Lectures on Logarithmic Algebraic Geometry

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1 Introduction

1.1 Motivation

Logarithmic geometry was developed to deal with two fundamental and related problems in algebraic geometry: compactification and degeneration. One of the key aspects of algebraic geometry is that it is essentially global in nature. Algebraic varieties can be compactified: any separated scheme S of finite type over a field k admits an open embedding $j\colon S \hookrightarrow T$, with T/k proper and S Zariski dense in T [30]. Since proper schemes are much easier to study than general schemes, it is often convenient to work with T even if it is the original scheme S that is of primary interest. It then becomes necessary to keep track of the complement $Z := T \setminus S$ and to study how functions, differential forms, sheaves, and other geometric objects on T behave near Z, and to somehow carry along the fact that it is S rather than T in which one is primarily interested, in a functorial way.

This compactification problem is related to the phenomenon of degeneration. A scheme S often arises as a moduli space, for example a space parameterizing smooth proper schemes of a certain type. If S is a "fine moduli space," there is a smooth proper morphism $f\colon U \to S$ whose fibers are the objects one wants to classify. One can then hope to find a compactification T of S such that the boundary points parametrize reasonably nice "degenerations" of the original objects. In this case there should be a proper and flat (but not smooth) $g\colon X \to T$ extending $f\colon U \to S$. Then one is left with the problem of comparing f to g and in particular of analyzing the behavior of g near $Y := X \setminus U$. Indeed, in many cases one can obtain important information about the original family f by studying the degenerate family over Z. A typical example is the compactification of the moduli stack of smooth curves by the moduli stack of stable curves.

The problems of compactification and degeneration are thus manifest in a diagram of the form:

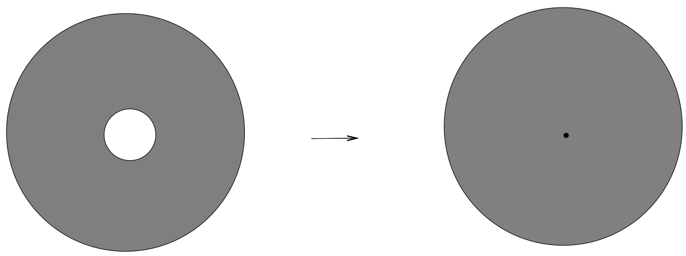
$$\begin{array}{ccccc} U & \hookrightarrow & X & \hookleftarrow & Y \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_{|Y}} \\ S & \hookrightarrow & T & \hookleftarrow & Z. \end{array}$$

The point of log geometry is that in many such cases there is a natural way to equip X and T with *log structures,* which somehow "remember" U and S and are compatible with g. Then $g\colon X \to T$ becomes a *morphism of log schemes* and inherits many of the nice features of f. In good cases it is *logarithmically smooth*, which makes it much easier to study than the underlying morphism of schemes g. The log structures on X and T restrict in a natural way to Y and Z, and the resulting morphism of log schemes $g_{|_Y}\colon Y \to Z$ still remembers useful information about $f\colon U \to S$.

Let us illustrate how log geometry works in the most basic case of a (possibly partial) compactification. Let $j\colon U \to X$ be an open immersion, with complementary closed immersion $i\colon Y \to X$. Then Y (and hence U) is determined by the sheaf $I_Y \subseteq \mathcal{O}_X$ consisting of those local sections of $\mathcal{O}_X$ whose restriction to Y vanishes, a sheaf of ideals of $\mathcal{O}_X$. However it is not Y but rather U that is our primary interest, so instead we consider the subsheaf $\mathcal{M}_{U/X}$ of $\mathcal{O}_X$ consisting of the local sections of $\mathcal{O}_X$ whose restriction to U is invertible. If f and g are sections of $\mathcal{M}_{U/X}$, then so is fg, but $f+g$ need not be. Thus $\mathcal{M}_{U/X}$ is not a sheaf of rings, but it is a sheaf of multiplicative *submonoids* of $\mathcal{O}_X$. Note that $\mathcal{M}_{U/X}$ contains the sheaf of units $\mathcal{O}_X^*$, and if X is integral, the quotient $\mathcal{M}_{U/X}/\mathcal{O}_X^*$ is just the sheaf of effective Cartier divisors on X with support in the complement Y of U in X. The morphism of sheaves of monoids $\alpha_{U/X}\colon \mathcal{M}_{U/X} \to \mathcal{O}_X$ (inclusion) is a *logarithmic structure.* In good cases this log structure "remembers" the inclusion $U \to X$ and furthermore satisfies a technical "coherence" condition which makes it manageable. In the category of log schemes, the open immersion j fits into a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\tilde{j}} & (X, \alpha_{U/X}) \\ & {}_{j}\searrow & \downarrow{\scriptstyle \tau_{U/X}} \\ & & X. \end{array}$$

This diagram provides a relative compactification of the open immersion j: the map $\tau_{U/X}$ is proper but the map $\tilde{j}$ somehow preserves much of the essential nature of the original open immersion j, and in good cases behaves like a local homotopy equivalence. We can imagine that the log structure $\alpha_{U/X}$ cuts away (or rather blows up) enough of X to make it look like U, but leaves enough of a boundary so that it remains compact. Indeed the log scheme $(X, \alpha_{U/X})$ behaves as an "algebraic variety with boundary." For example, in the case of the standard embedding of $\mathbf{G_m} \to \mathbf{A}^1$, the corresponding log scheme $(\mathbf{A}^1, \alpha)$ behaves very much like the complex plane with an (infinitely small) open disc about the origin removed, and one finds the following picture.



The morphism in this picture can be identified with the multiplication map $\mathbf{R}_{\geq} \times \mathbf{S}^1 \to \mathbf{C}$, *i.e.*, the real blowup of the origin in the plane. It plays the role of a proper model of the inclusion $\mathbf{G_m} \to \mathbf{A}^1$, whose homotopy theory it closely resembles. These ideas will be made more precise in section V.1, where we discuss "Betti realizations" of log schemes.

More generally, if X is any scheme, a *log structure* on X is a morphism of sheaves of commutative monoids $\alpha\colon \mathcal{M} \to \mathcal{O}_X$ inducing an isomorphism $\alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$. We do not require α to be injective. For example, let T be the spectrum of a discrete valuation ring R with maximal ideal m, residue field k, and fraction field K. Let $t := \operatorname{Spec}(k)$, let $\tau := \operatorname{Spec}(K)$, and let $i\colon t \to T$ (resp. $j\colon S := \tau \to T$) be the natural closed (resp. open) immersion. The procedure described in the previous paragraph associates to the open immersion j a log structure (called the "standard log structure") $\alpha_{S/T}\colon \mathcal{M}_{S/T} \to \mathcal{O}_T$. The stalk of $\alpha_{S,T}$ at t is the inclusion $R' \to R$, where $R' := R \setminus \{0\}$. A more exotic example (the "hollow log structure") is the map $R' \to R$ which is the inclusion on the group R^* of units of R but sends the maximal ideal m to $0 \in R$. The restrictions of these two log structures to t are isomorphic, and give a log structure

$$\alpha\colon i^*(\mathcal{M}_{S/T}) \to k,$$

where $i^*(\mathcal{M}_{S/T})$ is the quotient of R' by the group U of units congruent to 1 modulo m. Thus, if k is the residue field of R, there is an exact sequence

$$1 \to k^* \to i^*(\mathcal{M}_{S/T}) \to \mathbf{N} \to 0$$

and α is the inclusion on k^* and sends all other elements of $i^*(\mathcal{M}_{S/T})$ to 0. This example shows how sections m of the sheaf of monoid $\mathcal{M}$ keep track of the "ghosts of vanishing coordinates." In particular, $\alpha(m)$ can be zero, although $\mathcal{M}$ is often *integral,* (so that multiplication by m is injective). The tension between these behaviors accounts for many of the technical difficulties of log geometry, especially those involving fiber products.

The naturality of these constructions allow them to work in appropriate relative settings, for example in the context of semi-stable reduction. Thus, let X be a regular scheme, let T be the spectrum of a discrete valuation ring as above, and let $f\colon X \to T$ be a flat and proper morphism whose generic fiber X_τ/τ is smooth and whose special fiber is a reduced divisor with normal crossings. Then canonical log structures α_X and α_T associated as above to the open embeddings $X_\tau \to X$ and $\tau \to T$ fit into a morphism of log schemes:

$$f\colon (X, \alpha_X) \to (T, \alpha_T)$$

which is in fact "logarithmically smooth." The concept of smoothness for log schemes fits very naturally into Grothendieck's geometric deformation theory; furthermore Betti realizations of proper log smooth morphisms behave in some respects like topological fibrations [24] and [31]. The fact that this picture works so well both in topological and arithmetical settings is one of the main justifications for the theory.

The justification for the machinery of log geometry must lie in its applications to problems in outside the theory itself. A detailed discussion of any of these would be beyond the scope of this book, and we can only point readers to the literature. Historically, the first (and perhaps still most striking) of such application is to the proof of what used to be called the C_{st} conjecture in p-adic Hodge theory, due to Hyodo [22], Kato [28], Tsuji [41]. Faltings [10] [9], and others. Indeed, log geometry began as an attempt to discern what additional structure on the special fiber of a semi-stable reduction was needed to define a "limiting crystalline cohomology," in analogy to Steenbrink's construction of limiting mixed Hodge structures in the complex analytic context. In ℓ-adic cohomology, the main applications have been to the Bloch conductor formula [37] and higher dimensional Ogg-Shavarevich formulas [1] and Gabber's results on resolution, purity, and duality []. Log geometry has also been notably used to improve the theory of compactifications of moduli spaces of abelian varieties [35], K3 surfaces [34], and toric Hilbert schemes [36].

1.2 Roots

The development of logarithmic geometry, like that of any organism, began well before its official birth, and was preceded by many classical methods dealing with the problems of compactification and degeneration. These include most notably the theories of toroidal embeddings, of differential forms and equations with log poles and/or regular singularities, and of vanishing cycles and monodromy. Logarithmic geometry was influenced by all these ideas and provides a language which incorporates and extends them in functorial and systematic ways. We mention in particular the powerful and subtle notion of base change in log geometry, as well as logarithmic versions of de Rham, Betti, étale, and crystalline cohomology, which have had major applications to arithmetic algebraic geometry.

Logarithmic structures fit so naturally with the usual building blocks of schemes that it is possible, and in most (but not all) cases easy and natural, to adapt in a relatively straightforward way many of the standard techniques and intuitions of algebraic geometry to the logarithmic context. Log geometry seems to be especially compatible with infinitesimal properties, including Grothendieck's notions of smoothness, differentials, and differential operators. For example, if X is smooth over a field k and U is the complement of a divisor with normal crossings, then the resulting log scheme (constructed from the compactification log structure as above) is logarithmically smooth. The sheaf of K ahler differnetials of (X, α_X), constructed from Grothendieck's deformation-theoretic viewpoint, coincides with the classical sheaf of differential forms of X with log poles along $X \setminus U$. Furthermore, any toric variety (with the log structure corresponding to the dense open torus it contains) is log smooth, and the theory of toroidal embeddings is essentially equivalent to the study of (logarithmically) smooth log schemes over a field.

1.3 Goals

Our aim in these notes is to provide an introduction to the basic notions and techniques of log geometry, accessible to graduate students with a basic knowledge of algebraic geometry. We hope they will also be useful to researchers in other areas of geometry, to which we believe the theory can be profitably adopted, as has already been done in the case of complex analytic geometry. For the sake of concreteness we work systematically with schemes as locally ringed spaces, although it certainly would have been possible and profitable to develop the theory for complex analytic varieties, or for algebraic spaces or stacks. However, even in the case of schemes, it is quite valuable to work locally in the étale topology, and we shall allow ourselves to do so, without systematically using the language of topos theory which would certainly have been more powerful. (This approach is taken in the very thorough treatment in [13].)

Just as scheme theory starts with the study of commutative rings, log geometry starts with the study of commutative monoids. Much of this foundational material is already available in the literature, but we have decided to offer a self-contained presentation more directly suited to our purposes. In log geometry, in an apparent contrast with toric geometry, the study of the *category* of monoids, and in particular of *homomorphisms of monoids*, plays a fundamental role. This difference was part of our motivation for including this material, and we hope our treatment may be of interest apart from its applications to log geometry per se. Thus Chapter I begins with the study of projective and inductive limits in the category of monoids, and in particular with the construction of pushouts, which are analogous to tensor products in the category of rings. We then discuss monoid actions (the analog of modules in ring theory), ideals, localization, and the spectrum of a monoid, with its Zariski topology. After these preliminaries we turn to constructions more familiar from toric geometry, including basic results about finiteness, duality, and cones. Then we discuss monoid algebras and some facts about affine toric varieties. The final section of Chapter I is devoted to a deeper study of properties of morphisms and actions of monoids, and in particular certain analogs of flatness. Especially important is Kato's key concept of *exactness*, which we already encountered in subsection 1.1. An example of its importance is manifest in the "Four Point Lemma" 4.2.8, where exactness is needed to make fiber products of logarithmically integral log schemes behave well. *Integrality* and *saturation* of morphisms, which we discuss next, are refinements of the notion of exactness. We finish by showing how exact morphisms can be made integral and saturated by a suitable base change; a logarithmic version of semi-stable reduction. This material is more technical than the rest of our exposition and can be skipped over in a first reading.

Chapter II discusses sheaves of monoids on topological spaces. After disposing of some generalities, we define *monoschemes*, which are constructed by gluing together spectra of commutative monoids just as schemes are constructed by gluing together spectra of commutative rings. Our monoschemes are sometimes called "schemes over $\mathbf{F}_1$" in the literature [5] and are generalizations of the fans used to construct toric varieties. The main application we have of this concept is a notion of blowing up, or *monoidal transformation,* for monoschemes. Of special importance is Theorem 1.6.5, which explains how a homomorphism of monoids can be made exact by a monoidal transformation. We then discuss Kato's notions of *charts* and *coherence* for sheaves of monoids and morphisms between them.

With the preliminaries well in hand, we are ready in Chapter III to turn to logarithmic geometry per se, including two variants of the standard theory: *idealized log schemes* and *relatively coherent log structures*. After explaining the main definitions and properties of log structures, we discuss the somewhat delicate construction of fibered products. Then we investigate properties of morphisms of log schemes, returning again to the notions of exactness, integrality, and saturation, as well as the Frobenius morphism for log schemes in characteristic p.

Chapter IV is devoted to *logarithmic differentials* and *logarithmic smoothness*. We begin with a purely algebraic construction of Kähler differentials for (pre) log schemes, then explain its geometric meaning in terms of deformation theory. Then we discuss smoothness for logarithmic schemes, defined in terms of a log version of Grothendieck's infinitesimal lifting criterion. Although smooth morphisms in logarithmic geometry are much more complicated than in classical geometry, locally they admit nice toric models. We also discuss a somewhat provisional, but seemingly useful, notion of smoothness for relatively coherent log structures, which we call *relative smoothness.* Finally we mention briefly Kato's notion of *log regularity*, which is an absolute, rather than a relative, notion, useful for arithmetic algebraic geometry.

In Chapter V we discuss topology and cohomology. To provide a geometric intuition, we begin with the construction of the *Betti realization* X_{log} of a log scheme X over $\mathbf{C}$. After giving the definition and topological properties of X_{log}, we define the sheaf of rings $\mathcal{O}_X^{log}$ on X_{log} which is obtained by adjoining the logarithms of sections of $\mathcal{M}$ in a canonical way. Then we discuss logarithmic de Rham cohomology, beginning with an algebraic description of the *logarithmic de Rham complex* of a monoid algebra and some of the natural filtrations it carries, then the sheafification and globalization of these constructions. We end with some basic facts about de Rham cohomology, including the log Poincaré lemma, comparison between Betti and de Rham cohomologies, and the Riemann–Hilbert correspondence.

Our last chapter is a somewhat scattershot collection of additional topics that is meant to illustrate some applications and examples In particular, we show how log geometry can be used to give a very explicit view of vanishing cycles and monodromy.

1.4 Organization

The goals of this text are to introduce the reader to the basic ideas of log geometry and to provide a technical foundation for further work in the area. These goals are somewhat contradictory, in that a good deal of the foundational material depends on the algebra of monoids and the geometry of convex bodies, the study of which can impede the momentum toward the ultimate goals coming from algebraic geometry. Although a fair amount of this material can be found in the literature, we have decided to treat it carefully here, partly because the author himself wanted to become comfortable with it, and partly because the perspective from log geometry, in which morphisms play a central role, is not to be found there. We have grouped nearly all of this material in the first two chapters and consequently don't arrive at log geometry itself until chapter III, potentially discouraging a reader eager to try log geometry in some specific context. Such a reader may find it preferably to skip some of the earlier sections, returning to them as necessary; we hope our exposition will make this possible. In particular, the material on idealized monoids and log schemes and on relative coherence, concepts whose ultimate utility has not yet been convincingly demonstrated, can be skipped on a first reading. Probably the same is true of monoschemes, which are really just an alternative to the classical theory of fans from toric geometry. A hasty reader could try to get by reading only sections 1.1, 4.1, and 4.2 of Chapter I, and then sections 1.1 and 2.1 of Chapter II, before proceeding to Chapter III. On the other hand, readers whose primary interest is convex rather than log geometry, may find it interesting to concentrate on the material in Chapters I and II, since some of it may be new to them, especially section 4 of Chapter I.

1.5 Acknowledgements

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