R-algebras, homomorphisms, and roots

Here we consider only commutative rings.

Definition 1 *Let R be a (commutative) ring. An R-algebra is a ring homomorphism $\alpha_R: R \to A$. If $\alpha_A: R \to A$ and $\alpha_B: R \to B$ are R-algebras, a homomorphismof R-algebras from α_A to α_B is a ring homomorphism $\theta: A \to B$ such that $\theta \circ \alpha_A = \alpha_B$.*

In practice, one usually calls an R-algebra by the name of the codomain, *i.e.*, one says an "R-algebra A"instead of α_A. If A and B are R-algebras, it is convenient to use the notations $Mor(A, B)$ or even $Mor_A(B)$ for the set of R-algebra homomorphisms A to B.

For example, if R is a ring, then the ring $R[X]$ of polyonomials with coefficients in R has a natural structure of an R-algebra, via the homomorphism $R \to R[X]$ sending an element r to the polynomial $(r, 0, 0, \cdots,)$. Here is one reason why this is so important.

Theorem 1 *Let A be an R-algebra and let a be any element of A. Then there is a unique homomorphism of R-algebras:*

$$\theta_a: R[X] \to A \quad \text{(evaluation at } a\text{)}$$

sending X to a. This correspondence induces a natural bijection from the set A to the set of R-algebra homorphisms from $R[X] \to A$:

$$A \leftrightarrow Mor_{R[X]}(A).$$

Proof: This is really just a check of the definitions. Recall that if $p := (r_0, r_1, \cdots)$ is an element of $R[X]$, then

$$\theta_a(p) := \alpha_A(r_0) + \alpha_A(r_1)a + \alpha_A(r_2)a^2 + \cdots.$$

One checks from the definitions that θ_a is a ring homomorphism, that $\theta_a(r, 0, \cdots) = \alpha_A(r)$, and that $\theta_a(X) = a$. Finally, it is also clear that θ_a is uniquely determined by these propeties. $\square$

Notice that if $\phi: B \to B'$ is a homomophism of R-algebras, then composition with ϕ defines a map of sets:

$$\phi_*: Mor_A(B) \to Mor_A(B').$$

Similarly, if $\pi: A \to A'$ is a homomorphism of R-algebras, composition with π defines a map:

$$\pi^*: Mor_{A'}(B) \to Mor_A(B).$$

If π is surjective, then π^* is injective. Indeed, if θ and θ' are two homomorphisms $A' \to B$ and $\theta \circ \pi = \theta' \circ \pi$, then $\theta = \theta'$ if π is surjective. We can even determine the image of π^*:

Theorem 2 *Let $\pi: A \to A'$ be a surjective homomorphism of R-algebras, and let $I \subseteq A$ be the kernel of π. Then the image of π_* consists of the set of all homomoprhisms $\theta: A \to B$ such that $I \subseteq Ker(\theta)$.*

Proof: If θ is in the image of π, then $\theta = \theta' \circ \pi$ for some $\theta': A' \to B$. Hence if $x \in I$, $\theta(x) = \theta'(\pi(x)) = 0$ since $\pi(x) = 0$. On the other, suppose that $I \subseteq Ker(\theta)$. Choose some $a' \in A'$. Since π is surjective, we can choose some $a \in A$ with $\pi(a) = a'$. We would like to define $\theta'(a')$ to be $\theta(a)$, but it is not clear yet that this is independent of the choice of a. But if a_1 and a_2 are two such choices, then $x := a_1 - a_2$ belongs to $Ker(\pi) = I$ and hence also to $Ker(\theta)$, so $\theta(a_1) = \theta(a_2)$. Thus θ' really is well-defined, and it is easy to check from the surjectivity of π that θ' is an R-algebra homomorphism. □

Theorem 3 *Let p be a polynomial in $R[X]$ and let $A_p := R[X]/I(p)$, where $I(p)$ is the ideal consisting of all multiples of p. Then for any R-algebra B, there is a natural bijection:*

$$Mor_{A_p}(B) \leftrightarrow \{b \in B : p(b) = 0\}.$$

Proof: This is an immediate consequence of the previous results. Let $\pi: R[X] \to A_p$ be the natural projection. This is a surjective homomorphism of R-algebras, and its kernel consists of the the ideal I of multiplies of p. Thus by Theorem 2,

$$\pi_*: \mathrm{Mor}_{A_p}(B) \to \mathrm{Mor}_{R[X]}(B).$$

is injective, and its image consists of those homomoprhisms θ such that $I \subseteq \mathrm{Ker}(\theta)$. Using Theorem 1 we can identify a homomorphism $\theta: R[X] \to B$ with the element $b := \theta(X)$. Since I is the set of multiplies of p, $I \subseteq \mathrm{Ker}(\theta)$ iff $p \in \mathrm{Ker}(\theta)$ iff $\theta(p) = 0$. But $\theta(p) = p(b)$. Thus the element θ lies in the image of π_* if and only if the corresponding $b \in B$ is a root of p. □