Midterm I Solutions , February 17, 2010

Work each problem on a separate sheet of paper. Be sure to put your name, your section number, and your GSI's name on each sheet of paper. Also, at the top of the page, in the center, write the problem number, and be sure to put the pages in order. Write clearly: explanations (with complete sentences when appropriate) will help us understand what you are doing.

1. Suppose f and g are functions given by the following tables.

x	$f(x)$
1	2
2	3
3	1

x	$g(x)$
1	2
2	2
3	1

(a) (5pts) What are the domain and range of f and g?
Solution: The domains of f and g are both $\{1, 2, 3\}$. The range of f is this same set, and the range of g is the set $\{1, 2\}$.

(b) (10 pts) Compute the tables of the functions $f + g$ and $f \circ g$.
Solution:

x	$f(x)$	$g(x)$	$(f+g)(x)$	$(f \circ g)(x)$
1	2	2	4	3
2	3	2	5	3
3	1	1	2	2

(c) (5 pts) Is f^{-1} defined? If so, what is it?
Solution: Yes, f is injective and surjective, so f^{-1} exists. It is given by :

x	$f^{-1}(x)$
1	3
2	1
3	2

(d) (5pts) Is g^{-1} defined? If so, what is it?
Solution: No, g is not one-to-one (injective), so it has no inverse function. Alternative answer: g^{-1} is just a relation, not a function.

2. Use the limit laws to evalute the following limits. Explain each step.

(a) (5pts) $\lim_{x \to 1} \sqrt{x^2 + x + 2}$
Solution: The polynomial $x^2 + x + 2$ is continuous and the square root function is also continuous. Hence the composite function is continuous and we can evaluate the limit by direct substitution. The limit must be $\sqrt{1^1 + 1 + 2} = \sqrt{4} = 2$.

(b) (10 pts) $\lim_{x \to 0} x^2 \sin(1/x)$
Solution. We know that $-1 \leq \sin(1/x) \leq 1$. Since $x^2 \geq 0$, it follows that $-x^2 \leq x^2 \sin(1/x) \leq x^2$. Since $\mathrm{Lim}(-x^2) = \mathrm{Lim}(x^2) = 0$, the squeeze theorem implies that $\lim_{x \to 0} x^2 \sin(1/x) = 0$.

(c) (10 pts) $\displaystyle\lim_{h\to 0} \frac{(1+h)^{-2}-1}{h}$
Solution:

$$\begin{aligned}
\lim_{h\to 0} \frac{(1+h)^{-2}-1}{h} &= \lim_{h\to 0} \frac{1-(1+h)^{-2}}{h(1+h)^{-2}} \\
&= \lim_{h\to 0} \frac{-2h-h^2}{h(1+h)^{-2}} \\
&= \lim_{h\to 0} \frac{-2-h}{(1+h)^{-2}} \\
&= -2 \text{ (by direct subsitution)}
\end{aligned}$$

3. Let $f(x) := x^{-2}$.

(a) (10 pts) Find an expression for the slope of the secant line of the graph over the interval between 1 and $1+h$.
Solution: The secant slope is

$$\frac{f(1+h)-f(1)}{h} = \frac{(1+h)^{-2}-1}{h}$$

(b) (10 pts) Define (5 points) and compute (5 points) the slope of the tangent line at the point of the graph lying over 1.
Solution: By definition, the slope of the tangent line is the limit of the secant slopes, that is,

$$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}.$$

We just computed that this is -2.

4. (a) (10 pts) Let f be a real-valued function whose domain is a subset of the reals and let L be a real number. Give a precise definition of

$$\lim_{x\to a} f(x) = L.$$

Solution. This means that for every positive real number ϵ, there exists a positive real number δ such that $|f(x)-L| < \epsilon$ for all real numbers x such that $0 < |x-a| < \delta$.

(b) (10 pts) Let $f\colon \mathbf{R} \to \mathbf{R}$ be the function given by

$$f(x) := \begin{cases} 2x & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Prove directly from the definition that for every $a > 0$, $\lim_{x\to a} f(x) = 2a$. If you have trouble, then for partial credit (5 points) just prove this when $a = 1$.
Solution: Given $a > 0$, let δ be the minimum of a and $\epsilon/2$. Then δ is positive. Moreover, if $0 < |x-a| < \delta$, then since $\delta \leq a$, it follows that $|x-a| < a$ and hence that $x > 0$. Then $|f(x)-2a| = |2x-2a| = 2|x-a|$. Since $\delta \leq \epsilon/2$, we conclude that $2|x-a| < 2\epsilon/2 = \epsilon$, as required.

5. (10 pts) Show that there is a real number x such that $e^x = x^2$.
Solution: Let $f(x) := e^x - x^2$. Then f is continuous. Furthermore, $f(-1) = e^{-1} - 1 < 0$ and $f(1) = e - 1 > 0$. It follows from the intermediate value theorem that $f(x) = 0$ for some x in the interval $(0, 1)$.