Mathematics 254a Exercises

November 2, 2007

1. Determine when the class number of $\mathbf{Q}(\sqrt{m})$ is one when m is among

$$2, 3, 5, 7, 173, 437, -1, -23, -7, -11, -19, -43, -67, -163.$$

   (Just do some of these.)

2. Show that the ideal class group of $\mathbf{Q}(\sqrt{-23})$ is cyclic of order three.

3. Show that the ideal class group of $\mathbf{Q}(\sqrt{-39})$ is cyclic of order four, while the ideal class group of $\mathbf{Q}(\sqrt{-21})$ is the Klein four group.

4. Show that the class number of $\mathbf{Q}(\sqrt[3]{2})$ is one.

5. Let E be a numberfield with ring of integers $\mathcal{O}_E$. Show that there is a finite extensions E' of E such that for every ideal I of $\mathcal{O}_E$, the ideal $I_{\mathcal{O}'_E}$ is principal. Hint: If $I^n = (a\mathcal{O}_E)$, adjoin the nth root of a.

6. Let E be a real quadratic field, i.e. $E = \mathbf{Q}(\sqrt{m})$, where m is positive and square free. Then the group of units modulo ± 1 is cyclic, and so there is a unique positive unit which generates the free part. This unit is called the fundamental unit of E. Suppose $m \cong 2$ or 3 modulo 4, and let b be the smallest integer such that $mb^2 \pm 1$ is a square, say a^2, with $a > 0$. Show that $a + b\sqrt{m}$ is the fundamental unit of E. Find an analogous statement when $m \cong 1 \bmod 4$. Compute the fundamental unit e.g. with $m = 7$. If you have a computer, do some more.