Math 1A, Fall 2009 — M. Christ
Solutions for Midterm Exam 1

There were three versions of this exam, with slightly different numbers and/or functions. For most problems I'll give a solution for only one of the three versions.

(1a) Use limit rules to evaluate $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$.

Solution. For any $x > 0$ which is not equal to 9,

$$\frac{\sqrt{x}-3}{x-9} = \frac{\sqrt{x}-3}{x-9}\frac{\sqrt{x}+3}{\sqrt{x}+3} = \frac{x-3}{(x-3)(\sqrt{x}+3)} = \frac{1}{\sqrt{x}+3}.$$

Thus

$$\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9} = \frac{1}{\sqrt{x}+3} = \frac{\lim_{x\to 9} 1}{\lim_{x\to 9}(\sqrt{x}+3)} = \frac{1}{3+3} = \frac{1}{6}$$

using the sum, ratio, and direct substitution rules for limits. Note that when I used the ratio rule, the limit of the denominator was not 0. □

(1b) Let $f(x) = \sqrt{x}$, with its natural domain. Does $f'(9)$ exist?

Solution.

Yes. The definition of the derivative says that $f'(9)$ exists if $\lim_{x\to 9}\frac{f(x)-f(9)}{x-9}$ exists (and is a real number). In problem (1a) we showed that this limit does exist. □

(2a) Let $f(x) = \frac{(x-2)(x-4)(x-8)}{3(x-1)(x-2)(x-8)}$, with its natural domain. Find all asymptotes of the graph of f.

Solution. Vertical asymptote at $x = 1$. Two horizontal asymptotes at $y = \frac{1}{3}$ as $x \to \infty$ and as $x \to -\infty$. (There are no asymptotes at $x = 2$ and $x = 8$.) □

(2b) Find $\lim_{x\to 0} x^2 \sin(e^{1/x})$.

Solution. The limit is zero. Let $g(x) = x^2\sin(e^{1/x})$, $f(x) = -x^2$, $h(x) = x^2$. Then since $\sin(y) \in [-1,1]$ for any real number y, $f(x) \le g(x) \le h(x)$ for all $x \ne 0$. We know that $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} h(x)$, by the direct substitution property. Therefore the *Squeeze Theorem* says that $\lim_{x\to 0} g(x)$ also exists and equals 0. □

(3) Show that there is at least one real number x which satisfies $x^6 = 1 + \sin(x)$.

Solution. Define $f(x) = x^6 - 1 - \sin(x)$ for all real numbers x. We need to show that $f(x) = 0$ for at least one x. f is a continuous function, since sums of continuous functions are continuous, and all the basic functions (including polynomials and sin) are continuous. Consider the interval $[a, b] = [0, 2]$. $f(a) = 0 - 1 - 0 < 0$, while $f(b) = 2^6 - 1 - \sin(2) \ge 2^6 - 1 - 1 = 62 > 0$.

Therefore the Intermediate Value Theorem applies and says that $f(x) = 0$ for at least one x in $[0, 2]$. □

(4a) Let $r > 0$. How is $\log_r(2)$ defined?

Solution. It is the unique real number x which satisfies $r^x = 2$. *(Full credit for that answer.)*

(A more complete answer might also include: $\log_r$ is the inverse of the function r^x. The range of r^x is $(0,\infty)$, so the domain of $\log_r$ is $(0,\infty)$; thus 2 belongs to its domain.) □

(4b) Let $f(x) = \tan(x)$ with domain $(-\frac{\pi}{2}, 0)$. Does f have an inverse? If so, what are the domain and range of the inverse function?

Solution. For $x \in (-\frac{\pi}{2}, 0)$, $\cos(x) \neq 0$ so $\tan(x) = \frac{\sin(x)}{\cos(x)}$ is defined. tan is an increasing function which takes on all values in $(-\infty, 0)$ as x varies over $(-\frac{\pi}{2}, 0)$.

Thus f has an inverse function. Its domain is $(-\infty, 0)$, and its range is $(-\frac{\pi}{2}, 0)$. □

(4c) If some *vertical* line intersects a graph at more than one point, what does this say about the graph?

Solution. The graph is not the graph of any function. □

(4d) Simplify: $\ln(5e\sqrt{x})$, assuming that $x > 0$.

Solution. $\ln(5e\sqrt{x}) = \ln(5) + \ln(e) + \ln(\sqrt{x}) = \ln(5) + 1 + \frac{1}{2}\ln(x)$. □

(4e) If the domain of f contains $(-1, 1)$, and if f is continuous at 0, must $f'(0)$ exist? Either explain in words why it must exist, or give an example of a function for which it does not exist.

Solution. No. Example: $f(x) = |x|$. ($f(x) = \sqrt{|x|}$ is also an example. Full credit for any correct example, of course, whether we learned it in this course or not.) □

(5a) Let $f(x) = x^2$. Find $\delta > 0$ such that $|f(x) - 36| < \frac{1}{1000}$ whenever $|x - 6| < \delta$.

Solution. It will be useful to know that $|f(x) = 36| = |x^2 - 36| = |x - 6| \cdot |x + 6|$.

Define $\delta = \frac{1}{13{,}000}$. Note that $\delta < 1$. Therefore if $|x - 6| < \delta$ then $x \in (5, 7)$ and thus $|x + 6| < 13$.

Therefore if $|x - 6| < \delta$ then

$$|f(x) - 36| \leq |x - 6| \cdot |x + 6| \leq \delta \cdot 13 < \frac{13}{13,000} = \frac{1}{1000}.$$

□

(5b) Show, using the precise definition of a limit, that

$$\lim_{x \to \frac{1}{3}} (9x - \frac{1}{x}) = 0.$$

Solution. It will be useful to know that for any $x \neq 0$,

$$|9x - \frac{1}{x}| = \left|\frac{9x^2 - 1}{x}\right| = \frac{|3x - 1| \cdot |3x + 1|}{|x|} = \frac{3|x - \frac{1}{3}| \cdot |3x + 1|}{|x|}.$$

Let any $\varepsilon > 0$ be given. Define $\delta = \min(\frac{1}{6}, \frac{\varepsilon}{54})$. If $|x - \frac{1}{3}| < \delta$ then $\frac{1}{6} < x < \frac{1}{2}$, so $1/|x| \leq 6$, and $|3x + 1| \leq \frac{3}{2} + 1 = \frac{5}{2} < 3$.

Therefore if $0 < |x - \frac{1}{3}| < \delta$ then

$$|9x - \frac{1}{x}| < 3 \cdot 6 \cdot 3 \cdot |x - \tfrac{1}{3}| < 54\delta.$$

Since $\delta \leq \varepsilon/54$, this is $\leq \varepsilon$, and therefore $|9x - \frac{1}{x}| < \varepsilon$. □

(Full credit would be given for $\delta = \min(\frac{1}{6}, \frac{\varepsilon}{3 \cdot 6 \cdot 3})$.)