Pointwise convergence

Let X be a set, let Y be a topological space, and let Y^X denote the set of all functions $f\colon X \to Y$. Recall that for each $x \in X$ and each open set U in Y, $B(x, U) := \{f \in Y^X : f(x) \in U\}$. Then $\mathcal{B}$ is the set of finite intersections of sets of the form $B(x, U)$, $\mathcal{T}$ is the set of all unions of elements of $\mathcal{B}$, and $\mathcal{T}$ is a topology on Y^X. A net in $(Y^X, \mathcal{T})$ converges if and only if for each $x \in X$, the corresponding net in Y converges.

Now suppose Y is $\mathbf{R}$ with its usual topology and let $\mathcal{T}$ again denote the above topology on $\mathcal{F} := \mathbf{R}^X$.

1. If X is finite, $\mathcal{T}$ is the topology associated with the metric sending $f \in \mathcal{F}$ to $\sum_x |f(x)|$.

2. If X is countable, $\mathcal{T}$ is the topology associated with the metric sending $f \in \mathcal{F}$ to $\sum_n 2^{-n}|f(x_n)|$, for any enumeration $x.$ of X.

3. If X is uncountable, $\mathcal{T}$ is not associated with any metric, as it does not satisfy the first axiom of countability.

The following exercises elucidate, I hope, (3) above. Let $X = [0, 1]$, for $n \in \mathbf{Z}^+$, let $f_n(x) = \sin(2\pi n x)$, and let $E := \{f_n : n \in \mathbf{Z}^+\}$. (Thanks to G. Bergman and M. Christ for help with these.)

1. The constant function 0 is in the closure of E.
   Hint: We have to prove that for every $\epsilon > 0$ and every finite subset S of X, there exists an n such that $|f_n(x)| < \epsilon$ for all $x \in S$. For each $x \in X$, let $z := e^{2\pi i x}$, a complex number of absolute value 1, and note that $f_n(x)$ is the imaginary part of z^n. For each natural number m, let T_m denote the set of m-tuples of complex numbers of absolute value 1 Then it is enough to prove the following result:

   Theorem: Let m be a natural number, let $\epsilon > 0$ be any positive number, and let $z \in T_m$ be any point. Then there exists a natural number $n > 0$ such that $|z^n - 1| < \epsilon$.

   To prove this use the fact that T_m is compact to show that the sequence (z^k) has a convergent subsequence. Hence there exist two distinct numbers k, j with z^k and z^j within ϵ of each other.

2. The sequence $(f.)$ has no convergent subsequences. This follows from the following result:
   Theorem: Let $n.$ be any strictly increasing sequence of natural numbers and let s, t be any two elements of $[-1, 1]$. Then there exist an

$x \in [0,1]$ and subsequences $n'_.$ and $n''_.$ of $n.$ such that $f_{n'_k}(x)$ converges to s and $f_{n''_k}(x)$ converges to t.

Hint. First prove:
Claim: Let $J \subseteq [0,1]$ be a closed interval of length $\delta > 0$ and let y be in $[0,1]$. Then for every $n > 1/\delta$m there exists an $x \in J$ and a $k \in \mathbf{Z}$ such that $y = nx + k$. Hence for every $n > 1/\delta$ and every $\epsilon > 0$, there exists a closed interval $J' \subseteq J$ of positive length such that for every $x' \in J'$, there exists a $k \in \mathbf{Z}$ such that $|y - nx' - k| < \epsilon$.

Now if $n.$ is given, use the claim to construct by induction a subsequence $m.$ of $n.$ and a nested sequence $J.$ of closed subintervals such that for all i, $J_{i+1} \subseteq J_i$, the length of J_i is less than $1/i$, and for every $x \in J_i$ there exists k such that $|s - m_i x - k| < 1/i$ if i is odd, and such that $|t - m_i x - k| < 1/i$ if i is even.

Remark: The above argument in fact proves the following statement, which perhaps looks more natural. Let $n.$ be any strictly increasing sequence of natural numbers and let s, t be any two complex numbers of absolute value 1. Then there exist a complex number x of absolute value 1 and subsequences $n'_.$ and $n''_.$ of $n.$ such that $f_{n'_k}(x)$ converges to s and $f_{n''_k}(x)$ converges to t.