The circle remains unbroken

1. Prove that very compact connected subset of $\mathbf{R}$ is a closed interval.

2. Prove that S^1 is compact and connected. Let $e\colon [0, 2\pi] \to S^1$ be defined by $\theta \mapsto (\cos\theta, \sin\theta)$. You may use the fact that e is surjective and continuous, and that its restriction to $[0, 2\pi)$ is injective.

3. Suppose that x and y are two distinct points of S^1. Prove that S^1 can be written as a union of two sets A and B, with $A \cap B = \{x, y\}$, and each of which is compact and connected.

4. Let $f\colon S^1 \to \mathbf{R}$ be a continuous map. Prove that the image of f is a closed interval, say $[a, b]$. Prove that for each $c \in (a, b)$, there exist at least two points z of S^1 with $f(z) = c$.