Mathematics 254a Exercises

September 7, 2007

1. Let $\theta: A \to B$ be a ring homomorphism. Let $f: T \to S$ be the map $\operatorname{Spec} B \to \operatorname{Spec} A$ corresponding to θ. Prove that if I is an ideal of A and $Z(I)$ is the corresponding closed subset of S, then $f^{-1}(Z(I))$ is the set $Z(IB)$, where IB is the ideal of B generated by the image of I under θ. Show that if P is a prime ideal of A, the evident map $\operatorname{Spec} B \otimes k(P) \to \operatorname{Spec} B$ is injective, and its image is precisely the set of prime ideals of B which lie over A. (Recall that $k(P)$ is the fraction field of A/P.)

2. Let k be a field. Classify all the two dimensional k-algebras B. Show that B/k is isomorphic to a field, to a product of two copies of k, or to $k[e]/(e^2)$. How does this relate to our study of $\mathbf{Z} \to \mathbf{Z}[i]$?