Linear Algebra Midterm Exam Solutions November 17, 2008

Write clearly, with complete sentences, explaining your work. You will be graded on clarity, style, and brevity. If you add false statements to a correct argument, you will lose points. Be sure to put your name on every page.

1. Let V be an inner product space over a field F, where F is $\mathbf{R}$ or $\mathbf{C}$.

   (a) (10 pts) Let $\mathcal{L} := (v_1, \ldots, v_n)$ is a list in V.

      i. What does it mean to say that $\mathcal{L}$ is *orthonormal?*

      ii. Prove that every orthonormal list is linearly independent.

   Solution: The list is orthonormal if $(v_i|v_j)$ is zero if $i \neq j$ and is 1 if $i = j$. This implies that $\mathcal{L}$ is linearly independent, since if $\sum a_i v_i = 0$, then for every j, $0 = \sum a_i(v_i|v_j) = a_j$.

   (b) (10 pts) Let T be a linear operator on V, v an element of V, and λ an element of F.

      i. What does it mean to say that v is an *eigenvector* of T with eigenvalue λ?

      ii. Prove that if $\mathcal{L}$ is a list of nonzero eigenvectors of T corresponding to distinct eigenvalues, then $\mathcal{L}$ is linearly independent.

   Solution: v is an eigenvector with eigenvalue λ if $Tv = \lambda v$. Suppose $Tv_j = \lambda_j v_j$ and $\lambda_i \neq \lambda j$ for $i \neq j$ and each $v_j \neq 0$. We shall prove by induction on n that $\mathcal{L}$ is linearly independent. Assume $\sum a_i v_i = 0$. Applying T, we find that $\sum a_i \lambda_i v_i = 0$. Multiplying the first equation by λ_n and substracing give $\sum a_i(\lambda_i - \lambda_n)v_i = 0$. This equation only involves the first n vectors, so the induction hypothesis implies that each $a_i(\lambda_i - \lambda_n) = 0$. Since $\lambda_i \neq \lambda_n)$ for $i < n$, $a_i = 0$ for $i < n$. Then the first equation says $a_n v_n = 0$, and since $v_n \neq 0$, $a_n = 0$.

(c) (15 pts) Suppose in the context of the previous problem that V is a finite dimensional inner product space.

i. Prove directly from the definitions that if T is self adjoint, the list $\mathcal{L}$ is orthogonal.

ii. Give an example to show that this is not necessarily the case if T is not self adjoint.

iii. What happens if T is normal?

Solution:

i. We have $(Tv_i|v_j) = \lambda_i(v_i|v_j)$, and also $(Tv_i|v_j) = (v_i|T^*v_j) = (v_i|Tv_j) = \overline{\lambda}_j(v_i|v_j)$. Taking $i = j$ we see that $\lambda_i = \overline{\lambda}_j$, since $(v_i|vi) \neq 0$, and now taking $i =\neq j$ we get that $\lambda_i(v_i|v_j) = \lambda_j(v_i|v_j)$. Since $\lambda_i \neq \lambda_j$, this implies that $(v_i|v_j) = 0$.

ii. Consider the transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$ given by $T(x_1, x_2) = (x_1 + x_2, 2x_2)$. This takes $(1, 0)$ to $(1, 0)$. and it takes the vector $(1, 1)$ to the $(2, 2)$. Thus these two vectors are eigenvectors with distinct eigenvalues, but they are not orthogonal.

iii. In the normal case the result is still true, by the spectral theorem. We know that V is an orthogonal direct sum of the eigenspaces of T: $V \cong \oplus_\lambda Eig_\lambda$. It follows that if v and w belong to distinct eigenspaces, then they are orthogonal.

2. Let $V := \mathbf{R}^4$ and let W be the linear subspace consisting of all the vectors in V which are orthogonal to the vector $(1, 1, 1, 1)$.

   (a) (5 pts) Find an orthogonal basis for W.
   Solution: $u_1 := (1, -1, 0, 0), u_2 := (0, 0, 1, -1), u_3 := (1, 1, -1, -1)$ will work.

   (b) (5 pts) Find the vector in W which is closest to the vector $v := (1, 2, 3, 4)$.
   Solution: The closest vector is the orthogonal projection w of v onto W. We use the formula

   $$w = \sum \frac{(v|u_i)}{(u_i|u_i)} u_i.$$

   For $v = (1, 2, 3, 4)$, this gives

   $$w = (-1/2, 1/2, 0, 0) + (0, 0, -1/2, 1/2)+)(-1, -1, 1, 1)$$
   $$= (-3/2, -1/2, 1/2, 3/2).$$

   (c) (10 pts) Prove that your answer really is the closest.
   Solution: Note that $u := v - w = 5/2(1, 1, 1, 1)$ which really is orthogonal to W. Now if $w' in W$,

   $$||v - w'||^2 = ||v - w + w' - w'||^2 = ||v - w + w''||^2$$

   where $w'' := w' - w$. Since w'' is orthogonal to $u = v - w$, we have

   $$||v - w'||^2 = ||v - w||^2 + ||w''||^2 \geq ||v - w||^2$$

   with equality iff $w' = w$.

3. Let V be a finite dimensional inner product space over F.

   (a) (6 pts) What is the definition of the *adjoint* T^* of a linear operator T on V? What is the definition of a *positive* operator? A *unitary* operator?
   Solution: The adjoint of T is the unique operator T^* such that $(Tv|w) = (v|T^*w)$ for all v and w in V. T is positive if it is self-adjoint (or normal) and $(Tv|v) \geq 0$ for all v, and T is unitary if $TT^* = \text{id}$, or equivalenty, if $||Tv|| = ||v||$ for all v.

   (b) (5 pts) The polar decomposition theorem says that any operator T can be written as a composition $T = SR$, where S is unitary and R is positive. Show that T can also be written as a product $T = R'S'$, where R' is positive and S' is unitary. (Hint: use the standard form of polar decomposition theorem for a suitable operator.)
   Solution: We can write $T^* = S'R'$, where S' is unitary and R' is positive. Then $T = T^{**} = R'^*S'^* = R'S^{-1}$, and S^{-1} is again unitary.

   (c) (9 pts) Prove that if P is self adjoint and $P^2 = P$, then P is positive, directly from the definitions you gave.
   Solution: We have

$$(Pv|v) = (P^2v|v) = (P^*Pv|v) = (Pv|Pv) \geq 0.$$