into the formula (3.17), and the formulas for μ'' and μ are completely straightforward. ∎

Recall that I_Σ is the ideal of the supersingular locus of X/k. Let us also use I_Σ to denote the filtration it induces on the space of modular forms M_{m+2}:

$$I_\Sigma^j M_{m+2} := \Gamma(X, I_\Sigma^j \otimes \omega^{m+2}).$$

PROPOSITION 3.20. *Suppose that $k > m-p+1 > 0$ and let*

$$\alpha\colon H^1(X/k, A^k_\Phi \mathcal{E}_{X/k}) \to H^1(X, A^k \mathcal{E}_{X/k}) \cong M_{m+2}$$

be the map induced by the inclusion (3.4) *of $A^k_\Phi \mathcal{E}_{X/k}$ in $A^k \mathcal{E}_{X/k}$ and the isomorphism given by Theorem* (3.1):

$$H^1(X/k, A^k \mathcal{E}_{X/k}) \cong M_{m+2}.$$

1. *If $k > pm'$, α is injective and its image is $I_\Sigma^{\mu''(\bar{k})} M_{m+2}$.*

2. *If $m-p+1 < k \leqslant pm'$, the image of α contains $I_\Sigma^{\mu''(\bar{k})} M_{m+2}$ and is contained in $I_\Sigma^{\mu'(\bar{k})-m_0-1} M_{m+2}$, and there is a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_\Sigma^{\mu''(\bar{k})} M_{m+2} & \longrightarrow & H^1(X/k, A^k_\Phi \mathcal{E}_X) & \longrightarrow & I_\Sigma^{\mu'(\bar{k})}/I_\Sigma^{\mu(\bar{k})} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \delta} & & \\
0 & \longrightarrow & I_\Sigma^{\mu''(\bar{k})} M_{m+2} & \longrightarrow & I_\Sigma^{\mu'(\bar{k})-m_0-1} M_{m+2} & \longrightarrow & I_\Sigma^{\mu'(\bar{k})-m_0-1}/I_\Sigma^{\mu''(\bar{k})} & \longrightarrow & 0
\end{array}$$

where the map δ takes f to the class of $(-1)^{m_0+1} f^{(m_0+1)}$.

Proof. We shall deduce the top row from from the spectral sequence of the filtered complex $(A^k_\Phi \mathcal{E}^{\cdot}_{X/k}, A)$. Let $C^{\cdot} := \mathcal{E}^{\cdot}_{X/k}$, with its two filtrations A and A_Φ. To save space, we shall sometimes just write H to denote the hypercohomology of a complex of sheaves on X. Since k and $k-1$ are greater than $m-p$, Theorem (3.4) implies that for $k \leqslant i < m+1$, $\mathrm{Gr}^i_A A^k_\Phi C^{\cdot}$ is the complex

$$I_\Sigma^{\nu_i(k)} \otimes \omega^{2i-m} \to I_\Sigma^{\nu_{i-1}(k-1)} \otimes \omega^{2i-m-2} \otimes \Omega^1_{X/k} \cong I_\Sigma^{\nu_{i-1}(k-1)} \otimes \omega^{2i-m},$$

that

$$\mathrm{Gr}^{m+1}_A A^k_\Phi C^{\cdot} = A^{m+1} A^k_\Phi C^{\cdot} = I_\Sigma^{\nu_m(k-1)} \omega^m \otimes \Omega^1_{X/k}[-1]$$

and that $\mathrm{Gr}^i_A A^k_\Phi C^{\cdot}$ is zero if $i < k$. As we have observed above in the proof of (3.14), the sheaves in these complexes have no higher cohomology, and consequently the hypercohomology of the complex can be computed by simply taking the cohomology of the complex of global sections.

It follows that $H^0(\mathrm{Gr}^i_A A^k_\Phi C^{\cdot}) = 0$ for all i, and

$$H^1(A^{m+1} A^k_\Phi \mathcal{E}^{\cdot}_{X/k}) \cong I_\Sigma^{m'} M_{m+2}$$

Recall from Corollary (3.18) that $\nu_i(k) = \nu_{i-1}(k-1)$ unless i is divisible by p, and note that, since $m \geqslant i \geqslant k > m-p$, i can only be divisible by p when

$i = pm'$. Moreover, $(pm'-1)' = m'-1$ and by definition, $\nu_{pm'-1}(k-1) = \mu'(\bar{k})$. Furthermore, $\nu_m(k-1)$ is $\mu''(\bar{k})$. Thus the associated graded to the filtration A of $A^k_\Phi \mathscr{E}^\cdot_{X/k}$ has nontrivial cohomology in only two degrees:

$$H^1(\mathrm{Gr}^i_A\, A^k_\Phi C^\cdot) \cong \begin{cases} I_\Sigma^{\mu''(\bar{k})} M_{m+2} & \text{if} \quad i = m+1 \\ I_\Sigma^{\mu'(\bar{k})}/I_\Sigma^{\mu(\bar{k})} \otimes \omega^{2i-2-m} \otimes \Omega^1_{X/k} & \text{if} \quad k \leqslant m'p \quad \text{and} \quad i = m'p \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $H^1(A^{m'p} A^k_\Phi C^\cdot) = H^1(A^k_\Phi C^\cdot)$, and since $H^q(\mathrm{Gr}^i_A\, A^k_\Phi C^\cdot) = 0$ for $q \neq 1$, the rows of the following diagram are exact:

$$\begin{tikzcd} 0 \arrow[r] & H^1(A^{m+1} A^k_\Phi C^\cdot) \arrow[r] \arrow[d, "\cong"'] & H^1(A^{pm'} A^k_\Phi C^\cdot) \arrow[r] \arrow[d, "\cong"'] & H^1(\mathrm{Gr}^{pm'}_A\, A^k_\Phi C^\cdot) \arrow[r] \arrow[d, "\cong"'] & 0 \\ 0 \arrow[r] & I_\Sigma^{\mu''(\bar{k})} M_{m+2} \arrow[r] & H^1(A^k_\Phi C^\cdot) \arrow[r] & I_\Sigma^{\mu'(\bar{k})}/I_\Sigma^{\mu(\bar{k})} \arrow[r] & 0 \end{tikzcd}$$

To obtain the commutative diagram in part (2) of the proposition, consider the commutative diagram of exact sequences of complexes:

$$\begin{tikzcd} 0 \arrow[r] & A^{pm'+1} A^k_\Phi C^\cdot \arrow[r] \arrow[d, "\cong"'] & A^{pm'} A^k_\Phi C^\cdot \arrow[r] \arrow[d] & \mathrm{Gr}^{pm'}_A\, A^k_\Phi C^\cdot \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A^{pm'+1} A^k_\Phi C^\cdot \arrow[r] & A^{pm'} C^\cdot \arrow[r] & A^{pm'} C^\cdot / A^{pm'+1} A^k_\Phi C^\cdot \arrow[r] & 0 \end{tikzcd}$$

As we have seen, $H^1(A^m A^k_\Phi C^\cdot) \cong H^1(A^{pm'+1} A^k_\Phi C^\cdot)$. Furthermore, $m' \geqslant 1$, so $H^1(A^{pm'} C^\cdot) \cong H^1(A^{m+1} C^\cdot) \cong M_{m+2}$. Thus we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I_\Sigma^{\mu''(\bar{k})} M_{m+2} \arrow[r] \arrow[d] & H^1(A^k_\Phi C^\cdot) \arrow[r] \arrow[d, "\alpha"'] & I_\Sigma^{\mu'(\bar{k})}/I_\Sigma^{\mu(\bar{k})} \arrow[r] \arrow[d, "\delta"'] & 0 \\ 0 \arrow[r] & I_\Sigma^{\mu''(\bar{k})} M_{m+2} \arrow[r] & M_{m+2} \arrow[r] & M_{m+2}/M_{m+2} I_\Sigma^{\mu''(\bar{k})} \arrow[r] & 0 \end{tikzcd}$$

It remains to calculate the map δ, which we may do locally in a neighborhood of each supersingular point. Suppose that $z \in A^{pm'} A^k_\Phi C^1$ lifts the class of an element z of $I_\Sigma^{\mu'(\bar{k})}/I_\Sigma^{\mu(\bar{k})}$ via the projection in the diagrams above. Then z is homologous in the complex $C^\cdot$ to an element $\alpha(z)$ of $A^{m+1} C^1$, and $\delta(z)$ is by definition the class of $\alpha(z)$. Using our local coordinates, z can be written $z = f x^{[pm'-1]} y^{m_0+1} dt$, with $f \in I_\Sigma^{\mu'(\bar{k})}$. Consider the element

$$h := f x^{[pm']} y^{m_0} - f' x^{[pm'+1]} y^{m_0-1} + \cdots + (-1)^{m_0} f^{(m_0)} x^{[m]} \in \mathscr{E}_X.$$

The ith term here is $(-1)^i f^{(i)} x^{[pm'+i]} y^{m_0-i}$, and by (2.16), its differential is $(-1)^i\, dt$ times

$$f^{(i+1)} x^{[pm'+i]} y^{m_0-i} + f^{(i)} x^{[pm'+i-1]} y^{m_0-i+1} + u_i + v_i,$$

where $u_i \in bf^{(i)} A^{pm+i-1}\mathscr{E}$, $v_i \in bf^{(i)} A^{pm+i}\mathscr{E}$ and $\operatorname{ord}_t b \geqslant p-1$. If $i=0$,

$$\operatorname{ord}_t(f^{(i)} bu_i) \geqslant \mu'(\bar{k}) + p - 1 \geqslant \mu(\bar{k}),$$

so $\delta(u_0)=0$. In general,

$$\operatorname{ord}_t(bf^{(i)}) \geqslant p-1+v_{pm'-1}(k-1)-i \geqslant v_{pm'-1}(k-1) \geqslant v_{pm'}(k-1).$$

Thus $v_i\, dt$ and (for $i>0$) $u_i\, dt$ belong to $A^{pm'} A^k \mathscr{E}^{\cdot}$ for all i, and hence are annihilated by δ. Since the sum of the first two terms telescopes, $dh = z + (-1)^{m_0} f^{(m_0)} x^{[m]} + w$, where $\omega := \sum u_i + v_i$ and $\delta(\omega)=0$. The lemma follows.

For example, if $m=3$, then $H^1(X/k, A^4_{\Phi}\mathscr{E}) \to H^1(X/k, \mathscr{E})$ is injective, but $H^1(X/k, A^3_{\Phi}\mathscr{E}) \to H^1(X/k, \mathscr{E})$ is not. Indeed, the latter map corresponds to the case $\bar{k}=1$, and $\mu(1)=1$, $\mu'(1)=0$, and δ is the map which takes the derivative of the constant term of a function f. More generally, if $m_0=0$ and $m_1=1$, then $\mu'(1)=m'-1\equiv 0 \pmod{p}$, $\mu(1)=\mu'(1)+1$, and δ corresponds to taking the derivative of a pth power, hence is zero, and the map on cohomology is not injective. The general situation is described by the following proposition.

Proposition 3.21. *Let $\bar{k} := m+1-k$, and assume that $\bar{k} < p < m$. Then the map $H^1(X/k, A^k_{\Phi}\mathscr{E}) \to H^1(X/k, \mathscr{E})$ is injective in any of the following cases*:

- $\bar{k} \leqslant m_0$
- $\bar{k} \leqslant m_1 - 1$
- $m_1 = 0$.

If $m_1 \neq 1$, then the map fails to be injective when $m_0 < \bar{k} = m_1$.

Proof. If $\bar{k} \leqslant m_0$, then $k > pm'$, and the injectivity follows immediately from (3.20). Suppose that $m_0 < \bar{k} \leqslant m_0 + m_1$. Then $m_1 \geqslant 1$, so $\mu(\bar{k}) = m' - \bar{k} + m_0 + 1$, $\mu'(\bar{k}) = m' - \bar{k} + m_0$, and $\mu''(\bar{k}) \geqslant m' - \bar{k} = \mu'(\bar{k}) - m_0$. Hence the map δ can be viewed as differentiating an element of $\operatorname{Gr}^{\mu'(\bar{k})}_{I_\Sigma}(\mathscr{O}_X)$ m_0+1 times and reducing modulo $I^{\mu''(\bar{k})}$. In other words, it takes the class of $t^{m'-\bar{k}+m_0}$ to $(m'+m_0-\bar{k})(m'+m_0-\bar{k}-1)\cdots(m'-\bar{k})\, t^{m'-\bar{k}-1}$. Reducing modulo $t^{\mu''(\bar{k})}$ does not lose information because $\mu''(\bar{k}) \geqslant \mu'(\bar{k}) - m_0$. Since $m' \equiv m_1 \pmod{p}$ and $m_0 < \bar{k} < p$, this coefficient is nonzero if $\bar{k} < m_1$ and is zero if $\bar{k} = m_1$.

Finally, suppose that $m_1 = 0$. Then

$$\mu(\bar{k}) = \begin{cases} m' & \text{if} \quad m_0 + 1 \leqslant \bar{k} \leqslant m_0 + \mathrm{ord}_p(m'') + 2 \\ m_0 + m' + \mathrm{ord}_p(m'') - \bar{k} + 1 & \text{if} \quad m_0 + \mathrm{ord}_p(m'') + 2 \leqslant \bar{k} \end{cases}$$

$$\mu'(\bar{k}) = \begin{cases} m_0 + m' - \bar{k} & \text{if} \quad m_0 \leqslant \bar{k} \leqslant m_0 + m' \\ 0 & \text{if} \quad m_0 + m' \leqslant \bar{k} \end{cases}$$

Then

$$\delta\colon I_\Sigma^{\mu'(\bar{k})}/I_\Sigma^{\mu(\bar{k})} \to \mathcal{O}_X/I_\Sigma^{\mu''(\bar{k})}$$

preserves the I_Σ-adic filtration, and the associated graded map in degree i is just multiplication by $i(i-1)(i-2)\cdots(i-m_0)$. Since $m_0 < \bar{k} < p$ and $m' \equiv 0 \pmod p$, no $j \in [\mu'(\bar{k}) - m_0, \mu'(\bar{k})]$ is divisible by p, and consequently the map is again injective. ∎

As we observed above, Theorems (4.5) and (4.7) of [9] apply to $\mathcal{E}$. Thus, the filtration induced by A_Φ on $H^1(X/k, \mathcal{E})$ is finer than the mod p abstract Hodge filtration F_Φ of the F-crystal $H^1(X/W, \mathcal{E}, \Phi)$ and the two filtrations coincide in the range of degeneracy of the Hodge spectral sequence. The next results attempt to make the consequences explicit.

Corollary 3.22. *Let* $j := \max(m_0, m_1 - 1)$ *if* $m_1 \neq 0$ *and* $p - 1$ *if* $m_1 = 0$. *Then if* $k \geqslant m + 1 - j$, *the map*

$$H^1(X/k, A_\Phi^k \mathcal{E}) \to H^1(X/k, \mathcal{E})$$

is injective, and the dimension of its image is $\mu(\bar{k}) - \mu'(\bar{k}) - \mu''(\bar{k})$. *Furthermore,*

$$A_\Phi^k H^1(X/k, \mathcal{E}) = F_\Phi^k H^1(X/k, \mathcal{E})$$

if $k \geqslant m + 1 - j$.

It is again a little nicer to use parabolic cohomology instead of ordinary cohomology. As the former is a direct summand of the latter in our case, the Frobenius Hodge filtrations are strictly compatible. Moreover, parabolic cohomology is self dual of weight $m+1$, so that $h_{!*}^k(\Phi) = h_{!*}^{m+1-k}(\Phi)$.

Corollary 3.23. *Suppose that* $\bar{k} := m + 1 - k \leqslant j$, *where* j *is as in* (3.22); *and let* $d(m)$ *be the dimension of* $\Gamma(X, I_\infty \otimes \omega^{m+2})$, *the space of cusp forms of weight* $m + 2$, *and let* σ *be the number of supersingular points.*

Furthermore, for $\bar{k} \leqslant j$, where j is defined as in (3.22), *let* $\varepsilon(\bar{k})$ *be defined as follows.*

1. *If* $m_1 \neq 0$,

$$\varepsilon(\bar{k}) := \begin{cases} 1 & \text{if } \ \bar{k} \neq m_0 + 1 \\ 2 & \text{if } \ \bar{k} = m_0 + 1 \end{cases}$$

2. *If* $m_1 = 0$, $\varepsilon = \varepsilon' + \varepsilon''$, *where*

$$\varepsilon'(\bar{k}) := \begin{cases} 1 & \text{if } \ m_0 + 1 \leqslant \bar{k} \leqslant \operatorname{ord}_p(m'') + 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\varepsilon''(\bar{k}) := \begin{cases} -m' & \text{if } \ \bar{k} = 0 \\ 0 & \text{if } \ 1 \leqslant \bar{k} \leqslant \operatorname{ord}_p(m'') + 1 \\ 1 & \text{if } \ \operatorname{ord}_p(m'' + 2) \leqslant \bar{k}. \end{cases}$$

Then the Frobenius Hodge numbers of $H^1_{!*}(X/W, \mathscr{E})$ *are given by*

$$\begin{cases} d(m) + \sigma\varepsilon(0) = d(m) - \sigma m' = d(m' + m_0) & \text{if } \ \bar{k} = 0 \\ \sigma\varepsilon(\bar{k}) & \text{if } \ \bar{k} > 0. \end{cases}$$

In particular, suppose that $\bar{k} > 0$ *and that* m *and* n *are greater than* $p\bar{k}$, *that* $m \equiv n \pmod{p^{\bar{k}+1}}$, *and that* $\bar{k}$ *satisfies the conditions of* (3.20) *with respect to* m *and* n. *Then* $h^{\bar{k}}(\Phi_m) = h^{\bar{k}}(\Phi_n)$.

Proof. Let l^i be the dimension of $F^i_\Phi H^1_{!*}(X/k, \mathscr{E})$, so that the ith Frobenius Hodge number h^i is $l^i - l^{i+1}$. By duality, $h^{\bar{k}} = h^{m+1-\bar{k}} = l^{m+1-\bar{k}} - l^{m+1-(\bar{k}-1)}$. If $i \leqslant j$, then l^i is given by (3.22). The formula for the Hodge numbers then follows by using the explicit formulas for μ, μ', and μ''. It implies the stated p-adic continuity because of the fact that if $m \equiv n \pmod{p^{\bar{k}+1}}$, then $m_0 = n_0$ and $\bar{k} \geqslant \operatorname{ord}_p(m'')$ if and only if $\bar{k} \geqslant \operatorname{ord}_p(n'')$. The lower bound on m and n guarantees that the values of m' and n' do not affect the Hodge numbers. ∎

It is remarkable how closely the nature of the p-adic continuity of the Hodge numbers with respect to the weight resembles the pattern conjectured by Gouvea and Mazur for the Newton numbers (which presumably lies much deeper).

Just as an example, we include a table (TABLE I) showing some values of the $\varepsilon(\bar{k})$ for $m = 1, ..., 122$ for $p = 5$. An asterisk means that the value of the Hodge number is not determined by (3.23). The point of the table is to show that the p-adic continuity is not trivial.

TABLE I

	$\bar{k}$						$\bar{k}$						$\bar{k}$				
m	0	1	2	3	4	m	0	1	2	3	4	m	0	1	2	3	4
0	0	*	*	*	*	41	−8	1	2	*	*	82	−16	1	0	*	*
1	0	0	*	*	*	42	−8	1	1	*	*	83	−16	1	0	1	*
2	0	0	0	*	*	43	−8	1	1	1	*	84	−16	1	0	1	1
3	0	0	0	0	*	44	−8	1	1	1	0	85	−17	2	*	*	*
4	0	0	0	0	0	45	−9	2	1	1	*	86	−17	1	*	*	*
5	−1	*	*	*	*	46	−9	1	2	1	*	87	−17	1	1	*	*
6	−1	1	*	*	*	47	−9	1	1	2	*	88	−17	1	1	0	*
7	−1	1	0	*	*	48	−9	1	1	1	*	89	−17	1	1	0	1
8	−1	1	0	0	*	49	−9	1	1	1	1	90	−18	2	1	*	*
9	−1	1	0	0	0	50	−10	1	2	1	1	91	−18	1	2	*	*
10	−2	2	*	*	*	51	−10	0	2	2	1	92	−18	1	1	*	*
11	−2	1	*	*	*	52	−10	0	1	2	2	93	−18	1	1	1	*
12	−2	1	1	*	*	53	−10	0	1	1	2	94	−18	1	1	1	0
13	−2	1	1	0	*	54	−10	0	1	1	1	95	−19	2	1	1	*
14	−2	1	1	0	0	55	−11	*	*	*	*	96	−19	1	2	1	*
15	−3	2	1	*	*	56	−11	1	*	*	*	97	−19	1	1	2	*
16	−3	1	2	*	*	57	−11	1	0	*	*	98	−19	1	1	1	*
17	−3	1	1	*	*	58	−11	1	0	1	*	99	−19	1	1	1	1
18	−3	1	1	1	*	59	−11	1	0	1	1	100	−20	1	2	1	1
19	−3	1	1	1	0	60	−12	2	*	*	*	101	−20	0	2	2	1
20	−4	2	1	1	*	61	−12	1	*	*	*	102	−20	0	1	2	2
21	−4	1	2	1	*	62	−12	1	1	*	*	103	−20	0	1	1	2
22	−4	1	1	2	*	63	−12	1	1	0	*	104	−20	0	1	1	1
23	−4	1	1	1	*	64	−12	1	1	0	1	105	−21	*	*	*	*
24	−4	1	1	1	1	65	−13	2	1	*	*	106	−21	1	*	*	*
25	−5	1	2	1	1	66	−13	1	2	*	*	107	−21	1	0	*	*
26	−5	0	2	2	1	67	−13	1	1	*	*	108	−21	1	0	1	*
27	−5	0	1	2	2	68	−13	1	1	1	*	109	−21	1	0	1	1
28	−5	0	1	1	2	69	−13	1	1	1	0	110	−22	2	*	*	*
29	−5	0	1	1	1	70	−14	2	1	1	*	111	−22	1	*	*	*
30	−6	*	*	*	*	71	−14	1	2	1	*	112	−22	1	1	*	*
31	−6	1	*	*	*	72	−14	1	1	2	*	113	−22	1	1	0	*
32	−6	1	0	*	*	73	−14	1	1	1	*	114	−22	1	1	0	1
33	−6	1	0	1	*	74	−14	1	1	1	1	115	−23	2	1	*	*
34	−6	1	0	1	1	75	−15	1	2	1	1	116	−23	1	2	*	*
35	−7	2	*	*	*	76	−15	0	2	2	1	117	−23	1	1	1	0
36	−7	1	*	*	*	77	−15	0	1	2	2	118	−23	1	1	1	*
37	−7	1	1	*	*	78	−15	0	1	1	2	119	−23	1	1	1	0
38	−7	1	1	0	*	79	−15	0	1	1	1	120	−24	2	1	1	*
39	−7	1	1	0	1	80	−16	*	*	*	*	121	−24	1	2	1	*
40	−8	2	1	*	*	81	−16	1	*	*	*	122	−24	1	1	2	*

REFERENCES

1. P. Berthelot and A. Ogus, "Notes on Crystalline Cohomology," Annals of Mathematics Studies, Princeton Univ. Press, Princeton, NJ, 1978.
2. P. Deligne and L. Illusie, Cristaux ordinaires et coordonnés canoniques, *in* "Surfaces Algébriques," volume 868 of Lecture Notes in Mathematics, Vol. 868, Springer-Verlag, Berlin/New York, 1981.
3. G. Faltings, Crystalline cohomology on open varieties—Results and conjectures, *in* "The Grothendieck Festschrift," Vol. II, pp. 219–248, Birkhäuser, Basel, 1990.
4. G. Faltings, Integral crystalline cohomology over very ramified valuation rings, *J. Amer. Math. Soc.* **12**, No. 1 (1999), 117–144.
5. F. Q. Gouvêa and B. Mazur, Searching for p-adic eigenfunctions, *Math. Res. Lett.* **2**, No. 5 (1995), 515–536.
6. R. C. Gunning, Special coordinate coverings of riemann surfaces, *Math. Ann.* **170** (1967), 67–86.
7. P. Lorenzon, Indexed algebras associated to a log structure and a theorem of p-descent on log schemes, *Manuscr. Math.* **101** (2000), 271–299.
8. S. Mochizuki, "Foundations of p-adic Teichmüller Theory," Amer. Math. Soc., Providence, RI, 1999.
9. A. Ogus, Frobenius and the Hodge spectral sequence, *Adv. in Math.* (2001).
10. A. Ogus, F-crystals and Griffiths transversality, *in* "Proceedings of the International Symposium on Algebraic Geometry, Kyoto 1977," pp. 15–44, Kinokuniya, Tokyo, 1977.
11. A. Ogus, "F-Crystals, Griffiths Transversality, and the Hodge Decomposition,"Astérisque, Vol. 221, Soc. Math. de France, 1994.
12. A. J. Scholl, Modular forms and de Rham cohomology; Atkin Swinnerton–Dyer congruences, *Invent. Math.* **79**, No. 1 (1985), 49–77.
13. D. Ulmer, Slopes of modular forms, *Contemp. Math.* **174** (1994), 167–183.