Mathematics 1A, Spring 2010 — A. Ogus
Sample Midterm Exam #2

Instructions.
Closed book exam — No formula sheets or notes are permitted. Calculators and other electronic devices are not allowed. Turn cell phones off and stow them in backpacks/pockets/purses.
Show work and/or reasoning where indicated.

(1) Calculate $f'(x)$, using any method from this course. Show your steps.
(1a) $f(x) = x^{-1}e^{2x}$
Solution: by the product rule, $f'(x) = -x^{-2}e^{2x} + x^{-1}2e^{2x} = x^{-2}e^{2x}(2x-1)$
(1b) $f(x) = x^{\ln(x)}$
Solution: Let's use logarithmic differentiation. $\ln f(x) = \ln x \ln x$, so $f'(x)/f(x) = 2\ln x/x$. Then $f'(x) = 2x^{\ln x}x^{-1}\ln x$.
(1c) $f(x) = \sqrt{\arcsin(x)}$.
Soilution: This is $1/2 \arcsin(x)^{-1/2} \arcsin'(x) = 1/2 \arcsin(x)^{-1/2}(1-x^2)^{-1/2}$.
(1d) If $x^4 + xy + 2y^4 = 20$, find dy/dx when $(x, y) = (2, 1)$.
Solution: By implicit differentiation: $4x^3 + y + xy' = -8y^3y'$, so at $(2, 1)$ we have $32 + 1 + 2y' = -8y'$ so $y' = -33/10$.

(2a) Find the maximum value of $f(x) = x(x-1)^2$ on $[-1, 2]$, and determine all points in this interval where that value is attained. Show all steps; you will be graded on these steps, not merely on your answer. **Solution:**
$f(x) = x^3 - 2x^2 + x$ so $f'(x) = 3x^2 - 4x + 1 = (3x-1)(x-1)$ and $f''(x) = 6x - 4$. The critical point are at $x = 1/3$ and $x = 1$. Since $f''(1) = 1 > 0$ it is a local minimum and $f''(1/3) = -2$ so it is a local maximum. Then $f(1/3) = 1/3(-2/3)^2 = 4/27$. But $f(2) = 2$, so $(2, 2)$ it the unique maximum.
(2b) If x and y are positive numbers and $xy = 1$, what is the minimum possible value of $x + 2y$?
Show your steps.
Solution: Since $y = 1/x$, we are trying to minimize $f(x) = x + 2x^{-1}$ for $x > 0$. Then $f'(x) = 1 - 2x^{-2}$ and this has a unique zero, when $x = \sqrt{2}$. Since $f''(x) = 4x^{-3} > 0$, this is a local minimum of f and since it is unique, it is a global minimum. The value of f at this point is $2 + 2/\sqrt{2} = 2\sqrt{2}$.

(3) Suppose that f and f' are differentiable functions on an interval (a, b), $c \in (a, b)$, and $f'(c) = 0$. What can one conclude if $f''(c) > 0$?
Solution: This is vague, but I guess the answer is that f has a local minimum at c. What if $f''(c) = 0$? Answer: nothing.

(4) Let $f(x) = x^{1/3}$. What is the equation for the linearization (also known as the linear approximation) of f at 8?
Solution: $f'(x) = 1/3x^{-2/3}$, so when $x = 8$ this is $1/12$. Thus $\ell(x) = 1/12(x-8)+2$.

(5) Show that if $x > 1$, then $\ln(x) < x - 1$.

Solution:
Let $f(x) = (x - 1) - \ln x$. Then $f'(x) = 1 - 1/x$, which is positive if $x > 1$. Thus f is increasing on $(1, \infty)$. But $f(1) = 0$. It follows that $f(x) > 0$ if $x > 1$, which implies the claim.