Math 54 Spring 2005

Solutions to Homework

Section 10.1

Problems 1-13: Either solve the given boundary value problem or else show it has no solution.

1. $y^{''} + y = 0, y(0) = 0, y^{'}(\pi) = 1.$

   $r^2 + 1 = 0, so\ r = \pm i, so\ y = c_1 \cos x + c_2 \sin x$. This is the general solution. $y^{'} = -c_1 \sin x + c_2 \cos x$. Plug in the initial values. We get $c_1 = 0, c_2 = 1$, so the solution is $y = \sin x$.

3. $y^{''} + y = 0, y(0) = 0, y^{(}L) = 0.$

   $r^2 + 1 = 0, so\ r = \pm i, so\ y = c_1 \cos x + c_2 \sin x$. This is the general solution. Plug in the initial values. We get $c_1 = 0, c_1 \cos L + c_2 \sin L = 0,\ i.e.\ c_2 \sin L = 0$. If $L = n\pi$ for some integer n, then $\sin L = 0$ so c_2 can be anything and the solution is $y = c \sin x$, otherwise $\sin L \neq 0, so\ c_2 = 0$ and the only solution is $y = 0$.

5. $y^{''} + y = x, y(0) = 0, y(\pi) = 0.$

   First solve the homogeneous equation $y^{''} + y = 0. r^2 + 1 = 0, so\ r = \pm i, so\ y = c_1 \cos x + c_2 \sin x$. This is the general solution of the homogeneous equation. Now,we look for a particular solution of the initial equation of the form $ax + b$. Since $(ax + b)^{''} = 0$, plugging this in the equation gives $ax + b = x, so\ a = 1, b = 0$. So the general solution to the original equation is $y = c_1 \cos x + c_2 \sin x + x$. Now plug in the initial values. We get $c_1 = 0, -c_1 + \pi = 0$, which is not possible so the boundary value problem has no solution.

7. $y^{''} + 4y = \cos x, y(0) = 0, y(\pi) = 0.$

   First solve the homogeneous equation $y^{''} + 4y = 0. r^2 + 4 = 0, so\ r = \pm 2i, so\ y = c_1 \cos 2x + c_2 \sin 2x$. This is the general solution of the homogeneous equation. Now,we look for a particular solution of the initial equation of the form $a \cos x + b \sin x$. Plugging this in the equation gives $-a \cos x - a \sin x + 4a \cos x + 4b \sin x = \cos x, so\ 3a = 1, 3b = 0, i.e.\ a = 1/3, b = 0$. So the general solution to the original equation is $y = c_1 \cos 2x + c_2 \sin 2x + 1/3 \cos x$. Now plug in the initial values. We get $c_1 + 1/3 = 0, c_1 - 1/3 = 0$, which is not possible so the boundary value problem has no solution.

8. $y^{''} + 4y = \sin x, y(0) = 0, y(\pi) = 0.$

   First solve the homogeneous equation $y^{''} + 4y = 0. r^2 + 4 = 0, so\ r = \pm 2i, so\ y = c_1 \cos 2x + c_2 \sin 2x$. This is the general solution of the homogeneous equation. Now,we look for a particular solution of the initial equation of the form $a \cos x + b \sin x$. Plugging this in the equation gives $-a \cos x - a \sin x + 4a \cos x + 4b \sin x = \sin x, so\ 3a = 0, 3b = 1, i.e.\ a = 0, b = 1/3$. So the general solution to the original equation is $y = c_1 \cos 2x + c_2 \sin 2x + 1/3 \sin x$. Now plug in the initial values. We get $c_1 = 0, c_1 = 0$, so the solution is $y = c \sin 2x + 1/3 sinx$.

11. $x^2 y^{''} - 2xy' + 2y = 0, y(1) = -1, y(2) = 1.$

    If you have a differential equation of the form $ax^2 y^{''} + bxy' + cy = 0$, the substitution $t = \ln x, i.e.\mathbf{x} = e^x$ is useful. When you do the substitution, you have to change everything to the t variable. In particular the y'' means derivative with respect to x, and we have to change it to a derivative with respect to t. Here is how it can be done.

$dy/dx = (dy/dt)(dt/dx) = (dy/dt)(\frac{d(\ln x)}{dx}) = (dy/dt)(1/x)$.

$\frac{d^2(y)}{dx^2} = \frac{d(\frac{dy}{dx})}{dx} = \frac{d(\frac{dy}{dx})}{dt}\frac{dt}{dx} = \frac{d(\frac{dy}{dx})}{dt}\cdot(\frac{1}{x}) = \frac{d(\frac{dy}{dt}\cdot\frac{1}{x})}{dt}\cdot(\frac{1}{x}) = \frac{d}{dt}(dy/dx)(dt/dx) = \frac{d}{dt}(dy/dx)(1/x) =$ (by product rule)$= (\frac{d(\frac{dy}{dt})}{dt}\cdot\frac{1}{x}+\frac{d(\frac{1}{x})}{dt}\cdot\frac{dy}{dt})\cdot(\frac{1}{x}) = (\frac{d(\frac{dy}{dt})}{dt}\cdot\frac{1}{x}+\frac{d(e^{-t})}{dt}\cdot\frac{dy}{dt})\cdot(\frac{1}{x}) = (\frac{d(\frac{dy}{dt})}{dt}\cdot\frac{1}{x}+(-e^{-t})\cdot\frac{dy}{dt})\cdot(\frac{1}{x}) = (\frac{d(\frac{dy}{dt})}{dt}\cdot\frac{1}{x}+(-\frac{1}{x})\cdot\frac{dy}{dt})\cdot(\frac{1}{x}) = (\frac{d^2y}{dt^2}-\frac{dy}{dt})\cdot(\frac{1}{x^2})$.

When we plug these in the differential equation we get the following differential equation with respect to t: $(y''-y')-2y'+2y=0$ Let's solve this. $r^2-3r+2=0 so\ r=1,2 \Rightarrow y = c_1e^t+c_2e^{2t}$. By switching back to x we get $y=c_1x+c_2x^2$. Plugging in the initial values gives $c_1+c_2=-1, 2c_1+4c_2=1$ which gives $c_2=3/2, c_1=-5/2$ which gives us $y=-5/2x+3/2x^2$ as a solution of the boundary value problem.

13. $x^2y^{''}+5xy'+(4+\pi^2)y=0, y(1)=0, y(e)=0.$

    If you have a differential equation of the form $ax^2y^{''}+bxy'+cy=0$, the substitution $t = \ln x, i.e.\mathbf{x}=e^x$ is useful. (see 10.1.11 for details)

    When we do the substitution we get the following differential equation with respect to t: $(y''-y')+5y'+(4+\pi^2)y=t$ First solve the homogeneous equation. $r^2+4r+4+\pi^2=0 i.e.\ (r+2)^2 = -\pi^2 so\ r=-2\pm\pi i \Rightarrow y=e^{-2t}(c_1\cos\pi t+c_2\sin\pi t)$. This gives the general solution of the homogeneous equation. Now, find a particular solution. Look for one of the form $at+b$. Plugging thin in the equation gives $5a+(4+\pi^2)(at+b)=t\ so\ (4+\pi^2)a=1, 5a+(4+\pi^2)b = 0 \Rightarrow a=\frac{1}{(4+\pi^2)}, b=\frac{-5}{(4+\pi^2)^2}$. The general solution to the differential equation then is $y = e^{-2t}(c_1\cos\pi t+c_2\sin\pi t)+\frac{1}{(4+\pi^2)}t-\frac{5}{(4+\pi^2)^2}$ By switching back to x we get $y=\frac{1}{x^2}(c_1\cos\pi\ln x + c_2\sin\pi\ln x)+\frac{1}{(4+\pi^2)}lnx-\frac{5}{(4+\pi^2)^2}$. Plugging in the initial values gives $c_1-\frac{5}{(4+\pi^2)^2}=0, -\frac{1}{e^2}c_1+\frac{1}{(4+\pi^2)}-\frac{5}{(4+\pi^2)^2}=0$ which is not possible, so the boundary value problem has no solution.

15. $y''+\lambda y=0, y'(0)=0, y(\pi)=0.$

    $r^2+\lambda=0, so\ r=\pm\sqrt{-\lambda}$ Now consider two cases. First case $\lambda\le 0$ i.e. r is real and $y = c_1e^{\sqrt{-\lambda}x}+c_2e^{-\sqrt{-\lambda}x}$. Plug in the initial values. We get $\sqrt{-\lambda}c_1-\sqrt{-\lambda}c_2=0, c_1e^{\sqrt{-\lambda}\pi}+c_2e^{-\sqrt{-\lambda}\pi}$. Since the equations for c_1, c_2 are linearly independent, the only solution is $c_1=c_2=0$ so $y=0$ i.e. y is not an eigenvector.

    Second case: $\lambda>0$, i.e. r is purely imaginary. Then $r=\pm\sqrt{\lambda}i$ and the solution is $y = c_1\cos\sqrt{\lambda}x+c_2\sin\sqrt{\lambda}x$. $y'=-\sqrt{\lambda}c_1\sin\sqrt{\lambda}x+\sqrt{\lambda}c_2\cos\sqrt{\lambda}x$ Plugging in the initial values gives $\sqrt{\lambda}c_2=0, c_1\cos\sqrt{\lambda}\pi+c_2\sin\sqrt{\lambda}\pi\ i.e.\ c_2=0, c_1\cos\sqrt{\lambda}\pi=0$. Now, if $\sqrt{\lambda}\pi$ is not of the form $-\frac{\pi}{2}+n\pi$ for some integer n, then $\cos\sqrt{\lambda}\pi\neq 0\ so\ c_1=0\Rightarrow y=0$ so y is not an eigenvector. But if $\sqrt{\lambda}\pi$ is of the form $-\frac{\pi}{2}+n\pi$ then $\cos\sqrt{\lambda}\pi=0$ so c_1 can be anything. This happens when $\sqrt{\lambda}=n-1/2$ and $\lambda>0$ i.e. $\lambda=(n-1/2)^2$ for some positive integer n. These are the eigenvalues. The corresponding eigenfunctions are $y=c_1\sin{(n-1/2)x}$.