Principal Minors and Geometry

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An example: Spectral graph theory

Let Γ be a graph with

- vertex set $Q_0 = \{v_1, \ldots, v_n\}$
- edge set $Q_1 = \{e_{i,j} \mid \overline{v_i v_j} \in \Gamma\}$.

The graph Laplacian of an undirected graph is a (symmetric) matrix

$$\Delta(\Gamma)_{i,j} = \begin{cases} -1 & \text{if } i \neq j \text{ and } e_{i,j} \in Q_1 \\ 0 & \text{if } i \neq j \text{ and } e_{i,j} \notin Q_1 \\ deg(v_i) & \text{if } i = j \end{cases}$$

The principal minors of $\Delta(\Gamma)$ are invariants of the graph, in fact:

Theorem (Kirchoff's Matrix-Tree theorem (∼1850's))

Any $(n-1) \times (n-1)$ principal minor of $\Delta(\Gamma)$ counts the number of spanning trees of Γ.

An example: Spectral graph theory

There are many generalizations of the Matrix-Tree Theorem, such as

Theorem (Matrix-Forest Theorem)

Let $\Delta(\Gamma)_S^S$ be the principal minor of $\Delta(\Gamma)$ indexed by S. Then $\Delta(\Gamma)_S^S =$ number of spanning forests of Γ rooted at vertices indexed by S.

The $\Delta(\Gamma)_S^S$ are graph invariants. The relations among principal minors are then also relations among these graph invariants.

Question

When does there exist a graph Γ with invariants $[v] \in \mathbb{P}^{2^n-1}$ specified by the principal minors of a symmetric matrix $\Delta(\Gamma)$?

Further Questions

- Holtz and Schneider, D. Wagner, ... : When is it possible to prescribe the principal minors of a symmetric matrix?
- Equivalently, when can you prescribe all the eigenvalues of a symmetric matrix and all of its principal submatrices?
- Algebraic reformulation: What is the defining ideal of the algebraic variety of principal minors of symmetric matrices?
- For $n \geq 3$ this is an overdetermined problem : $\binom{n+1}{2}$ versus 2^n.

Examples: 2×2 case

Define a (homogeneous) map:
φ : symmetric matrices $\to$ principal minors:

$$\varphi\left(\begin{pmatrix} a & c \\ c & b \end{pmatrix}, t\right) = [t^2, ta, tb, ab - c^2]$$

When can we go backwards? Given $[w, x, y, z]$ is there a 2×2 matrix that has these principal minors? Need to solve: (WLOG assume $t = w = 1$)

$$\begin{aligned} x &= a \\ y &= b \\ z &= ab - c^2 \quad \Rightarrow \quad c = \pm\sqrt{xy - z} \end{aligned}$$

$$\varphi\left(\begin{pmatrix} x & \pm\sqrt{xy-z} \\ \pm\sqrt{xy-z} & y \end{pmatrix}, 1\right) = [1, x, y, z]$$

Conclude: Even in the $n \times n$ case, the 0×0, 1×1, and 2×2 minors *determine* a symmetric matrix up to the signs of the off-diagonal terms.

Principal Minors: 3×3 symmetric matrices

$$\varphi\left(\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{pmatrix}, t\right) = \begin{matrix} t^3 \\ t^2 x_{11} \\ t^2 x_{22} \\ t(x_{11}x_{22} - x_{12}^2) \\ t^2 x_{33} \\ t(x_{11}x_{33} - x_{13}^2) \\ t(x_{22}x_{33} - x_{23}^2) \\ (x_{11}x_{22}x_{33} + 2x_{12}x_{13}x_{23} - x_{11}x_{23}^2 \end{matrix}$$

Given $[X_{000}, X_{100}, X_{010}, X_{110}, X_{001}, X_{101}, X_{011}, X_{111}]$ is there a matrix that maps to it?

Count parameters: 7 versus 8 - there must be some relation that holds!

First result

Theorem (Holtz-Sturmfels '07)

All relations among the principal minors of a 3×3 matrix are generated by ... this beautiful degree 4 homogeneous polynomial:

$$\begin{aligned}(X_{000})^2(X_{111})^2 + (X_{100})^2(X_{011})^2 + (X_{010})^2(X_{101})^2 + (X_{110})^2(X_{001})^2 \\ + 4X_{000}X_{110}X_{101}X_{011} + 4X_{100}X_{010}X_{001}X_{111} \\ - 2X_{000}X_{100}X_{011}X_{111} - 2X_{100}X_{010}X_{011}X_{101} \\ - 2X_{000}X_{010}X_{101}X_{111} - 2X_{100}X_{001}X_{110}X_{011} \\ - 2X_{000}X_{001}X_{110}X_{111} - 2X_{001}X_{010}X_{101}X_{110}\end{aligned}$$

– Cayley's hyperdeterminant of format $2 \times 2 \times 2$.

It is invariant under the action of $\mathfrak{S}_3 \ltimes SL(2) \times SL(2) \times SL(2)$!

The Variety of Principal Minors of Symmetric Matrices

- The variety of principal minors of $n \times n$ symmetric matrices, Z_n, is defined by the principal minor map

$$\varphi : \mathbb{P}(S^2\mathbb{C}^n \oplus \mathbb{C}) \dashrightarrow \mathbb{P}(\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2) = \mathbb{P}\mathbb{C}^{2^n}$$

$$\begin{aligned}[A,t] \mapsto [&t^n, t^{n-1}\Delta_{10\dots0}(A), t^{n-1}\Delta_{010\dots0}(A), t^{n-2}\Delta_{110\dots0}(A),\\ &t^{n-1}\Delta_{0010\dots0}(A), t^{n-2}\Delta_{1010\dots0}(A), t^{n-2}\Delta_{0110\dots0}(A),\\ &\qquad\qquad\qquad\qquad \dots \quad \dots, \Delta_{1\dots1}(A)]\end{aligned}$$

  where $\Delta_I(A)$ is the principal minor of A with rows indicated by I.

- Q: Given a vector v of length 2^n, how can you tell whether or not it arose in this way?
- A: Test whether v satisfies all the relations in $\mathcal{I}(Z_n)$.

Hidden Symmetry

Theorem (Landsberg, Holtz-Sturmfels)

Z_n is invariant under the action of $G = \mathfrak{S}_n \ltimes SL(2)^{\times n}$.

- *Fact:* A variety $X \subset \mathbb{P}^N$ is a G-variety $\Leftrightarrow$ the ideal $\mathcal{I}(X)$ is a G-module.
- Z_n is a subvariety of $\mathbb{P}(V_1 \otimes \cdots \otimes V_n)$, where each $V_i \simeq \mathbb{C}^2$.
- KEY POINT: We must study $\mathcal{I}(Z_n) \subset \mathrm{Sym}(V_1^* \otimes \cdots \otimes V_n^*)$ as a G-module!
- Mantra: "Each irreducible module is either in or out!"

Slight Detour: A Geometric Proof of Symmetry

- For non-degenerate $\omega \in \bigwedge^2 \mathbb{C}^n$, the Lagrangian Grassmannian is $Gr_\omega(n,2n) = \{E \in Gr(n,2n) \mid \omega(v,w) = 0\ \forall v,w \in E\}$.
- $Gr_\omega(n,2n)$ is a homogeneous variety for $Sp(2n)$.
- $Gr_\omega(n,2n)$ is the image of the rational map:

$$\begin{array}{ccc} \psi : \mathbb{P}(S^2\mathbb{C}^n \oplus \mathbb{C}) & \dashrightarrow & \mathbb{P}\Gamma_n \simeq \mathbb{P}^{\binom{2n}{n} - \binom{2n}{n-2} - 1} \\ \{symmetric\ matrix\} & \mapsto & \{vector\ of\ all\ nonredundant\ minors\} \end{array}$$

- The connection: Z_n is a linear projection of $Gr_\omega(n,2n)$.
- Can use this projection to *find* symmetries of Z_n as a subgroup of $Sp(2n)$.
- Try to find projections of homogeneous varieties to study other G-varieties (later in the talk).

Multilinear Algebra

- $S^d(V_1^* \otimes \cdots \otimes V_n^*)$ = homogeneous degree d polynomials on 2^n variables. It is a module for $G = SL(V_1) \times \cdots \times SL(V_n)$.
- If we choose a basis $\{x_i^0, x_i^1\}$ of $V_i^* \simeq \mathbb{C}^2$ for each i, then $V_1^* \otimes \cdots \otimes V_n^*$ has the induced basis $x_1^{\epsilon_1} \otimes \cdots \otimes x_n^{\epsilon_n} =: X_I$.
- Then G acts on $V_1^* \otimes \cdots \otimes V_n^*$ by change of basis in each factor: If $g = (g_1, \ldots, g_n) \in G$, then

$$g.X_I = (g_1.x_1^{\epsilon_1}) \otimes \cdots \otimes (g_n.x_n^{\epsilon_n}),$$

  and acts on $S^d(V_1^* \otimes \cdots \otimes V_n^*)$ by the induced action:

$$g.(X_I X_J \ldots X_K) = (g.X_I)(g.X_J) \ldots (g.X_K)$$

- We have defined the action on a basis of each module, so we can just extend by linearity to get the action on the whole module.

Representation Theory

- Want to study $\mathcal{I}_d(Z_n) \subset S^d(V_1^* \otimes \cdots \otimes V_n^*)$.
- Each irreducible $\mathfrak{S}_n \ltimes SL(2)^{\times n}$-module in $S^d(V_1^* \otimes \cdots \otimes V_n^*)$ is isomorphic to one indexed by partitions π_i of d of the form :

$$S_{\pi_1} S_{\pi_2} \dots S_{\pi_n} := \bigoplus_{\sigma \in \mathfrak{S}_n} S_{\pi_{\sigma(1)}} V_1^* \otimes S_{\pi_{\sigma(2)}} V_2^* \otimes \cdots \otimes S_{\pi_{\sigma(n)}} V_n^*$$

- Can use the combinatorial information $\pi_1, \dots, \pi_n$ to construct the module.
- If M is an irreducible G-module, then $M = \{G.v\}$, some vector v - use this as often as possible.
- This gives a finite list of vectors to test for ideal membership!
- Also gives a way to produce many polynomials in $\mathcal{I}(Z_n)$ from one polynomial.

An Example

The module $S_{(2,2)}V \subset V^{\otimes 4}$ is one dimensional, and every vector is a scalar multiple of

$$h = 2X_{0011} - X_{1001} - X_{1010} - X_{0101} - X_{0110} + 2X_{1100}$$

To find a polynomial in $S_{(2,2)}V_1 \otimes S_{(2,2)}V_2 \otimes S_{(2,2)}V_3$, we need to compute $h \otimes h \otimes h$ in $V_1^{\otimes 4} \otimes V_2^{\otimes 4} \otimes V_3^{\otimes 4}$, but we want a polynomial in $S^4(V_1 \otimes V_2 \otimes V_3)$, so we just permute

$$V_1^{\otimes 4} \otimes V_2^{\otimes 4} \otimes V_3^{\otimes 4} \to (V_1 \otimes V_2 \otimes V_3)^{\otimes 4}$$

and symmetrize

$$(V_1 \otimes V_2 \otimes V_3)^{\otimes 4} \to S^4(V_1 \otimes V_2 \otimes V_3)$$

An Example

Finally, we get the result

$$\begin{aligned}(X_{000})^2(X_{111})^2 + (X_{100})^2(X_{011})^2 + (X_{010})^2(X_{101})^2 + (X_{110})^2(X_{001})^2 \\ + 4X_{000}X_{110}X_{101}X_{011} + 4X_{100}X_{010}X_{001}X_{111} \\ - 2X_{000}X_{100}X_{011}X_{111} - 2X_{100}X_{010}X_{011}X_{101} \\ - 2X_{000}X_{010}X_{101}X_{111} - 2X_{100}X_{001}X_{110}X_{011} \\ - 2X_{000}X_{001}X_{110}X_{111} - 2X_{001}X_{010}X_{101}X_{110}\end{aligned}$$

In fact, this is Cayley's hyperdeterminant of format $2 \times 2 \times 2$!
It's an irreducible degree 4 polynomial on 8 variables.
It is invariant under the action of $\mathfrak{S}_3 \ltimes SL(2) \times SL(2) \times SL(2)$.
It generates the module $S_{(2,2)}S_{(2,2)}S_{(2,2)}$.
It is the single equation defining the hypersurface Z_3.

Rephrasing of Previous Results

Theorem (Holtz-Sturmfels)

$\mathcal{I}(Z_3)$ is generated in degree 4 by $S_{(2,2)}S_{(2,2)}S_{(2,2)}$ (Cayley's Hyperdeterminant of format $2 \times 2 \times 2$).

Theorem (H-S)

$\mathcal{I}(Z_4)$ is generated in degree 4 by $S_{(4)}S_{(2,2)}S_{(2,2)}S_{(2,2)}$ (A hyperdeterminantal module).

Remark: $S_{(4)}S_{(2,2)}S_{(2,2)}S_{(2,2)}$ is the span of the G-orbit of the $2 \times 2 \times 2$ hyperdeterminant on the variables X_{***0}.

Conjecture (H-S)

$\mathcal{I}(Z_n)$ is generated in degree 4 by $S_{(4)} \dots S_{(4)}S_{(2,2)}S_{(2,2)}S_{(2,2)}$ (the hyperdeterminantal module).

A Limit of the Computer's Usefulness

- For $n = 3$: A single irreducible degree 4 polynomial on 8 variables cuts out the irreducible hypersurface in $\mathbb{P}^7$.
- For $n = 4$: 20 degree 4 polynomials on 16 variables.
  Macaulay2 $\Rightarrow$ the ideal is prime and has the correct dimension.
  But Z_4 is an irreducible variety + commutative algebra $\Rightarrow \square$.
- For $n = 5$: 250 degree 4 polynomials on 32 variables.
  Sadly, the computer melted.
- For $n = 6$: 2500 degree 4 polynomials on 64 variables. ☹
- For $n = n$: $\binom{n}{3}5^{n-3}$ degree 4 polynomials on 2^n variables. ☹ ☹
  What can we say in general without the computer?

New Results

Theorem (-)

Let $HD := \{\mathfrak{S}_n \ltimes SL(2)^{\times n}.hyp_{123}\} = S_{(4)} \dots S_{(4)} S_{(2,2)} S_{(2,2)} S_{(2,2)}$. *The variety* Z_n *is cut out set-theoretically by the hyperdeterminantal module.*

$$\mathcal{V}(HD) = Z_n.$$

- To prove that $Z_n \subset \mathcal{V}(HD)$, show that *hyp* (a highest weight vector for the irreducible module HD) vanishes on every point of Z_n. This follows from the 3×3 case.
- To prove that $Z_n \supset \mathcal{V}(HD)$, need a geometric understanding of zero-sets of modules with similar properties to HD.

Outline of proof of main theorem

- Want to show $\mathcal{V}(HD) \subset Z_n$ - do induction on n. For $z \in \mathcal{V}(HD)$, attempt to construct a matrix $A \in S^2\mathbb{C}^n$ so that $A \mapsto z \in \mathcal{V}(HD)$.

- Have already seen: the 0×0, 1×1 and 2×2 principal minors of a symmetric matrix determine the matrix up to the signs of the off-diagonal terms.

- For $n \geq 4$ can show that if two symmetric matrices have the same $0 \times 0 \dots 3 \times 3$ principal minors, then 4×4 principal minors agree also. Then iterate.

- We show that points in $\mathcal{V}(HD)$ have essentially the same property: *i.e.* if $z, w \in \mathcal{V}(HD)$ and $z_I = w_I$ for all $I \neq [1, \dots, 1]$ then $z = w$.

- Main Tool: a geometric characterization of augmented modules.

Characterizing the zero set of $\mathcal{V}(HD)$ via augmentation

- Notice that for case n, $HD = \underbrace{S_{(4)} \dots S_{(4)}}_{n-3} S_{(2,2)} S_{(2,2)} S_{(2,2)}$ and for case $n+1$, $HD = \underbrace{S_{(4)} \dots S_{(4)}}_{n-2} S_{(2,2)} S_{(2,2)} S_{(2,2)}$ is still degree 4.
- What can we say about zero set of an augmented ideal $\mathcal{V}(\mathcal{I}_d(X) \otimes S^d V^*)$ based on $\mathcal{V}(\mathcal{I}_d(X))$?

Lemma (inspired by Landsberg-Manivel lemma on prolongation)

Let $X \subset \mathbb{P}W$ and let $\tilde{X} = \mathcal{V}(\mathcal{I}_d(X))$ (notation).

$$\mathcal{V}(\mathcal{I}_d(X) \otimes S^d V^*) = \bigcup_{L \subset \tilde{X}} \mathbb{P}(L \otimes V),$$

where $L \subset \tilde{X}$ are linear subspaces.

What does this buy us?

Consequence

Assume that $HD = \bigoplus_i HD_{\hat{i}} \otimes S^4 V_i \subset S^4(V_1 \otimes \cdots \otimes V_n)$ *and* $V_i \simeq \mathbb{C}^2$, *then*

$$\mathcal{V}(HD) = \cap_{i=1}^n \left(\bigcup_{L \subset V(HD_{\hat{i}})} \mathbb{P}(L \otimes V_i) \right).$$

- Suppose $z \in \mathcal{V}(HD) = \mathcal{V}(\bigoplus_i HD_{\hat{i}} \otimes S^4 V_i)$, and assume for induction that $\mathcal{V}(HD_{\hat{i}}) \simeq Z_{n-1}$.
- Then our geometric realization gives n different expressions for z,

$$z = \varphi([A^{(i)}, t^{(i)}]) \otimes x_i^0 + \varphi([B^{(i)}, s^{(i)}]) \otimes x_i^1,$$

  where $A^{(i)}, B^{(i)} \in S^2\mathbb{C}^{n-1}$ and $\{x_i^0, x_i^1\} = V_i$.
- We can use this information (+ technical details) to build an $n \times n$ matrix A so that $\varphi([A, t]) = z$, and this proves the theorem.

The tangential variety to the Segre product

- The Segre Variety, i.e the variety of rank one tensors is $Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n) = \{v_1 \otimes \cdots \otimes v_n] \mid v_i \in V_i\} \subset \mathbb{P}(V_1 \otimes \cdots \otimes V_n)$.

- If $X \subset \mathbb{P}^N$ is a smooth variety, define the tangential variety $\tau(X) \subset \mathbb{P}(V)$ by
$$\tau(X) := \cup_{x \in X} \tilde{T}_x X$$

- $\tau(Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)) = \{[\sum_{i=1}^n v_1 \otimes \cdots \otimes v_i' \otimes \cdots \otimes v_n] \mid v_i, v_i' \in V_i\}$.

- $\tau(Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1))$ is a $(SL(2)^{\times n}) \ltimes \mathfrak{S}_n$-variety.
dim $= 2n << \binom{n+1}{2} \Rightarrow$ too small to be equal to Z_n for $n \geq 4$.

- $\tau(Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1)) \subset Z_n$ for $n \geq 3$, with equality for $n = 3$.

Exclusive rank

The standard notion of rank is destroyed by the $SL(2)^{\times n}$ action.

For a matrix A, the minor $\Delta^I_J(A)$ is said to be *exclusive* if $I \cap J = \emptyset$, *i.e.* the minor has no coincidental row and column indices.

The matrix A has *exclusive-rank (E-rank)* $\leq k$ if all of its $k+1 \times k+1$ exclusive minors vanish. (Laplace expansion implies uniqueness.)

Proposition

The variety of principal minors of symmetric matrices with E-rank $\leq k$ is $(SL(2)^{\times n}) \ltimes \mathfrak{S}_n$-invariant.

Idea of proof: Can use the projection of the Lagrangian Grassmannian just like the case of Z_n. Find that each exclusive minor is fixed by the action of $SL(2)^{\times n}$ when viewed as a subgroup of $SP(2n)$ acting on the space of all minors. This symmetry "survives" the projection to Z_n.

Principal minors of low E-rank matrices

Proposition

The image of the matrices with E-rank-0 under φ is

$$Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1).$$

The image of the symmetric matrices with E-rank ≤ 1 under φ is

$$\tau(Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1)).$$

Rough idea of proof: It is easy to show that a vector of principal minors of an E-rank-1 matrix is a point on the tangential variety.
To go the other way, we show that every point on the tangential variety is in the $SL(2)^{\times n}$-orbit of the set of principal minors of rank-1 symmetric matrices (usual rank).
The set of principal minors of E-rank ≤ 1 symmetric matrices is an irreducible $SL(2)^{\times n}$-invariant variety of the same dimension $\Rightarrow \square$.

The Landsberg-Weyman Conjecture

Let V_i be complex vector spaces and let V_i^* be their dual spaces.

Conjecture (Conjecture 7.6. Landsberg-Weyman)

$I(\tau(Seg(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*)))$ is generated by the quadrics in $S^2(V_1 \otimes \cdots \otimes V_n)$ which have at least four $\bigwedge^2$ factors, the cubics with four $S_{2,1}$ factors and all other factors $S_{3,0}$, and the quartics with three $S_{2,2}$'s and all other factors $S_{4,0}$.

Theorem (-)

$\tau(Seg(\mathbb{P}V_1^* \times \cdots \times \mathbb{P}V_n^*))$ is cut out set-theoretically by the cubics in $S^3(V_1 \otimes \cdots \otimes V_n)$ with four $S_{2,1}$ factors and all other factors $S_{3,0}$, and the quartics in $S^4(V_1 \otimes \cdots \otimes V_n)$ with three $S_{2,2}$'s and all other factors $S_{4,0}$.

Proof of the Landsberg-Weyman Conjecture

- A standard argument: Because all of the modules of polynomials occurring have partitions with no more than 2 parts, it suffices to prove the case with all $\mathbb{P}^1$'s.
- The degree four equations are actually the hyperdeterminantal module HD! So by using the result $\mathcal{V}(HD) = Z_n$, we can proceed by showing that $\tau(Seg(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1))$ is precisely the subvariety of Z_n cut out by the cubics in the ideal: $S_{2,1}S_{2,1}S_{2,1}S_{2,1}S_3 \dots S_3$.
- We directly computed the cubics and pulled them back to the space of symmetric matrices via the principal minor map.
- The result was the set of 2×2 exclusive minors! But we just showed that the image of the E-rank-1 symmetric matrices under the principal minor map is the tangential variety.

Thanks!