Final Exam Solutions—May 12, 2010

1. (10 pts) Complete the following sentence. (Give the *precise* definition.) Let f be a function from $\mathbf{R}$ to $\mathbf{R}$ and let a and L be numbers. Then $\lim_{x \to a} f(x) = L$ if ....
Solution: ...for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

2. Evaluate the following limits. You may use any technique covered in our course, but explain what you are doing.

(a) (3 pts) $\lim_{x \to 0} \left(\frac{\sin^2 x}{1 - \cos x} \right)$.
Solution: The numerator and denominator both approach 0, so we can apply L'Hôpital's rule. Thus

$$\lim_{x \to 0} \frac{\sin^2 x}{1 - \cos x} = \lim_{x \to 0} \frac{2 \sin x \cos x}{\sin x} = \lim_{x \to 0} 2 \cos x = 2.$$

(b) (4 pts) $\lim_{n \to \infty} s_n$, where

$$s_n := \sum_{i=1}^{n} \frac{1}{n + 10i} = \frac{1}{n + 10} + \frac{1}{n + 20} + \cdots + \frac{1}{11n}.$$

Solution: Let $\Delta x = 10/n$. Then

$$s_n = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 + \frac{10i}{n}} = \frac{\Delta x}{10} \sum_{i=1}^{n} \frac{1}{1 + i\Delta x}.$$

Except for the factor $1/10$, this is a Riemann sum corresponding to the function $1/x$ over the interval $[1, 11]$. It follows that the limit is

$$\frac{1}{10} \int_1^{11} \frac{dx}{x} = \frac{\ln 11}{10}.$$

(c) (3 pts) $\lim_{x \to \infty} \sqrt{x^2 + 3x} - x$.
Solution:

$$\lim_{x \to \infty} \sqrt{x^2 + 3x} - x = \lim_{x \to \infty} \frac{(\sqrt{x^2 + 3x} - x)(\sqrt{x^2 + 3x} + x)}{(\sqrt{x^2 + 3x} + x)}$$

$$= \lim_{x \to \infty} \frac{x^2 + 3x - x^2}{(\sqrt{x^2 + 3x} + x)} = \lim_{x \to \infty} \frac{3x}{(\sqrt{x^2 + 3x} + x)}$$

$$= \lim_{x \to \infty} \frac{3}{(\sqrt{1 + 3x^{-1}} + 1)} = 3/2.$$

3. Let f be a function defined on an interval $[a, b]$.

(a) (5 pts) Define the *average slope* of f over $[a, b]$ and the *instantaneous slope* of f at a point $c \in (a, b)$.
Solution: The average slope of f over $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}.$$

The instantaneous slope of f at c is

$$\operatorname*{Lim}_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

(b) (5 pts) Let $f(x) = 1/x^2$ on the interval $[1, 3]$. Compute the average slope over the interval $[1, 3]$ and the instantaneous slope at $x = 2$.
Solution: The average slope is

$$\frac{1/9 - 1}{3 - 1} = -4/9.$$

The instanenous slope is given by

$$f'(2) = -2(2^{-3}) = -1/4$$

4. (10 pts) Let f be a function continuous on $[0, 10]$ and differentiable on $(0, 10)$. Suppose that $|f'(x)| \leq 2$ for all $x \in (0, 10)$ and that $f(0) = 1$. How large and how small could $f(10)$ possibly be? Prove your result, naming any theorem your proof uses.
Solution: According to the mean value theorem, there is a point $c \in (0, 10)$ such that $f'(c)10 = f(10) - f(0)$, so that

$$f(10) = 10f'(c) + 1.$$

Since $|f'(c) \leq 2$, $f(10)$ is at most 21 and at least -19.

5. For the following problems, show your work, but you do not need to give proofs or even explanations. However, if your work is clear, you are more likely to receive partial credit.

(a) (1 pt) Find $f'(x)$ if $f(x) = \sin(\cos(x))$.
Solution:

$$f'(x) = \cos(\cos x)\cos'(x) = -\cos(\cos x)\sin x,$$

by the chain rule.

(b) (2 pts) Find $f'(x)$ if $f(x) = x^x$ (here $x > 0$).
Solution: Let's use logarithmic differentiation:

$$\begin{aligned}
\ln f(x) &= x \ln x \\
\frac{f'(x)}{f(x)} &= \ln x + \frac{x}{x} = 1 + \ln x \\
f'(x) &= x^x(1 + \ln x)
\end{aligned}$$

(c) (2 pts) Find the slope of the curve defined by $e^y = \sin x + e$ at the point $(0, 1)$.
Solution: Using implicit differentiation, we get $e^y y' = \cos x$. At $(0, 1)$ this becomes $ey' = 1$. Thus $y' = 1/e$ at this point.

(d) (1 pt) $\int_0^1 x^{10} dx$

Solution:

$$\int_0^1 x^{10} dx = \frac{x^{11}}{11}\Big|_0^{11} = \frac{1}{11}$$

(e) (1 pt) $\int_0^1 (x+1)^{10} dx$

Solution: Let $u = x + 1$. Then $du = dx$ and

$$\int_0^1 (x+1)^{10} dx = \int_1^2 u^{10} du = 2^{10}/11 - 1/11 = \frac{1023}{11}$$

(f) (1 pt) $\int_{-1}^1 \sin(x^3) dx$.

Solution: This function is odd, so the answer is 0.

(g) (2 pts) Compute the derivative of

$$F(x) := \int_0^{\arcsin x} \sin^2 t dt.$$

Solution: Let $g(x) := \int_0^{\arcsin x} \sin^2 t dt.$. Then $F(x) = g(\arcsin(x))$, and by the chain rule and the fundamental theorem:

$$F'(x) = g'(\arcsin x) \arcsin'(x)$$

$$= \sin^2(\arcsin x)(1 - x^2)^{-1/2} = \frac{x^2}{\sqrt{1 - x^2}}.$$

6. The economy is beginning to recover, and you land a job computing reciprocals. Your job is to give decimal approximations of x^{-1} for numbers x between 10 and 11. You come up with a remarkably simple formula, linear in x, which does the job for you.

(a) (3 pts) What is your formula?

Solution: $f'(x) = -x^{-2}$, so the tangent approximation is

$$\ell(x) = -.01(x - 10) + .1 = -.01x + .2$$

(b) (4 pts) Your boss asks you how accurate your answer is. Prove to him that:

i. Your answer is correct to at least three decimal places.

Solution:

$$\frac{1}{x} - \ell(x) = \frac{1}{x} + \frac{x - 10}{100} - \frac{10}{100}$$

$$= \frac{100 + x(x - 10) - 10x}{100x} = \frac{(x - 10)^2}{100x}.$$

So if $10 < x < 11$, this is less than .001.

ii. Your answer is never too large.

Solution: The function $1/x$ is concave up. Hence the tangent line lies below the curve.

(c) (3 pts) The boss is so impressed that he gives you a more difficult task. How close does x need to be to 10 to guarantee that your answer is correct to $2n + 1$ decimal places?

Solution: By the formula in part i, the error is bounded by $.001(x - 10)^2$. So if x is within $n - 1$ decimal places of 10, the error will be less $2n - 2 + 3 = 2n + 1$ decimal places.

Remark: This problem was perhaps not clearly worded. When I wrote 'correct to three decimal places" I didn't mean that the first three decimal places are correct, but rather that the error is less that 10^{-3}.

7. (10 pts) Describe the graph of the function $f\colon \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) := \frac{x}{x^2+1}.$$

Show the x and y-interecepts, minimum and maximum values of f, where the function is increasing and decreasing, horizontal and vertical asymptotes, inflection points, and where the function is concave up or down.

Solution:

$$f(x) = \frac{x}{x^2+1}$$
$$f'(x) = \frac{1-x^2}{(x^2+1)^2}$$
$$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}$$

(a) The only x-intercept is the origin, and this is also the only y-interecept.

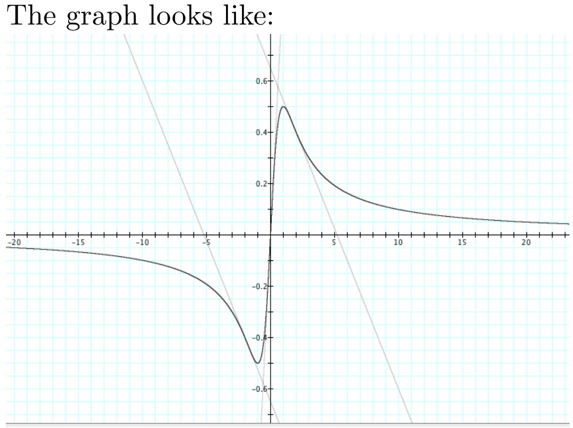
(b) The mimimum is at $(-1,-1/2)$ and the maximum at $(1,1/2)$.

(c) The inflection points are at the orgin and $(\pm\sqrt{3}, \pm\sqrt{3}/4)$.

(d) The function is concave down on $(\infty, -\sqrt{3})$ and $(0, \sqrt{3})$ and concave up everywhere else.

(e) There are no vertical asymptotes, and the line $y = 0$ is a horizontal asymptote.

(f) The graph looks like:



8. A piece of wire 24 cm long is cut into two pieces. One piece is bent into a rectangle which is three times as wide as it is high and the other into a square.

(a) (5 pts) What should the height of the rectangle be to maximize the area enclosed by the two pieces?

(b) (5pts) To minimize it?

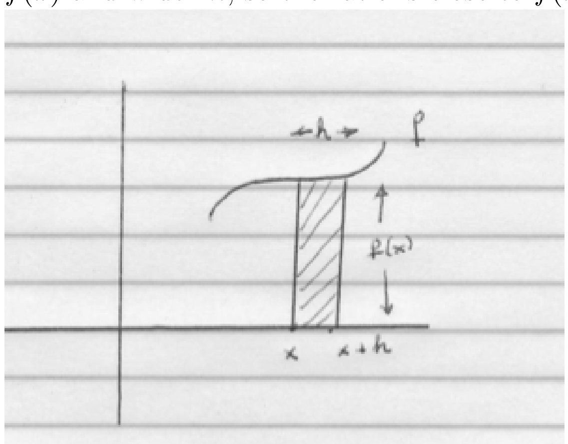
Solution: Let x be the height of the rectangle. Then the width is $3x$ and the perimeter is $8x$. If y is the length of the side of the square, we must have $4y+8x=24$, so $y=6-2x$. Then the area is

$$A(x) = 3x^2 + y^2 = 3x^2 + (6-2x)^2, \quad 0 \le x \le 3.$$
$$A'(x) = 6x - 4(6-2x) = 14x - 24$$

This has a minimum when $x = 12/7$ and a maximum when $x = 0$.

9. (a) (5 pts) State the fundamental theorem of calculus (part I) very carefully.
Solution: Let f be a continuous function on a closed interval $[a,b]$. For $x \in (a,b)$, let $g(x) := \int_a^x f$. Then g is differentiable, and $g'(x) = f(x)$.

(b) (5 pts) Write a brief sentence or two and draw a picture explaining the idea of the proof.
Solution: By definition, $g'(x) = \operatorname{Lim}_{h\to 0} \dfrac{g(x+h)-g(x)}{h} = h^{-1}\int_x^{x+h} f$.
The integral is approximately the area of the rectangle of height $f(x)$ and width h, so the ratio is close to $f(x)$.



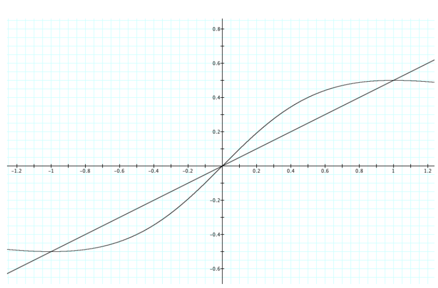
10. Integration

(a) (4 pts) Find the (unoriented) area of the bounded region cut out by the graph of the function f of problem 7 and the line $y = x/2$.
Solution: The curve meets the line at the origin and when $x = \pm 1$. By symmetry, the total area is twice the area over the interval $[0,1]$. This is the area under the curve minus the area of the triangle cut out by the line. The area of the curve is

$$\int_0^1 \frac{x}{x^2+1}dx = 1/2\ln(x^2+1)\Big|_0^1 = 1/2\ln 2.$$

The area of the triangle is $1/4$. Thus the area on the right side is $1/2\ln 2 - 1/4$ and the total area is

$$A = \ln 2 - 1/2.$$



(b) (6 pts) Your bank pays interest at the rate of 10% per year, compounded continuously. You start with \$1,000 at the beginning of the year.

i. How much money do you have after t years?
Solution:

$$y(t) = 1000e^{.1t}$$

ii. What is the average balance in your account over the course of the first two years?
Solution:

$$Av = 1/2\int_0^2 1000e^{.1t}dt = 5000e^{.1t}\Big|_0^2 = 5000(e^{.2} - 1)$$