Theorem: Let X and Y be compact topological spaces. Then $X \times Y$ is compact.

Proof: Let $\mathcal{B}$ be the set of all subsets of $X \times Y$ of the form $U \times V$, where $U \subseteq X$ and $V \subseteq Y$ are open. It is enough to prove that every subset $\mathcal{C}$ of $\mathcal{B}$ which covers $X \times Y$ contains a finite subset which covers $X \times Y$. Let $\mathcal{C}$ be such a cover. For each $(x, y) \in X \times Y$, there is a set $W_{xy} \in \mathcal{C}$ such that $(x, y) \in W_{xy}$. Write $W_{xy} = U_{xy} \times V_{xy}$. Fix y, and let $\mathcal{C}_y := \{U_{xy} : x \in X\}$. This is an open cover of X, and hence there is a finite subset $\mathcal{C}'_y := \{U_{x_1 y} \ldots U_{x_n y}\}$ which covers X. (Note: this n depends on y, but this won't matter in the end.) Let $V_y := V_{x_1,y} \cap \cdots V_{x_n y}$, which is an open neighborhood of y. The set of all $V_y : y \in Y$ is thus an open cover of Y, and since Y is compact, there is a finite set $\{V_{y_1}, \cdots V_{y_m}\}$ which covers Y. Now I claim that the set $\mathcal{C}' := \{W_{x_i y_j \; 1 \leq i \leq n, 1 \leq j \leq m} \subset \mathcal{C}\}$ covers $X \times Y$. (Note that there are only finitely many y'_j here so we can use a single n which works for all of them.) To prove this, suppose $(x, y) \in X \times Y$. We know that for some j, $y \in V_{y_j}$. Furthermore, $\mathcal{C}'_{y_j}$ covers X, so there is some i such that $x \in U_{x_i y_j}$. But $V_{y_j} \subseteq V_{x_i y_j}$. Hence $(x, y) \in U_{x_i y_j} \times V_{x_i y_j}$.