We use the notation in the problem.

1. Suppose γ is monotone increasing. Then for any partition P, $V_P(\gamma) = \sum_i \gamma(x_i) - \gamma(x_{i-1}) = \gamma(b) - \gamma(a)$. Thus $V(\gamma) = \gamma(b) - \gamma(a)$, so γ is of bounded variation. If γ is monotone decreasing, we can apply a similar argument. Suppose γ is Lipschitz, with constant M. Then for any P, $V_P(\gamma) \leq \sum_i M(x_i - x_{i-1}) \leq M(b-a)$, and γ is of bounded variation, with $V(\gamma) \leq M(b-a)$.

2. Suppose that γ_1 and γ_2 are functions, c_1 and c_2 are constants, and P is a partition. Then if $\gamma := c_1\gamma_1 + c_2\gamma_2$,

$$\begin{aligned} V_P(\gamma) &= \sum_i |c_1\gamma_1(x_i) + c_2\gamma_2(x_i) - c_1\gamma_1(x_{i-1}) - c_2\gamma_2(x_{i-1}) \\ ) &\leq \sum |c_1|(\gamma_1(x_i) - c_1\gamma_1(x_{i-1})| + \sum_i |c_1|(\gamma_1(x_i) - c_1\gamma_1(x_{i-1})| \\ &\leq |c_1|V_P(\gamma_1) + |c_2|V_P(\gamma_2) \end{aligned}$$

Hence if γ_1 and γ_2 are of bounded variation, so is γ, and $V(\gamma_1+\gamma_2) \leq V(\gamma_1)+V(\gamma_2)$. Since it is also true that $V(c\gamma) = |c|V(\gamma)$ for any γ, V defines a seminorm on BV. Finally, if $V(\gamma) = 0$, then for any partition P, $\gamma(x_i) = \gamma(x_{i-1})$ for any i, hence γ is constant, and in particular if $\gamma(a) = 0$, $\gamma = 0$. Thus V defines a norm on BV_0.

3. It follows from the previous parts that any function which the difference between two increasing function has bounded variation. For the converse, we proceed as in the hint. Suppose γ is of bounded variation and for $x \in [a, b]$, let γ_x denote the restriction of γ to $[a, x]$. Then for any partition P of $[a, x]$, $P' := P \cup \{b\}$ is a partition of I and $V_P(\gamma_x) \leq V_{P'}(\gamma) \leq V(\gamma)$. Thus γ_x is of bounded variation, and $V(\gamma_x) \leq V(\gamma)$. Let $\alpha(x) := V(\gamma_x)$. If $x' > x$, the same argument shows that $V(\gamma_x) \leq V(\gamma_{x'})$ so α is an increasing function. Furthermore, if $x' \geq x$, and if P is a partition of $[0, x]$, and $P' := P \cup \{x'\}$, then $V_{P'}(\gamma_{x'}) = V_P(\gamma_x)+|\gamma(x')-\gamma(x)| \geq V_P(\gamma_x)+\gamma(x')-\gamma(x)$. Hence $V(\gamma_{x'}) \geq V_P(\gamma_x)$ for all P, and hence $V(\gamma_{x'}) \geq V(\gamma_x)(\gamma_x)+\gamma(x')-\gamma(x)$. This shows that $\alpha(x')-\gamma(x') \geq \alpha(x)-\gamma(x)$. In other words, $\beta := \alpha-\gamma$ is also a monotone increasing function, and hence $\gamma = \alpha - \beta$ is the difference between two such functions.

4. Suppose γ is of bounded variation and f is continuous. Write $\gamma = \alpha - \beta$, where α and β are increasing. We would like to define $\int f d\gamma$ to be $\int f d\alpha - \int f d\beta$. Note that if also $\gamma = \alpha' - \beta'$, then $\alpha + \beta' = \alpha' + \beta$,

hence $\int f d\alpha + \int f d\beta' = \int f d\alpha' + \int f d\beta$ and hence $\int f d\alpha - \int f d\beta = \int f d\alpha' - \int f d\beta'$. Thus the expression for $\int f d\gamma$ is independent of the choices and is well-defined. Now if P is any marked partition of I, note that

$$S(P, \gamma) := \sum_i f(p_i)\Delta_i\gamma = S(P, \alpha) - S(P, \beta).$$

For sufficiently fine P, $S(P, \alpha)$ and $S(P, \beta)$ become arbitrarily close to $\int f d\alpha$ and $\int f d\beta$, so $S(P, \gamma)$ becomes arbitrarily close to $\int f d\gamma$. On the other hand,

$$|S(P, \gamma)| \leq \sum_i |f(x_i)|\Delta_i\gamma| \leq \sum_i ||f|||\Delta_i\gamma| \leq ||f||V(\gamma).$$

Since this holds for all P, it follows that $|f d\gamma| \leq ||f||V(\gamma)$.