# EACA EXTENDED ABSTRACT: ARE ALL SECANT VARIETIES OF SEGRE PRODUCTS ARITHMETICALLY COHEN-MACAULAY?

LUKE OEDING

Abstract. The general problem of arithmetic Cohen-Macaulayness for secant varieties of Segre products is investigated. A inductive procedure based on the work of Landsberg and Weyman is given. New computational results are presented for rank 4 tensors of format $3 \times 3 \times 4$, together with a certain generalization.

## Introducción/Introduction

After finding many equations in low degree, the question then remains: What is the maximal degree of minimal defining equations of a given secant variety, and when do the known equations suffice? This question is well studied in some cases such as monomial ideals, for curves, and in some infinite dimensional cases, however the general question is still very open. It may be possible to obtain upper bounds (via Castelenuovo-Mumford regularity, for example), but these computations are often also difficult. Another approach undertaken by Aschenbrenner and Hillar, Draisma and Kutler, Sam and Snowden, and others is to investigate these questions in an infinite dimensional setting. This method has been used to determine when certain ideals are "Noetherian up to symmetry" and in turn, this can provide a (non-constructive) guarantee that tensors of bounded rank are defined by equations in bounded degree not depending on the number of tensor factors. This method, however, does not typically give an explicit bound. The varying degrees of success of these approaches are indicators that secant varieties of Segre products may all share particularly nice structural property that governs the behavior of their ideals, namely the Cohen-Macaulay property. This article is focused on uncovering this ground truth about secant varieties.

Another way to know when the given equations generate a prime ideal that might be available is if the variety is arithmetically Cohen-Macaulay (aCM), i.e. if the depth of its coordinate ring is equal to the codimension. If the variety is aCM, one can determine if the given ideal agrees with the ideal of the variety by checking if it (a) cuts the variety out set-theoretically, and (b) is generically reduced. Inspired by recent evidence and classical results we offer the following:

**Conjecture 0.1.** *Every secant variety of a Segre product is arithmetically Cohen-Macaulay.*

In addition to the implicitization motivation, this conjecture is interesting because aCM varieties have many nice properties, such as being equi-dimensional, and being aCM says that the singularities of the variety are mild. Thus knowing whether a variety is aCM or not gives insight into how complicated that variety is. Evidence for Conjecture 0.1 includes the following cases where a secant variety of a Segre product is known to be aCM: Segre varieties themselves, when the secant variety fills its ambient space or is a hypersurface, determinantal varieties, subspace varieties, as well as several other special cases.

Geramita made the symmetric version of Conjecture 0.1, ( see [9, p55] ), and Kanev proved special cases:

**Theorem 0.2** (Kanev, [10]). *$\sigma_s(\nu_d\mathbb{P}^n)$ is aCM if either $d = 2$, or $n = 1$ or $s \leq 2$.*

The dimension, degree, and singular locus are also known. A local, non-symmetric version of Kanev's result is the following result obtained by the author together with Michalek and Zwiernik:

**Theorem 0.3** (Michalek-Oeding-Zwiernik [12]). *$\sigma_2(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m})$ is covered by open normal toric varieties. In particular $\sigma_2(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m})$ is locally Cohen-Macaulay.*

Theorem 0.4 below pushed the computational boundaries and confirms Conjecture 0.1 in a new case. The inspiration came from B. Sturmfels's "algebraic fitness session" at the Simon's Institute, Fall 2014, and the following results: It is known classically that $2\times2\times2\times2$ tensors are defective in rank 3. Bocci and Chiantini showed that $2 \times 2 \times 2 \times 2 \times 2$ tensors are not identifiable in rank 5 — the generic tensor of that format has exactly 2 decompositions [3]. Further, Bocci-Chiantini-Ottaviani showed that for $\geq 6$ factors, the Segre is almost always $k$-identifiable, This generated interest in finding the equations for the boundary case of 5 binary factors of rank 5.

**Theorem$^\star$ 0.4** (Oeding-Sam [13]). *The affine cone of $\sigma_5(\mathrm{Seg}(\mathbb{P}^{1^{\times 5}}))$ is a complete intersection of a degree 6 and a degree 16 equation.*

The star refers to the use of careful numerical, sometimes probabilistic computations in the proofs. In particular, $\sigma_5(\mathrm{Seg}(\mathbb{P}^{1^{\times 5}}))$ is aCM. These computations took approximately *two weeks of human/computer time.*

**An adaptation of Weyman's geometric technique.** Landsberg and Weyman applied a partial desingularization together with a mapping cone argument to show the following.

**Theorem 0.5** (Landsberg–Weyman [11]). *Suppose $X := \sigma_r(\mathrm{Seg}(\mathbb{P}^{{r-1}^{\times d}}))$ is aCM, with "a resolution by small partitions." If $n_i \geq r-1$ for all $1 \leq i \leq d$, then $\sigma_r(\mathrm{Seg}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_d}))$ is aCM and its ideal is generated by those inherited from $X$ and the $(r+1)\times(r+1)$-minors of flattenings.*

This begs the question about the cases of $\sigma_r(\mathrm{Seg}(\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_d}))$ when $n_i < r-1$ for some $i$. The concept of resolutions via small partitions (which appeared as a technical curiosity in [11]. An $R$-module $M$ has a resolution by small partitions if the Schur modules which occur in the $G$-equivariant resolution are indexed by partitions that fit inside prescribed sized boxes. More specifically, suppose $A_i' \subseteq A_i$ for $1 \leq i \leq n$ and let $S_\pi A$ denote the Schur module associated to the multi-partition $\pi$. A $G$-variety $Y$ has an $(s_j)$-small resolution if every module $S_\pi A$ occurring in the resolution has the property

$$\text{for each } j \text{ the first part of } \pi^j \text{ is not greater than } s_j.$$

Let $\widehat{a_j} := \frac{a_1\dots a_n}{a_j}$, $\widehat{r_j} := \frac{r_1\dots r_n}{r_j}$, $G = \mathrm{GL}(A_1)\times\cdots\times\mathrm{GL}(A_n)$ and $G' = \mathrm{GL}(A_1')\times\cdots\times\mathrm{GL}(A_n')$.

**Theorem 0.6** (Oeding, (adapted from [11]))**.** *If a $G'$-variety $Y$ is an aCM with an resolution that is $(\widehat{r_j} - r_j)$-small for every $j$ for which $0 < r_j < a_j$, then $\overline{G.Y}$ is aCM.*

*Moreover one obtains a (not necessarily minimal) resolution of $\overline{G.Y}$ that is $(s_j)$-small with*

$$s_j = \max_\pi \begin{cases} \widehat{a_j} - r_j, & \text{if } r_j < a_j \\ \widehat{a_j} - \widehat{r_j} + \pi_1^j, & \text{if } r_j = a_j, \end{cases}$$

*where the* max *is taken over all multi-partitions $\pi$ occurring in the resolution of $\mathbb{C}[Y]$.*

Applying the geometric technique to Strassen's degree 9 hypersurface one obtains the following.

**Proposition 0.7.**

| | | |
|---|---|---|
| $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$ | *is aCM, deg. 9 hypersurface* | *[Strassen]* |
| $\mathrm{GL}(4).\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$ | *is aCM and* codim *3 in* | $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$. |
| $\mathrm{GL}(4)^{\times 2}.\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$ | *is aCM and* codim *4 in* | $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3)$. |
| $\mathrm{GL}(4)^{\times 3}.\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$ | *is aCM and* codim *5 in* | $\sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$. |
| $\mathrm{GL}(4).\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$ | *is aCM and* codim *1 in* | $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3)$. |
| $\mathrm{GL}(4)^{\times 2}.\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$ | *is aCM and* codim *2 in* | $\sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$. |
| $\mathrm{GL}(4).\sigma_4(\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3)$ | *has* codim *1 in* | $\sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$. |

It remains to determine if one may lift the aCM property further. If possible, this will complete a major step forward, since in particular it would solve the salmon conjecture [1,2,5,6].

Let $R = \mathbb{C}[A \otimes B \otimes C]$ and $G = \mathrm{GL}(A) \times \mathrm{GL}(B) \times \mathrm{GL}(C)$. Using Galetto's HighestWeights package [7] in Macaulay2, we determined the $G$-module structure of the minimal free resolution of $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$, and verified that it is aCM with a resolution by small partitions (we omit the details for space reasons). Daleo and Hauenstein have also numerically verified that this variety is Cohen-Macaulay [4]. After checking that the ideal is generically reduced, one obtains the following.

**Theorem 0.8.** *The secant variety $\sigma_4(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3))$ is arithmetically Cohen-Macaulay. Its prime ideal is minimally generated by the 10 degree 6 Landsberg-Manivel equations, and the 20 degree 9 equations inherited from Strassen's equation.*

Applying the adaptation of the Landsberg–Weyman inheritance result, and the G-module structure one has:

**Theorem 0.9.** *The secant variety $\sigma_4(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^n))$ is arithmetically Cohen-Macaulay for all $n \geq 0$. Its prime ideal is minimally generated by $\binom{n+2}{2}$ degree 6 (Landsberg-Manivel) equations (nontrivial when $n \geq 3$), and $\binom{n+3}{3}$ degree 9 (Strassen) equations (nontrivial when $n \geq 2$).*

Special cases of Conjecture 0.1 are also quite interesting. In the case $s = 2$ [15] that resolved the Garcia-Stillman-Sturmfels Conjecture [8] that the $3 \times 3$ minors of flattenings minimally define the ideal of $\sigma_2$, and it is known that the variety is locally Cohen-Macaulay [12], so it is expected that the aCM property should also hold, and it would be a nice addition to the story. Moreover, the connection to cumulants found in [16] and [12] simplified much of the computational efforts.

If in the case $s = 3$ one can prove that $\sigma_3$ is aCM for any number of factors, this would improve Yang Qi's recent result that $\sigma_3$ is defined set-theoretically by Strassen's degree 4 equations [14] together with flattenings by showing that his result holds even ideal-theoretically.

In addition it is anticipated that insights discovered in a case-by-case study will also prove useful in special cases that currently command much attention outside of pure mathematics. The matrix multiplication tensor for $3\times 3$-matrices, denoted $\mathrm{MM}_3$, lies in $\sigma_s(\mathbb{P}^8\times\mathbb{P}^8\times\mathbb{P}^8) \subset \mathbb{P}^{728}$ for some $14 \leq s \leq 21$. It would be a major result in Computational Complexity to determine $s$ exactly. Settling the case $s = 4$ and 3 tensor factors of Conjecture 0.1 would resolve the salmon conjecture for Phylogenetics. If one knew that they were arithmetically Cohen-Macaulay, then this would allow more tools from Commutative Algebra to be used in their study.

## References


1. E. Allman, *Open problem: Determine the ideal defining* $Sec_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$, http://www.dms.uaf.edu/ eallman/salmonPrize.pdf, 2010.
2. D. J. Bates and L. Oeding, *Toward a salmon conjecture*, Exp. Math. **20** (2011), no. 3, 358–370, arXiv:1009.6181.
3. C. Bocci and L. Chiantini, *On the identifiability of binary Segre products*, J. Algebraic Geom. **22** (2013), no. 1, 1–11.
4. N. S. Daleo and J. D. Hauenstein, *Numerically deciding the arithmetically Cohen-Macaulayness of a projective scheme*, Journal of Symbolic Computation **72** (2016), 128–146.
5. S. Friedland, *On tensors of border rank l in* $\mathbb{C}^{m\times n\times l}$, Linear Algebra Appl. **438** (2013), no. 2, 713–737.
6. S. Friedland and E. Gross, *A proof of the set-theoretic version of the salmon conjecture*, J. Algebra **356** (2012), 374–379.
7. F. Galetto, *HighestWeights: a package for free resolutions and modules with a semisimple lie group action*, preprint (2014), http://www.mast.queensu.ca/~galetto/research.htm.
8. L. Garcia, M. Stillman, and B. Sturmfels, *Algebraic geometry of Bayesian networks*, J. Symbolic Comput. **39** (2005), no. 3-4, 331–355.
9. A. V. Geramita, *Inverse systems of fat points: Waring's problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals*, The Curves Seminar at Queen's, Vol. X (Kingston, ON, 1995), Queen's Papers in Pure and Appl. Math., vol. 102, Queen's Univ., Kingston, ON, 1996, pp. 2–114.
10. V. Kanev, *Chordal varieties of Veronese varieties and catalecticant matrices*, J. Math. Sci. (New York) **94** (1999), no. 1, 1114–1125, Algebraic geometry, 9.
11. J.M. Landsberg and J. Weyman, *On the ideals and singularities of secant varieties of Segre varieties*, Bull. Lond. Math. Soc. **39** (2007), no. 4, 685–697.
12. M. Michalek, L. Oeding, and P. Zwiernik, *Secant cumulants and toric geometry*, International Mathematics Research Notices **2015** (2015), no. 12, 4019–4063, arXiv:1212.1515.
13. L. Oeding and S. V Sam, *Equations for the fifth secant variety of Segre products of projective spaces*, Experimental Mathematics **25** (2016), 94–99, arXiv:1502.00203.
14. Y. Qi, *Equations for the third secant variety of the Segre product of n projective spaces*, preprint (2013), arXiv:1311.2566.
15. C. Raicu, *Secant varieties of Segre-Veronese varieties*, Algebra Number Theory **6-8** (2012), 1817–1868.
16. B. Sturmfels and P. Zwiernik, *Binary cumulant varieties*, Ann. Comb. **17** (2013), no. 1, 229–250, arXiv:1103.0153.



Auburn University
*E-mail address*: oeding@auburn.edu