Let $I =: [a,b]$ let γ be a function $I \to R$ and let $P := (x_0, \cdots x_n)$ be a partition of I. Define the *variation* $V_P(\gamma)$ of γ over P to be the sum $\sum_i |\gamma(x_i) - \gamma(x_{i-1})|$. Then γ is said to be *of bounded variation* if $\{V_P(\gamma)\}$, as P ranges over the set of all partitions, is bounded, and in this case we define $V(\gamma)$ to be the least upper bound. Let $BV(I)$ denote the set of all functions of bounded variation, and prove the following:

1. If γ is monotone, $\gamma \in BV(I)$. If γ is Lipschitz, for example if γ' exists and is bounded, then $\gamma \in BV(I)$.

2. $BV(I)$ is a linear subspace of $B(I)$. If $V(\gamma) = 0$, f is constant. If $BV_0(I)$ is the set subset of all $\gamma \in BV(I)$ such that $\gamma(a) = 0$, then V defines a norm on $BV_0(I)$.

3. $BV(I)$ is exactly the set of all functions on I which can be written as a difference of two monotone functions. (Hint: If $\gamma \in BV(I)$ and $x \in I$, then the restriction γ_x of γ to $[a,x]$ is of bounded variation, and $V(\gamma_x)$ is an increasing function of x.

4. It follows from (c) that we can define, for any continuous function f on I and any function γ of bounded variation on I, the integral $\int f d\gamma$. Prove that $|\int f d\gamma| \leq ||f|| V(\gamma)$.