Symmetrization of Principal Minors

Principal Minor Assignment Problem

Question (Principal Minor Assignment Problem)

Given $v \in \mathbb{C}^{2^n}$, does there exist an $n \times n$ matrix A such that v is the vector of all principal minors of A?

There's a simple test if only we knew generators of the ideal of relations amongst principal minors.

Applications outside of geometry

- Spectral graph theory.
- Probability theory - covariance of random variables.
- Statistical physics - determinantal point processes.
- Matrix theory - P-matrices, GKK-τ matrices.

Interesting problem, see [Borodin-Rains], [Kenyon-Pemantle], [Lin-Sturmfels], [Holtz-Sturmfels], [Rising-Kulesza-Taskar]...

Principal Minor Coordinates on $(\mathbb{C}^2)^{\otimes n}$ and $S^n\mathbb{C}^2$

$(\mathbb{C}^2)^{\otimes n} = \operatorname{span}\{D_I \mid I \subset [n]\}$, with an action of $\mathrm{GL}(2)^{\times n}$.
Coordinate functions on the variety of principal minors:

$$\begin{array}{ccc} \mathbb{C}^{n\times n} & \to & (\mathbb{C}^2)^{\otimes n} \\ A & \mapsto & (D_I(A)) = (\Delta_I(A)). \end{array}$$

$S^n\mathbb{C}^2 = \operatorname{span}\{d_s \mid 0 \leq s \leq n\}$, with an action of $\mathrm{GL}(2) \hookrightarrow_\Delta \mathrm{GL}(2)^{\times n}$.

Get $S^n\mathbb{C}^2 \hookrightarrow (\mathbb{C}^2)^{\otimes n}$ by seting $d_s = D_I = D_J$ whenever $|I| = |J| = s$.

This process is called *symmetrization*.
Coordinate functions on the variety of symmetrized principal minors: Assume A is such that $D_I(A) = D_J(A)$ whenever $|I| = |J| = s$.

$$\begin{array}{ccc} \mathbb{C}^{n\times n} & \to & S^n\mathbb{C}^2 \\ A & \mapsto & (d_k(A)) = (\Delta_{[k]}(A)). \end{array}$$

For this whole talk, $D_\emptyset = d_0 = 1$.

Cycle-sums and principal minors (Following Lin-Sturmfels)

Another set of coordinate functions on $(\mathbb{C}^2)^{\otimes n}$ and $S^n\mathbb{C}^2$.

Definition

For $A \in \mathbb{C}^{n\times n}$ and $I \subset [n]$ the cycle-sum C_I is

$$C_I(A) := \sum_{\{i_1,\dots,i_k\}=I, \quad i_1=\min I} a_{i_1,i_2}a_{i_2,i_3}\cdots a_{i_{k-1},i_k}a_{i_k,i_1}.$$

Example

The first few cycle-sums are the following.

$$\begin{aligned}
C_\emptyset(A) &= 1\\
C_{\{1\}}(A) &= a_{1,1}\\
C_{\{1,2\}}(A) &= a_{1,2}a_{2,1}\\
C_{\{1,2,3\}}(A) &= a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{3,2}a_{2,1}\\
C_{\{1,2,3,4\}}(A) &= a_{1,2}a_{2,3}a_{3,4}a_{4,1} + a_{1,3}a_{3,2}a_{2,4}a_{4,1} + a_{1,4}a_{4,2}a_{2,3}a_{3,1}\\
&\quad + a_{1,2}a_{2,4}a_{4,3}a_{3,1} + a_{1,3}a_{3,4}a_{4,2}a_{2,1} + a_{1,4}a_{4,3}a_{3,2}a_{2,1}
\end{aligned}$$

Proposition ([Prop. 4, Lin-Sturmfels])

Fix $n \in \mathbb{Z}^+$, and rings $R_C = \mathbb{C}[C_S \mid S \subset [n]]$ and $R_D = \mathbb{C}[D_S \mid S \subset [n]]$.
We have a (lower triangular) non-linear isomorphism of rings given by

$$D_S = \sum_{S_1 S_2 \cdots S_k \in \Pi_S} (-1)^{|S|-k}\, C_{S_1} C_{S_2} \cdots C_{S_k}, \tag{1}$$

$$C_S = \sum_{S_1 S_2 \cdots S_k \in \Pi_S} (-1)^{|S|-k} (k-1)!\, D_{S_1} D_{S_2} \cdots D_{S_k}, \tag{2}$$

where Π_S is the lattice of set-partitions on S, and $D_\emptyset = C_\emptyset = 1$.

Lin and Sturmfels' proof.

The transition $R_D \to R_C$ is Leibnitz's formula.
$R_C \to R_D$ follows by Möbius inversion on the lattice of set-partitions [Prop. 3.7.1, Stanley]. □

Symmetrized cycle-sums and principal minors

Set $C_I = C_J = c_s$ whenever $|I| = |J| = s$.
Get another set of coordinate functions on $S^n(\mathbb{C}^2)$: $\{c_i \mid 0 \leq i \leq n\}$.

Example

$$\begin{aligned}
d_1 &= c_1\\
d_2 &= c_1^2 - c_2\\
d_3 &= c_1^3 - 3c_1c_2 + c_3\\
d_4 &= c_1^4 - 6c_1^2c_2 + 3c_2^2 + 4c_1c_3 - c_4\\
d_5 &= c_1^5 - 10c_1^3c_2 + 15c_1c_2^2 + 10c_1^2c_3 - 10c_2c_3 - 5c_1c_4 + c_5\\
d_6 &= c_1^6 - 15c_1^4c_2 + 45c_1^2c_2^2 + 20c_1^3c_3 - 15c_2^3 - 60c_1c_2c_3 - 15c_1^2c_4 + 10c_3^2 + 15c_2c_4 + 6c_1c_5 - c_6
\end{aligned}$$

$$\begin{aligned}
c_1 &= d_1\\
c_2 &= d_1^2 - d_2\\
c_3 &= 2d_1^3 - 3d_1d_2 + d_3\\
c_4 &= 6d_1^4 - 12d_1^2d_2 + 3d_2^2 + 4d_1d_3 - d_4\\
c_5 &= 24d_1^5 - 60d_1^3d_2 + 30d_1d_2^2 + 20d_1^2d_3 - 10d_2d_3 - 5d_1d_4 + d_5\\
c_6 &= 120d_1^6 - 360d_1^4d_2 + 270d_1^2d_2^2 + 120d_1^3d_3 - 30d_2^3 - 120d_1d_2d_3 - 30d_1^2d_4 + 10d_3^2 + 15d_2d_4 + 6d_1d_5 - d_6
\end{aligned}$$

Symmetrized cycle-sums and principal minors

Proposition

Symmetrized cycle sums and principal minors transform as

$$d_s = \sum_{\alpha \vdash s} (-1)^{s-|\alpha|} \, p_\alpha \, c^\alpha, \tag{3}$$

$$c_s = \sum_{\alpha \vdash s} (-1)^{s-|\alpha|} \, (|\alpha| - 1)! \, p_\alpha \, d^\alpha, \tag{4}$$

where

$$p_\alpha = \frac{s!}{1!^{m_1} m_1! 2!^{m_2} m_2! \cdots s!^{m_s} m_s!}.$$

is the number of set-partitions of $[s]$ *with type*$(\alpha) = (m_1, \ldots, m_s)$.

Proof.

We simply combine the symmetrized terms in (1) and (2) to get (3) and (4).
The formula for p_α is [Eq. 3.37, Stanley]. □

Examples: 2×2 symmetric matrices

Define a map φ : symmetric matrices $\to$ principal minors:

$$\begin{array}{rcl} \varphi\colon S^2\mathbb{C}^2 & \to & \mathbb{C}^2 \otimes \mathbb{C}^2 \\ \begin{pmatrix} a & c \\ c & b \end{pmatrix} & \mapsto & = [1, a, b, ab - c^2] \end{array}$$

When can we go backwards? Given $[w, x, y, z]$ is there a 2×2 matrix that has these principal minors? Need to solve: (WLOG assume $w = 1$)

$$\begin{aligned} x &= a \\ y &= b \\ z &= ab - c^2 \quad \Rightarrow \quad c = \pm\sqrt{xy - z} \end{aligned}$$

Then

$$\varphi \begin{pmatrix} x & \pm\sqrt{xy - z} \\ \pm\sqrt{xy - z} & y \end{pmatrix} = [1, x, y, z]$$

Conclude: Even in the $n \times n$ case, the 0×0, 1×1, and 2×2 minors determine a symmetric matrix up to the signs of the off-diagonal terms.

Principal Minors: 3×3 symmetric matrices

$$\varphi \colon S^2\mathbb{C}^2 \quad \to \quad \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

$$\varphi \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{pmatrix} =$$

$$\left[\begin{array}{l} D_{\emptyset} = 1, \\ D_{\{1\}} = x_{11}, \\ D_{\{2\}} = x_{22}, \\ D_{\{1,2\}} = (x_{11}x_{22} - x_{12}^2), \\ D_{\{3\}} = x_{33}, \\ D_{\{1,3\}} = (x_{11}x_{33} - x_{13}^2), \\ D_{\{2,3\}} = (x_{22}x_{33} - x_{23}^2), \\ D_{\{1,2,3\}} = \left(x_{11}x_{22}x_{33} - x_{11}x_{23}^2 - x_{22}x_{13}^2 - x_{33}x_{12}^2 + 2x_{12}x_{13}x_{23}\right) \end{array} \right]$$

Given $[D_{\emptyset}, D_{\{1\}}, D_{\{2\}}, D_{\{1,2\}}, D_{\{3\}}, D_{\{1,3\}}, D_{\{2,3\}}, D_{\{1,2,3\}}]$ is there a matrix that maps to it?
Count parameters: 6 versus 7 - there must be some relation that holds!

3×3 principal minors of symmetric matrices

Theorem (Holtz-Sturmfels 2007, Cayley 1845)

All relations among the principal minors of a 3×3 matrix are generated by Cayley's hyperdeterminant of format $2 \times 2 \times 2$:

$$\begin{gathered} Det := D_\emptyset^2 D_{\{1,2,3\}}^2 + D_{\{1\}}^2 D_{\{2,3\}}^2 + D_{\{2\}}^2 D_{\{1,3\}}^2 + D_{\{3\}}^2 D_{\{1,2\}}^2 \\ +4\left(D_{\{1\}} D_{\{2\}} D_{\{3\}} D_{\{1,2,3\}} + D_\emptyset D_{\{1,2\}} D_{\{1,3\}} D_{\{2,3\}}\right) \\ -2\begin{pmatrix} D_{\{1\}} D_{\{2\}} D_{\{1,3\}} D_{\{2,3\}} + D_{\{1\}} D_{\{1,2\}} D_{\{3\}} D_{\{2,3\}} + D_{\{2\}} D_{\{1,2\}} D_{\{3\}} D_{\{1,3\}} \\ +D_\emptyset D_{\{1\}} D_{\{2,3\}} D_{\{1,2,3\}} + D_\emptyset D_{\{2\}} D_{\{1,3\}} D_{\{1,2,3\}} + D_\emptyset D_{\{1,2\}} D_{\{3\}} D_{\{1,2,3\}} \end{pmatrix}. \end{gathered}$$

In cycle-sums C_I the $2 \times 2 \times 2$ hyperdeterminant is

$$Det = -4C_{\{1,2\}} C_{\{1,3\}} C_{\{2,3\}} + C_{\{1,2,3\}}^2,$$

see [Sturmfels-Zwiernik] since in this case cycle-sums correspond to binary cumulants.

Symmetrized principal minors

[Grinshpan, Kaliuzhnyi-Verbovetskyi, Woerdeman] studied the symmetrized principal minor problem in relation to a question on determinantal representations of multivariate polynomials.

For $A \in \mathbb{C}^{n\times n}$, $\Delta_I(A)$ the principal minor of A with row/column sets I.
A has symmetrized principal minors if $\Delta_I(A) = \Delta_J(A)$ when $|I| = |J|$.

Setting $D_S = d_{|S|}$ and $D_\emptyset = 1$, the hyperdeterminant symmetries to

$$SDet = -3d_1^2d_2^2 + 4d_1^3d_3 + 4d_2^3 - 6d_1d_2d_3 + d_3^2,$$

the discriminant of the cubic $1 + 3d_1x + 3d_2x^2 + d_3x^3$.

A curious fact: Notice that

$$c_2 \; = \; d_1^2 - d_2, \quad \text{and} \quad c_3 \; = \; 2d_1^3 - 3d_1d_2 + d_3,$$

In cycle sums

$$SDet = -4c_2^3 + c_3^2$$

(the syzygy amongst the covariants of the binary cubic with unit constant term).

General matrices with symmetrized cycle-sums

Proposition

Let $A \in \mathbb{C}^{n\times n}$ be a matrix with the symmetrized cycle-sums property.

1. **Diagonal Modification**: *The matrix $A - \lambda I_n$ has symmetrized cycle-sums; its diagonal entries are $c_1(A - \lambda I_n) = \lambda - c_1(A)$; and the k-th symmetrized cycle-sum $c_k(A - c_1 I_n) = c_k(A)$ for all $k \geq 2$.*
2. **Diagonal Similarity**: *For any nonsingular $n \times n$ diagonal matrix D, the diagonal conjugation DAD^{-1} preserves all cycle-sums; so $c_k(DAD^{-1}) = c_k(A)$ for all $k \geq 1$.*
3. **Homogeneity**: *For nonzero $\lambda \in \mathbb{C}$, λA still has symmetrized cycle-sums, and $c_k(\lambda A) = \lambda^k c_k(A)$ for all $k \geq 1$.*
4. **Permutation Similarity**: *For any permutation matrix P, the permutation conjugation PAP^{-1} also has the symmetrized cycle-sums property, and $c_k(PAP^{-1}) = c_k(A)$ for all $k \geq 1$.*

Use this symmetry to put the matrix A in the nicest possible format.

Symmetric matrices with symmetrized cycle-sums

Proposition

If A is symmetric, then A is conjugate to

$$\lambda \mathbb{1}_n + \mu I_n, \quad \textit{for } \lambda, \mu \in \mathbb{C},$$

where $\mathbb{1}_n$ denotes the $n \times n$ all-ones matrix. We have the following parameterizations:

$$\begin{aligned} d_k(\lambda \mathbb{1}_n + \mu I_n) &= (\mu - \lambda)^{k-1} \cdot (\mu + k \cdot \lambda) \ , \\ c_k(\lambda \mathbb{1}_n + \mu I_n) &= (k-1)! \cdot \lambda^k \ . \end{aligned}$$

The variety of symmetrized principal minors of symmetric matrices is toric.

Theorem (Huang-Oeding)

Let $\mathcal{J}_n^\circ$ denote the ideal $Z_n^\circ \cap S^n\mathbb{C}^2 \cap U_{c_0=1}$. If $n=3$ then $\mathcal{J}_n^\circ$ is prime, and generated by a single equation,

$$\mathcal{J}_3^\circ = \langle -4c_2^3 + c_3^2 \rangle.$$

For $n \geq 4$ $\mathcal{J}_n^\circ$ has two components in its primary decomposition. One primary component has radical $\langle c_s \mid 2 \leq s \leq n \rangle$. The other component is prime, and generated by the following $\frac{(n-3)n}{2}$ binomial quadrics:

$$\{(i+j-1)!c_ic_j - (i-1)!(j-1)!c_{i+j} \mid 2 \leq i \leq j \leq n, \quad i+j \leq n\}$$

$$\cup \left\{(k-1)!(l-1)!c_ic_j - (i-1)!(j-1)!c_kc_l \;\middle|\; \substack{2\leq i,j,k,l\leq n,\ i+j=k+l,\\ i<k,\ i\leq j,\ k\leq l} \right\}.$$

Symmetrized principal minors of skew-symmetric matrices

Proposition

If A is skew-symmetric, A is conjugate to

$$\lambda \mathbb{1}_n^{\wedge}\,, \quad \text{or} \quad \lambda \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix} \text{ (for } n=4 \text{ only)}, \qquad \text{for} \quad \lambda \in \mathbb{C},$$

where $\mathbb{1}_n^{\wedge}$ denotes the $n \times n$ skew-symmetric matrix with 1's above the diagonal. We have the following parameterizations:

$$d_k(\mathbb{1}_n^{\wedge}) = 1 \quad \text{for} \;\; k \geq 2,$$

$$c_k(\mathbb{1}_n^{\wedge}) = (-1)^{s/2} E_{k-1}, \qquad \text{where } E_k \text{ is the Euler number.}$$

Note: if A is skew-symmetric and has symmetrized cycle-sums $c_{2k+1} = d_{2k+1} = 0$.

Theorem (Huang-Oeding)

Suppose $n \geq 3$ and let $\mathcal{J}_n^{\wedge}$ denote the ideal of relations among the symmetrized cycle-sums of even sized cycles for a generic skew-symmetric matrix $A \in \bigwedge^2 \mathbb{C}^n$. $\mathcal{J}_4^{\wedge}$ decomposes as the intersection of two prime components

$$\mathcal{J}_4^{\wedge} \;=\; \langle -2c_2^2 + c_4 \rangle \quad \cap \quad \langle -6c_2^2 - c_4 \rangle.$$

$\mathcal{J}_5^{\wedge}$ has primary decomposition with two minimal primes:

$$\langle -2c_2^2 + c_4 \rangle \quad \text{and} \quad \langle c_2, c_4 \rangle$$

When $n \geq 5$ we have either $d_s = 0$ for all s, or $d_{2k} = 1$ and $d_{2k+1} = 0$ for all $k \leq n/2$. The cycle-sum relations can be deduced from this.

The proof of the first cases is by direct computation in `Macaulay2`. The general case is proved by induction using Schur complements.

Theorem (Huang-Oeding)

If A is general, then

- *If $n \geq 3$, and $c_1 = c_2 = 0$, then one of the following holds*
  1. *A is conjugate to a strictly upper triangular matrix, where*
  $$c_1 = c_2 = \cdots = c_n = 0.$$
  2. *A is conjugate to an matrix representing an n-cycle and*
  $$c_1 = c_2 = \cdots = c_{n-1} = 0, \qquad c_n \neq 0.$$
- *If $c_2 \neq 0$ and $c_1 = c_3 = 0$, then A is conjugate to a skew-symmetric matrix with symmetrized principal minors.*
- *if $c_1 = 0$, and $c_2 c_3 \neq 0$, then ...*

Theorem (Huang-Oeding (Continued))

- *if* $c_1 = 0$*, and* $c_2 c_3 \neq 0$*, then* A *is conjugate to* $\lambda T_n(x)$*, where* $T_n(x)$ *is the following Toeplitz matrix for* $x \in \mathbb{C}^*$*:*

$$T_n(x) := \begin{pmatrix} 0 & 1 & x & x^2 & \cdots & x^{n-2} \\ -1 & 0 & 1 & x & \cdots & x^{n-3} \\ -\frac{1}{x} & -1 & 0 & 1 & \cdots & x^{n-4} \\ -\frac{1}{x^2} & -\frac{1}{x} & -1 & 0 & \cdots & x^{n-5} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{x^{n-2}} & -\frac{1}{x^{n-3}} & -\frac{1}{x^{n-4}} & -\frac{1}{x^{n-5}} & \cdots & 0 \end{pmatrix},$$

where the (i,j) *entry of* $T_n(x)$ *is exactly* $\operatorname{sgn}(j-i) \cdot x^{j-i-\operatorname{sgn}(j-i)}$*. Moreover* $\lambda^2 = -c_2$ *and* $\lambda^3(x - \frac{1}{x}) = c_3$*, and*

$$c_s(T_n(x)) = x^{-s} E_{s-1}(-x^2),$$

where $E_n(x)$ *is the* n*-th Eulerian polynomial.*

$$d_s(T_n(x)) = \frac{(x^2)^{s-1} + (-1)^s}{x^{s-2}(x^2+1)}, \quad \text{or} \quad (x^2+1) d_s(x \cdot T_n(x)) = x^{2s} + (-1)^s x^2.$$

Theorem (Huang-Oeding)

Let $n \geq 3$ and suppose $A \in \mathbb{C}^{n\times n}$ has symmetrized cycle-sums. Let $\mathcal{J}_n$ denote the ideal of relations among the symmetrized cycle-sums of A.
$\mathcal{J}_3$ is empty.
$\mathcal{J}_3$ is empty. $\mathcal{J}_4$ decomposes as the intersection of two prime components:

$$\langle 2c_2^3 + c_3^2 - c_2c_4\rangle \qquad \text{and} \qquad \langle c_3, 6c_2^2 + c_4\rangle.$$

When $n \geq 5$, $\mathcal{J}_n$ has two components: one with radical $\langle c_2, \dots, c_n\rangle$ (with complicated scheme structure), and the ideal generated by the maximal minors of

$$\begin{pmatrix} d_0 & d_1 & d_2 & \dots & d_{n-2} \\ d_1 & d_2 & d_3 & \dots & d_{n-1} \\ d_2 & d_3 & d_4 & \dots & d_n \end{pmatrix}.$$

Computational Experiments

Here are the results of our tests for $S^2\mathbb{C}^n$.

n	I_1	I_2	$\sqrt{I_2}$	time
3	1 cubic	n/a	n/a	< 0.1 sec.
4	2 quadrics	1 linear, 1 quadric, 2 cubics	$\langle c_2, c_3, c_4 \rangle$	< 0.1 sec.
5	5 quadrics	1 linear, 3 quadrics, 3 cubics	$\langle c_2, c_3, c_4, c_5 \rangle$	< 0.2 sec.
6	9 quadrics	1 linear, 6 quadrics, 4 cubics	$\langle c_2, \dots, c_6 \rangle$	0.6 sec.
7	14 quadrics	1 linear, 10 quadrics, 5 cubics	$\langle c_2, \dots, c_7 \rangle$	13 sec.
8	20 quadrics	1 linear, 15 quadrics, 6 cubics	$\langle c_2, \dots, , c_8 \rangle$	8762 sec.

Computations done on a Server: 24 1.6GHz processors (not all are used at all times in M2) and 141GB of RAM.

Computational Experiments

Here are the results of our tests for $\mathbb{C}^{n\times n}$.

n	I_1	I_2	$\sqrt{I_2}$	time
3	n/a	n/a	n/a	< 0.1 sec.
4	1 cubic	1 linear, 2 quadric	1 linear, 2 quadric	< 1 sec.
5	4 cubics	1 linear, 2 quadrics, 2 cubics, 1 quartic, 1 quintic	$\langle c_2, c_3, c_4, c_5 \rangle$	5000 sec.