Hyperdeterminants of Polynomials

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Quadratic polynomials, matrices, and discriminants

Square matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$$

determinant $a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$:
vanishes when A is singular -
e.g. columns linearly dependent.

Quadratic polynomial

$f = ax^2 + bx + c$

discriminant $\Delta(f) = ac - b^2/4$:

vanishes when f is singular

$\Delta(f) = 0 \iff f$ has a repeated root.

Notice can associate to f a symmetric matrix

$$A_f = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

and $\det(A_f) = \Delta(f)$.

Quadratic polynomials, matrices, and discriminants

Square matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \dots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,n} \end{pmatrix}$$

determinant $det(A)$:
vanishes when A is singular -
e.g. columns linearly dependent.

Quadratic polynomial $f =$

$$\sum_{1 \le i \le n} a_{i,i} x_i^2 + \sum_{1 \le i < j \le n} a_{i,j} 2 x_i x_j$$

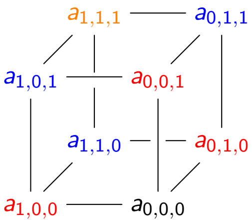
$\updownarrow$ Symmetric matrix $A_f =$

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ a_{1,2} & a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \dots & \vdots \\ a_{1,n} & a_{2,n} & a_{3,n} & \dots & a_{n,n} \end{pmatrix}$$

discriminant $\Delta(f)$:
vanishes when f is singular
i.e. has a repeated root.

If f is quadratic, then $det(A_f) = \Delta(f)$.

The symmetrization of the determinant of a matrix
= the determinant of a symmetrized matrix.

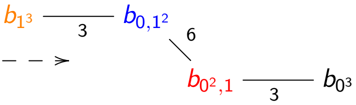


Binary cubic polynomial $f =$

$$b_{0^3}x_0^3 + 3b_{0^21}x_0^2x_1 + 3b_{01^2}x_0x_1^2 + b_{1^3}x_1^3$$

$\updownarrow$ Symmetric $2 \times 2 \times 2$ tensor

identify same colors



$$\begin{aligned}\mathrm{Det}(A) = {} & (a_{000})^2(a_{111})^2 + (a_{100})^2(a_{011})^2 \\ & + (a_{010})^2(a_{101})^2 + (a_{001})^2(a_{110})^2 \\ & - 2a_{000}a_{100}a_{011}a_{111} - 2a_{100}a_{010}a_{011}a_{101} \\ & - 2a_{000}a_{010}a_{101}a_{111} - 2a_{100}a_{001}a_{011}a_{110} \\ & - 2a_{000}a_{001}a_{110}a_{111} - 2a_{010}a_{001}a_{101}a_{110} \\ & + 4a_{000}a_{011}a_{101}a_{110} + 4a_{001}a_{010}a_{100}a_{111}\end{aligned}$$

$$\begin{aligned}\Delta(f) = {} & (b_{0^3})^2(b_{1^3})^2 + 3(b_{0^21})^2(b_{01^2})^2 \\ & - 6b_{0^3}b_{0^21}b_{01^2}b_{1^3} - 6(b_{0^21})^2(b_{01^2})^2 \\ & + 4b_{0^3}(b_{01^2})^3 + 4(b_{0^21})^3 b_{1^3}\end{aligned}$$

Discriminant vanishes when f is singular.

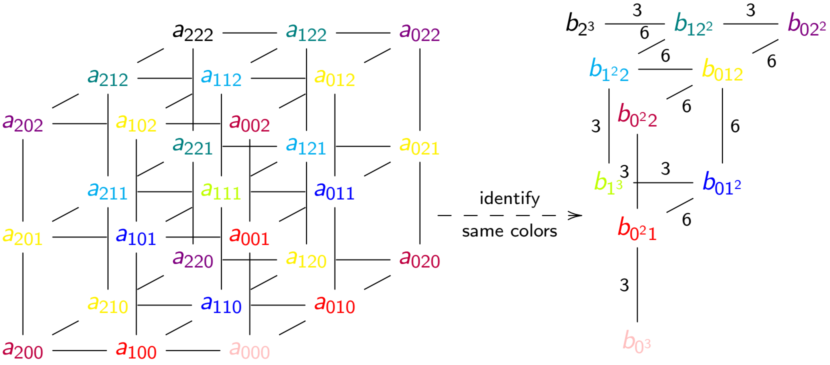
Again $\mathrm{Det}(A_f) = \Delta(f)$.

$2 \times 2 \times 2$ hyperdeterminant vanishes when A is singular.

...from wikipedia...

> 3. The algebra of invariants of the cubic form $ax^3 + 3bx^2y + 3cxy^2 + dy^3$ is a polynomial algebra in 1 variable generated by the discriminant $D = 3b^2c^2 + 6abcd - 4b^3d - 4c^3a - a^2d^2$ of degree 4.

Ottaviani: $3 \times 3 \times 3$ Hyperdeterminant and plane cubics



Specialize the variables, and use Schläfli's method to express the symmetrized (degree 36) hyperdeterminant. Boole's formula $\deg(\Delta_{(d),n}) = (n)(d-1)^{n-1}$ says $\mathit{degree}(\Delta(f)) = 3 \cdot 2^2 = 12$.

The determinant of f factors(!)

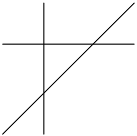
$$\mathrm{Det}(A_f) = \Delta(f) \cdot (AR)^6 \qquad (\mathit{deg}\ 36) = (\mathit{deg}\ 12) \cdot (\mathit{deg}\ 4)^6$$

Geometric Version of Ottaviani's Example

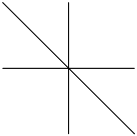
Hyperdeterminant vanishes = discriminant · Aronhold^6
Dual of Segre = Dual of Veronese · (dual of something?)^6

Classical geometry and Aronhold's invariant:

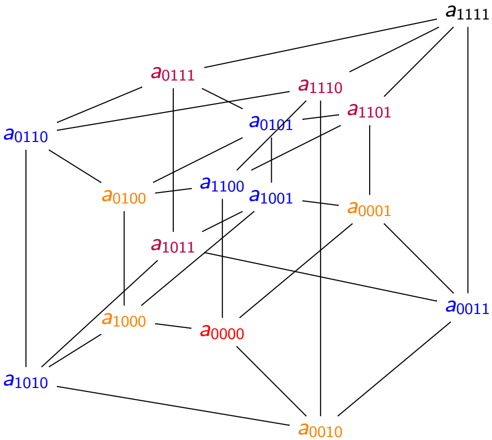
$\mathcal{V}(AR) =$ Fermat cubics $\sigma_3(v_3(\mathbb{P}^2))$
$= \overline{\{x^3 + y^3 + z^3 \mid [x], [y], [z] \in \mathbb{P}^2\}}$.

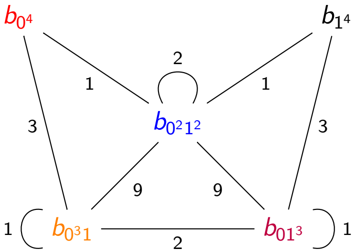


$\mathcal{V}(AR)^\vee =$ completely reducible cubics
$Chow_{1,1,1} = \overline{\{\xi\gamma\zeta \mid [\xi], [\gamma], [\zeta] \in (\mathbb{P}^2)^\vee\}}$.



$2 \times 2 \times 2 \times 2$ hyperdeterminant and quartic curves





The discriminant of binary quartics has degree 6 - and is associated to the $3-9-9-3$ chain. What is the other stuff?

The $2 \times 2 \times 2 \times 2$ hyperdeterminant has degree 24.

Schläfli's method yields

$$\mathrm{Det}(A_f) = \Delta(f)(cat)^6 \qquad (deg\ 24) = (deg\ 6) \cdot (deg\ 3)^6$$

One large example

The $3 \times 3 \times 3 \times 3 \times 3$ hyperdeterminant has degree 68688 and when symmetrized, splits as

$$\mathrm{Det}(A_f) = (\text{deg } 48)(\text{deg } 312)^{10}(\text{deg } 108)^{20}(\text{deg } 384)^{30}(\text{deg } 480)^{60}(\text{deg } 192)^{120}$$

And we can identify all the components geometrically.

Geometry: Segre-Veronese and Chow varieties

Let $V \cong \mathbb{C}^n$. Let $\lambda \vdash d$ with $\#\lambda = s$.

$$\begin{array}{rcll} \mathrm{Seg}_\lambda : \mathbb{P}V^{\times s} & \xrightarrow{|\mathcal{O}(\lambda)|} & \mathbb{P}\left(S^{\lambda_1}V \otimes \cdots \otimes S^{\lambda_s}V\right) & \subseteq \mathbb{P}\left(V^{\otimes d}\right) \\ ([a_1], \ldots, [a_s]) & \mapsto & [a_1^{\lambda_1} \otimes \cdots \otimes a_s^{\lambda_s}]. & \end{array}$$

The image is the *Segre-Veronese variety*, denoted $\mathrm{Seg}_\lambda\left(\mathbb{P}V^{\times s}\right)$.
Pieri formula implies for all $\lambda \vdash d$, there is an inclusion

$$S^d V \subset S^{\lambda_1}V \otimes \cdots \otimes S^{\lambda_s}V.$$

Since $\mathrm{GL}(V)$ is reductive, $\exists!$ G-invariant complement W^λ:

$$W^\lambda \oplus S^d V = S^{\lambda_1}V \otimes \cdots \otimes S^{\lambda_s}V.$$

Project from this complement

$$\begin{array}{rcl} \pi_{W^\lambda} : \mathbb{P}\left(S^{\lambda_1}V \otimes \cdots \otimes S^{\lambda_s}V\right) & \dashrightarrow & \mathbb{P}S^d V \\ [a_1^{\lambda_1} \otimes \cdots \otimes a_s^{\lambda_s}] & \mapsto & a_1^{\lambda_1} \cdots a_s^{\lambda_s}. \end{array}$$

The *Chow variety*, $\mathrm{Chow}_\lambda(\mathbb{P}V) = \pi_{W^\lambda}\left(\mathrm{Seg}_\lambda(\mathbb{P}V^{\times s})\right)$.
Notice $\mathrm{Chow}_\lambda(\mathbb{P}V) = \mathrm{Seg}_{\sigma(\lambda)}(\mathbb{P}V^{\times s})$ for any permutation $\sigma \in \mathfrak{S}_s$.

Dual Varieties, Hyperdeterminants and Projections

Incidence

$X \subset \mathbb{P}V$ $\qquad \mathcal{I} = \{(x, H) \mid T_x X \subset H\} \subset \mathbb{P}V \times \mathbb{P}V^*$

$$X \leftarrow \mathcal{I} \rightarrow X^\vee$$

$X^\vee$ - variety of tangent hyperplanes to X. Usually a hypersurface.
Intuitively, $X^\vee$ is not a hypersurface only if X has too many lines.

Theorem (GKZ)

If $X = \mathrm{Seg}_\mu(\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_t-1})$, then $X^\vee$ is a hypersurface iff

$$2n_i \leq \sum_{j=1}^{s} n_j \quad \textit{for all } i \textit{ such that } \lambda_i = 1.$$

$\mathrm{Seg}((\mathbb{P}^{n-1})^{\times t})^\vee = \mathcal{V}(\mathrm{Det}(A))$, hyperdet. hypersurface in $\mathbb{P}^{n^t-1}$

$\mathrm{Seg}_\mu((\mathbb{P}^n)^{\times t})^\vee = \mathcal{V}(\mathrm{Det}_\mu(A))$, μ-hyperdet. hypersurface in $\mathbb{P}^{\binom{n+\mu_1}{\mu_1}\cdots\binom{n+\mu_s}{\mu_t}}$.

$\mathrm{Seg}_{(d)}(\mathbb{P}^n)^\vee = \mathcal{V}(\Delta(f))$, discriminant hypersurface in $\mathbb{P}^{\binom{n+d}{d}}$.

Projections and Duals of Chow Varieties

Lemma (GKZ)

Let $X \subset \mathbb{P}V$, $W \subset V$, $X \not\subset \mathbb{P}W$. *Let* $\pi_W \colon \mathbb{P}V \dashrightarrow \mathbb{P}(V/W)$ *projection.*

$$\text{Then } \pi_W(X)^\vee \subseteq X^\vee \cap \mathbb{P}W^\perp.$$

If $\pi_W(X) \cong X$ *then* $=$ *holds.*

Lemma

Let $X \subset \mathbb{P}V$, $U \oplus W = V$. *Then*

$$(X \cap \mathbb{P}U)^\vee \subseteq X^\vee \cap \mathbb{P}(V/U)^\perp.$$

$\mathbb{P}(V/U)^\perp \cong \mathbb{P}U^*$.

Fact: If $\lambda \prec \mu$, then $\mathrm{Seg}_\lambda(\mathbb{P}V^{\times s}) = \mathrm{Seg}_\mu(\mathbb{P}V^{\times t}) \cap S^{\lambda_1}V \otimes \cdots \otimes S^{\lambda_l}V^*$

Projections and Duals of Chow Varieties

Lemma (GKZ)

$$\pi_W(X)^\vee \subseteq X^\vee \cap \mathbb{P}W^\perp$$

Lemma

$$(X \cap \mathbb{P}U)^\vee \subseteq X^\vee \cap \mathbb{P}U^*.$$

Fact: If $\lambda \prec \mu$, then $\mathrm{Seg}_\lambda(\mathbb{P}V^{\times s}) = \mathrm{Seg}_\mu(\mathbb{P}V^{\times t}) \cap S^{\lambda_1}V \otimes \cdots \otimes S^{\lambda_s}V^*$

Proposition (O.)

If $\lambda \prec \mu$, *then* $\mathrm{Chow}_\lambda(\mathbb{P}V)^\vee \subseteq \mathrm{Seg}_\mu(\mathbb{P}V^{\times t})^\vee$

Question: What about the other inclusion?

Main Results

The more general result for Segre-Veronese varieties:

Theorem (O.)

Let μ be a partition of $d \geq 2$, and V be a complex vector space of dimension $n \geq 2$. Then

$$\mathrm{Seg}_{\mu}\left(\mathbb{P}V^{\times t}\right)^{\vee} \cap \mathbb{P}\left(S^{d}V^{*}\right) = \bigcup_{\lambda \prec \mu} \mathrm{Chow}_{\lambda}\left(\mathbb{P}V\right)^{\vee},$$

where $\lambda \prec \mu$ is the refinement partial order. In particular,

$$\mathcal{V}(Sym(\Delta_{\mu,n})) = \prod_{\lambda \prec \mu} \Xi_{\lambda,n}^{M_{\lambda,\mu}}$$

where $\Xi_{\lambda,n}$ is the equation of $\mathrm{Chow}_{\lambda}\left(\mathbb{P}V\right)^{\vee}$ when it is a hypersurface in $\mathbb{P}(S^{d}V^{*})$, and the multiplicity $M_{\lambda,\mu}$ is the number of partitions μ that refine λ.

Theorem (O.)

The $n^{\times d}$-hyperdeterminant of a polynomial (degree $d \geq 2$, $n \geq 2$ variables) splits as the product

$$\mathcal{V}(Sym(Det(A))) = \prod_{\lambda} \Xi_{\lambda,n}^{\mathcal{M}_\lambda},$$

where $\Xi_{\lambda,n}$ is the equation of the dual variety of the Chow variety $\mathrm{Chow}_\lambda \mathbb{P}^{n-1}$ when it is a hypersurface in $\mathbb{P}^{\binom{n-1+d}{d}-1}$, $\lambda = (\lambda_1, \dots, \lambda_s)$ is a partition of d, and the multiplicity $M_\lambda = M_{\lambda,1^d} = \binom{d}{\lambda_1,\dots,\lambda_s}$ is the multinomial coefficient.

Which dual varieties of Chow varieties are hypersurfaces?

Theorem (O.)

Suppose $d \geq 2$, $\dim V = n \geq 2$ and $\lambda = (\lambda_1, \dots, \lambda_s) = (1^{m_1}, \dots, p^{m_p})$. Then $\mathrm{Chow}_\lambda (\mathbb{P}V)^\vee$ a hypersurface with the only exceptions

- $n = 2$ *and* $m_1 \neq 0$
- $n > 2$, $s = 2$ *and* $m_1 = 1$ *(so $\lambda = (d-1, 1)$).*

If $n = 2$, get closed formula for degrees of the duals of Chow varieties.

Theorem (O.)

The degree of $\mathrm{Chow}_\lambda(\mathbb{P}^1)^\vee$ *with* $\lambda = (1^{m_1}, 2^{m_2}, \ldots, p^{m_p})$, $m_1 = 0$ *and* $m = \sum_i m_i$ *is*

$$(m+1)\binom{m}{m_2, \ldots, m_p} 1^{m_2} 2^{m_3} \cdots (p-1)^{m_p}$$

If $n \geq 2$ get a *recursion* to compute all degrees of the duals of Chow varieties.

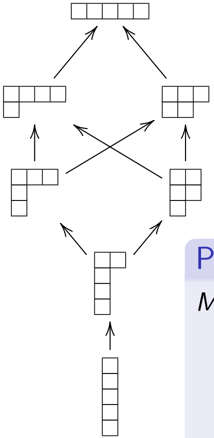
Theorem (O.)

Suppose $\dim V \geq 2$. *Let* d_λ *denote* $\deg(\mathrm{Chow}_\lambda(\mathbb{P}V)^\vee)$ *when it is a hypersurface and* 0 *otherwise. Then the vector* $(d_\lambda)_\lambda$ *is the unique solution to*

$$\deg(\mathrm{Det}_\mu) = \sum_{\lambda \prec \mu} d_\lambda M_{\lambda,\mu}.$$

The degree of Det_μ is given by a generating function (see [page 454,GKZ]), the multiplicities $M_{\lambda,\mu}$ are computable counting number of refinements.

Combinatorics: Partitions and Refinement



	(5)	(3,2)	(3,1,1)	(2,2,1)	$(2,1^3)$	(1^5)
(5)	1					
(4,1)	1					
(3,2)	1	1				
(3,1,1)	1	1	2			
(2,2,1)	1	2		2		
$(2,1^3)$	1	4	6	6	6	
(1^5)	1	10	20	30	60	120

Proposition

$M_{\lambda,\mu}$:= number of partitions μ that refine λ.

- *$M_{(d),\mu} = 1$ for all $|\mu| = d$.*
- *$M_{\lambda,\mu} = 0$ if $s > t$ or if $s = t$ and $\lambda \neq \mu$*
  i.e. $(M_{\lambda,\mu})_{\lambda,\mu}$ is lower triangular.
- *If $\lambda = (1^{m_1}, 2^{m_2}, \ldots, p^{m_p})$, then $M_{\lambda,\lambda} = m_1! \cdots m_p!$.*
- *$M_{\lambda,1^d} = \binom{d}{\lambda} := \binom{d}{\lambda_1,\ldots,\lambda_s} = \frac{d!}{\lambda_1!\cdots\lambda_s!}$, the multinomial coefficient.*

Combinatorics: Naively Counting Refinements

Proposition

Let $B(t,s) :=$ *all surjective maps*

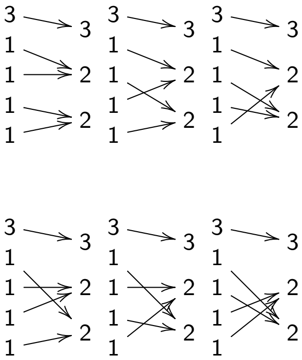
$$\phi\colon \{1,\dots,t\} \longrightarrow \{1,\dots,s\},$$

and let $\chi(a,b) = \delta_{a,b}$. *Then*

$$M_{\lambda,\mu} = \sum_{\phi \in B(t,s)} \prod_{i=1}^{s} \chi\left(\lambda_i, \sum_{j \in \phi^{-1}(i)} \mu_j\right).$$

Any better formulas?

For example, $M_{(3,2,2),(3,1,1,1,1)} = 6$.
Only ϕ contribute non-zero to $\mathcal{M}_{\lambda,\mu}$:



Note that this construction accounts for the ambiguity in the location of the 2's in the partition $(3,2,2)$.

Combinatorics: generating functions

Ring of symmetric functions:
$\bigwedge[x] = \bigwedge[x_1, x_2, \dots]$.
$p_\lambda \in \bigwedge[x]$ - power-sum symmetric functions,

$$p_\lambda(x) = \prod_i (x_1^{\lambda_i} + x_2^{\lambda_i} \dots)$$

$\lambda = (\lambda_1, \dots, \lambda_s) \vdash d$.

$m_\mu \in \bigwedge[x]$ - monomial symmetric functions,

$$m_\mu(x) = \sum_{\sigma \sim} x^{\sigma.\mu},$$

sum over distinct permutations σ of $\mu = (\mu_1, \mu_2, \dots, \mu_s, 0, \dots)$ and $x^\mu = x_1^{\mu_1} \dots x_s^{\mu_s}$.

Proposition (Thanks to Mark Haiman)

Suppose λ, μ, p_λ and m_μ are as above. Then the number of refinements matrix $(M_{\lambda,\mu})$ is the change of basis matrix

$$p_\lambda(x) = \sum_{\mu \vdash d} M_{\lambda,\mu} m_\mu(x). \tag{1}$$

The matrix $(M_{\lambda,\mu})$ can be quickly computed in any computer algebra system.
Compare the coefficients - $M_{\lambda,\mu}$ is the coefficient on the monomial x^μ in (1).

Generating function from GKZ for degree of A-discriminants [GKZ]

$$\sum_{\kappa} N(\kappa;\mu) z^{\kappa} = \frac{1}{\left[\prod_i (1+z_i) - \sum_j \mu_j z_j \prod_{i\neq j}(1+z_i)\right]^2},$$

where $N(\kappa;\mu) = \deg(\Delta_{\kappa,\mu})$ is the degree of $\mathrm{Seg}_{\mu}(\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_t})^{\vee}$ and $\kappa \in \mathbb{Z}^t_{>0}$. We can now compute the degree of the duals of the Chow varieties via the following generating function.

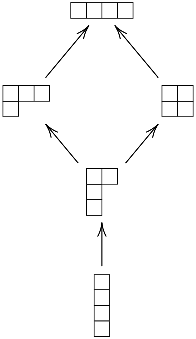
Theorem (O.)

Suppose $\dim V \geq 2$. *Let* d_λ *denote* $\deg(\mathrm{Chow}_\lambda(\mathbb{P}V)^{\vee})$ *when it is a hypersurface and* 0 *otherwise. Let* $\Delta_{\mu,n}$ *denote the equation of the hypersurface* $\mathrm{Seg}_{\mu}(\mathbb{P}V^{\times t})^{\vee}$. *The degrees* d_λ *are computed by*

$$\sum_{\mu} \deg(\Delta_{\mu,n}) m_\mu(x) = \sum_{\lambda} d_\lambda p_\lambda(x),$$

where m_μ *and* p_λ *are respectively the monomial and power sum symmetric functions.*

Example $d = 4$, $n = 3$



Consider the system $M_{\lambda,\mu} d_\lambda = D_\mu$

$$\begin{pmatrix} 1 & & & \\ 1 & & & \\ 1 & 2 & & \\ 1 & 2 & 2 & \\ 1 & 6 & 12 & 24 \end{pmatrix} \begin{pmatrix} d_{(4)} \\ d_{(2,2)} \\ d_{(2,1,1)} \\ d_{(1^4)} \end{pmatrix} = \begin{pmatrix} D_{(4)} \\ D_{(3,1)} \\ D_{(2,2)} \\ D_{(2,1,1)} \\ D_{(1^4)} \end{pmatrix} = \begin{pmatrix} 27 \\ 27 \\ 129 \\ 225 \\ 1269 \end{pmatrix}.$$

The unique solution is

$$(d_{(4)}, d_{(2,2)}, d_{(2,1,1)}, d_{(1^4)}) = (27, 51, 48, 15).$$

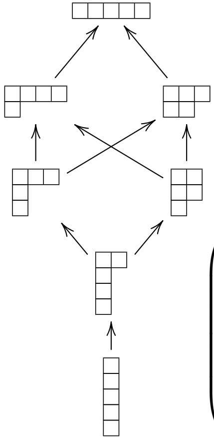
The symmetrized hyperdeterminant of format $3^{\times 4}$ splits as:

$$Sym(Det(A)) = discrim \cdot (\mathrm{Chow}^\vee_{2,2})^6 \cdot (\mathrm{Chow}^\vee_{2,1,1})^{12} \cdot (\mathrm{Chow}^\vee_{1,1,1,1})^{24}$$

$$\deg 1269 = (\deg 27) \cdot (\deg 51)^6 \cdot (\deg 48)^{12} \cdot (\deg 15)^{24}.$$

The other μ-discriminants have the same factors, with different multiplicities encoded by $M_{\lambda,\mu}$.

Example $d = 5$, $n = 3$



Consider the system $M_{\lambda,\mu} d_\lambda = D_\mu$

$$\begin{pmatrix} 1 & & & & & \\ 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 1 & 2 & & & \\ 1 & 2 & & 2 & & \\ 1 & 4 & 6 & 6 & 6 & \\ 1 & 10 & 20 & 30 & 60 & 120 \end{pmatrix} \begin{pmatrix} d_{(5)} \\ d_{(3,2)} \\ d_{(3,1,1)} \\ d_{(2,2,1)} \\ d_{(2,1^3)} \\ d_{(1^5)} \end{pmatrix} = \begin{pmatrix} D_{(5)} \\ D_{(4,1)} \\ D_{(3,2)} \\ D_{(3,1,1)} \\ D_{(2,2,1)} \\ D_{(2,1^3)} \\ D_{(1^5)} \end{pmatrix} = \begin{pmatrix} 48 \\ 48 \\ 360 \\ 576 \\ 1440 \\ 7128 \\ 68688 \end{pmatrix}$$

The unique solution is

$$(d_{(5)}, d_{(3,2)}, d_{(3,1,1)}, d_{(2,2,1)}, d_{(2,1^3)}, d_{(1^5)}) = (48, 312, 108, 384, 480, 192).$$

The first case we consider is $n = 2$ and $d = 7$ and solve

$$\begin{pmatrix} 1 & & & \\ 1 & & & \\ 1 & 1 & & \\ 1 & 1 & & \\ 1 & & 1 & \\ 1 & 1 & 1 & \\ 1 & 3 & 1 & \\ 1 & & 2 & \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 3 & 2 \\ 1 & 6 & 5 & 6 \\ 1 & 3 & 3 & 6 \\ 1 & 5 & 7 & 7 \\ 1 & 11 & 15 & 50 \\ 1 & 21 & 35 & 210 \end{pmatrix} \begin{pmatrix} d_{(7)} \\ d_{(5,2)} \\ d_{(4,3)} \\ d_{(3,2,2)} \end{pmatrix} = \begin{pmatrix} D_{(7)} \\ D_{(6,1)} \\ D_{(5,2)} \\ D_{(5,1,1)} \\ D_{(4,3)} \\ D_{(4,2,1)} \\ D_{(4,1^3)} \\ D_{(3,3,1)} \\ D_{(3,2,2)}, \\ D_{(3,2,1,1)} \\ D_{(3,1^4)} \\ D_{(2,2,2,1)} \\ D_{(2,2,1^3)} \\ D_{(2,1^5)} \\ D_{(1^7)} \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \\ 36 \\ 36 \\ 48 \\ 72 \\ 120 \\ 84 \\ 144 \\ 216 \\ 480 \\ 336 \\ 720 \\ 2016 \\ 6816 \end{pmatrix}.$$

complicated poset

The unique solution to $M_{\lambda,\mu} d_\lambda = D_\mu$ is

$$(d_{(7)}, d_{(5,2)}, d_{(4,3)}, d_{(3,2,2)}) = (12, 24, 36, 24).$$

Next we use only a relevant lower-triangular sub-matrix of $M_{\lambda,\mu}$, when $d = 8$ and $n = 2$ where here we have omitted several rows that are unnecessary for computing the degrees of Ξ_λ.

$$\begin{pmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & & 1 & & & & \\ 1 & & & 2 & & & \\ 1 & 2 & & 4 & 2 & & \\ 1 & 1 & 2 & & & 2 & \\ 1 & 4 & & 6 & 12 & & 120 \\ 1 & 28 & 56 & 70 & 420 & 560 & 2520 \end{pmatrix} \begin{pmatrix} (d_{(8)} \\ d_{(6,2)} \\ d_{(5,3)} \\ d_{(4,4)} \\ d_{(4,2,2)} \\ d_{(3,3,2)} \\ d_{(2,2,2,2)}) \end{pmatrix} = \begin{pmatrix} (D_{(8)} \\ D_{(6,2)} \\ D_{(5,3)} \\ D_{(4,2,2)} \\ D_{(3,3,2)} \\ D_{(2,2,2,2)} \\ D_{(1^8)}) \end{pmatrix} = \begin{pmatrix} 14 \\ 44 \\ 62 \\ 116 \\ 656 \\ 848 \\ 60032 \end{pmatrix}.$$

The unique solution to $M_{\lambda,\mu} d_\lambda = D_\mu$ is

$$(d_{(8)}, d_{(6,2)}, d_{(5,3)}, d_{(4,4)}, d_{(4,2,2)}, d_{(3,3,2)}, d_{(2,2,2,2)}) = (14, 30, 48, 27, 36, 48, 5).$$

Thanks!