1. Let G be a cyclic group with only one generator. Then G has at most two elements. To see this, note that if g is a generator for G, then so is g^{-1}. If G has only one generator, it must be the case that $g = g^{-1}$. But then $g^2 = e$. Since g generates G, it follows that G has at most two elements.

2. Let H be a nonempty finite subset of a group G. Prove that if H is closed, then it is a subgroup. To see this, suppose $h \in H$. Since H is closed, $h^n \in H$ for every positive integer n, and since H is finite, there exist n, m with $n > m$ such that $h^n = h^m$. Then $h^{n-m} = e$, so $e \in H$. Furthermore, $h^{-1} = h^{n-m-1} \in H$. Thus H is a subgroup.

3. Suppose a and b are elements of a group G and that ab has order n. Then ba also has order n. To see this, note that $(ba)^n = a^{-1}(ab)^n a$. Thus if $(ab)^n = e$, then $(ba)^n = e$. The converse holds by symmetry, and so ab and ba have the same order.

4. Suppose G is a group which has only finitely many subgroups. We want to prove that G is finite. First of all, recall that if $g \in G$, then $< g >$ is a cyclic group containing g. If $< g >$ is infinite, then $< g >$ is isomorphic to $\mathbf{Z}$. But the group $\mathbf{Z}$ has infinitely many subgroups (one for each natural number), and then G would also have infinitely many subgroups, a contradiction. Hence each $< g >$ is finite. Since G has only finitely many subgroups, and since each of these is finite, and since every element of G is contained in a finite group, G has only finitely many elements.

5. The group V_4 has the property that every proper subgroup is cylic, but it itself is not cyclic.

6. Suppose that G is a group and $a \in G$ is the unique element of order 2. Then $ax = xa$ for all $x \in G$. To see this, let $b := xax^{-1}$. Then $a = x^{-1}bx$. Since $a \neq e$, $b \neq a$, and an easy calculation shows that $b^2 = e$, so $\langle b \rangle$ has order 2. Since a is unique, $a = b$, and this implies that $ax = xa$.

7. Let G be a cyclic group of order n, written multiplicatively, and let m be a divisor of n. Consider the set of all x in G such that $x^m = e$.

This is clearly a subgroup H of G, since G is commutative. But every subgroup of G is cyclic, and if g generates G, $H = \langle g^d \rangle$ for some divisor d of n. Since $g^d \in H$, $g^{dm} = e$, and hence n divides dm. On the other hand, if $d' := n/m$, $g^{d'} \in H$, so d divides d'. Write $dm = an$ and $d' = bd$. Then $n = bmd = ban$, so $b = a = 1$. This implies that $d = d'$, so H has m elements.