

Polynomials and polynomial functions

Let \mathbf{F} be a field. Recall that $\mathbf{F}^{\mathbf{N}}$ is the vector space of all sequences $a := (a_0, a_1, \dots)$ with $a_i \in \mathbf{F}$, with addition and scalar multiplication defined coordinatewise. For each natural number d , let \mathcal{P}_d denote the set of all sequences a such that $a_i = 0$ for $i > d$. Check that \mathcal{P}_d is a linear subspace of $\mathbf{F}^{\mathbf{N}}$, and that $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \cdots$. Now let $\mathcal{P} := \cup\{\mathcal{P}_d : d \in \mathbf{N}\}$, *i.e.*, \mathcal{P} is the set of all a such that there exists some d (possibly depending on a) such that $a_i = 0$ for all $i > d$. Check that \mathcal{P} is also a linear subspace of $\mathbf{F}^{\mathbf{N}}$.

We can think of a sequence (a_0, a_1, \dots) in \mathcal{P} as specifying the coefficients of a polynomial (with coefficients in the field \mathbf{F}). More specifically:

Definition: Let a be an element of \mathcal{P} , say a in \mathcal{P}_d . Then \tilde{a} is the function $\mathbf{F} \rightarrow \mathbf{F}$ such that $\tilde{a}(x) = a_0 + a_1x + a_2x^2 + \cdots + a_dx^d$ for all $x \in \mathbf{F}$.

Note that if $d' \geq d$, then a also belongs to $\mathcal{P}_{d'}$, and

$$a_0 + a_1x + a_2x^2 + \cdots + a_dx^d = a_0 + a_1x + a_2x^2 + \cdots + a_dx^d + \cdots + a_{d'}x^{d'},$$

since $a_i = 0$ for all $i > d$. Thus \tilde{a} is independent of the choice of d . This means that we have a well-defined mapping

$$\epsilon: \mathcal{P} \rightarrow \mathbf{F}^{\mathbf{F}}, \text{ sending each } a \text{ to } \epsilon(a) := \tilde{a}.$$

An element $a \in \mathcal{P}$ is called a *polynomial*; really then a polynomial is just a list of coefficients, all but finitely many of which are zero. Associated to such a list a we have formed a function \tilde{a} , which we call a *polynomial function*. Now recall that we also have operations on $\mathbf{F}^{\mathbf{F}}$ which make it into a vector space over \mathbf{F} . Fortunately, the next proposition tells us that these operations are compatible with the operations.

Proposition: If $a, b \in \mathcal{P}$ and $c \in \mathbf{F}$, then $\epsilon(a + b) = \epsilon(a) + \epsilon(b)$ and $\epsilon(ca) = c\epsilon(a)$.

Write out the proof of this proposition. It says that addition of polynomial functions can be computed by adding the corresponding coefficients, and similarly for multiplication of a polynomial function by a constant.

Axler in his book doesn't really distinguish between the sequence of coefficients a and the corresponding polynomial function \tilde{a} . This is because for him, \mathbf{F} is always either \mathbf{R} or \mathbf{C} , in which case one can recover the coefficients from the polynomial function. However for the field \mathbf{F}_2 , this is not the case. For example, the polynomial $a := (0, 1, 1, 0, 0, \dots)$ gives rise to the same function as the polynomial $b := (0, 0, 0, \dots)$. This is because $\tilde{a}(x) = x + x^2$, which is 0 if x is any element of \mathbf{F}_2 .