Applications of Secant Varieties and their Equations

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Applications of Secant Varieties (Highlights)

Secant varieties are classical objects in Algebraic Geometry, but have shown up naturally in applications such as:

- Algebraic Complexity Theory:
  Bounding the complexity of algorithms via membership (or non-membership) in a given secant variety.
- Signal Processing:
  Determining the decomposition of tensors into sums of simpler tensors is important for blind identification of under-determined mixtures, a broadly used concept in applications.
- Algebraic Statistics
  Finding invariants of statistical models for evolution (phylogenetics). The salmon prize and a ubiquitous example for this talk.

Complexity of Matrix Multiplication

Let $A = \mathbb{C}^{n^2}$, $B = \mathbb{C}^{n^2}$, and $C = \mathbb{C}^{n^2}$ be spaces of matrices.
Then we can express matrix multiplication as a bilinear map

$$\phi : A \times B \to C \quad \text{or equivalently as a tensor} \quad \phi \in A \otimes B \otimes C^*$$

Have $\phi = \sum_{k=1}^{r} a_k \otimes b_k \otimes \gamma_k$, where $r = \#$ of multiplications in this expression of ϕ. The minimum such r is the *rank* of ϕ.
The min r so that $\phi \in S^r\big(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C^*)\big)$ is the *border rank* of ϕ.

Theorem (Lickteig)

For all $n \neq 3$,

$$\dim S^{r-1}(Seg(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1})) = \min\{r(3n-2)-2, n^3-1\}$$

Theorem (Strassen)

$S^3(Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2))$ *is a degree 9 hypersurface.*

Complexity of Matrix Multiplication

Theorem (Lickteig)

The border rank of $n \times n$ matrix multiplication is no smaller than $\frac{3n^2}{2} + \frac{n}{2} - 1$.

Theorem (Bläser)

The rank of $n \times n$ matrix multiplication is no smaller than $\frac{5n^2}{2} - 3n$.

For $n = 2$ Lansberg showed that the border rank is 7. For $n = 3$, the rank is between 19 and 23, whereas the border rank is between 14 and 21.

Open question: Find better bounds on the rank and border rank of ϕ.

One approach is to study secant varieties and related auxiliary varieties to gain information about ϕ.

Symmetric Tensor Decomposition

Let V be a complex vector space. $Sym^d(V)$ = symmetric tensors. Rank-1 symmetric tensors = Veronese:

$$v_d : \mathbb{P}V \to \mathbb{P}(Sym^d(V))$$

$$[v] \mapsto [v \otimes v \otimes \cdots \otimes v]$$

The *symmetric rank* of a tensor $A \in Sym^d(V)$ is the minimum r such that $A = \sum_{k=1}^r v_k \otimes \cdots \otimes v_k$, with $v_k \in V$.

However, the *rank* of A is the minimum r such that $A = \sum_{k=1}^r v_1^k \otimes \cdots \otimes v_d^k$, where the v_i^k can all be different vectors in V.

Open Question: (P. Comon) Is the symmetric rank of a symmetric tensor equal to its rank?

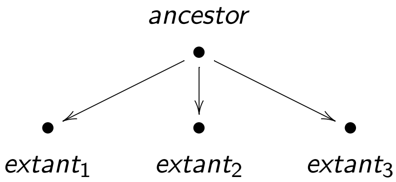
Geometric version for border rank:
$\mathbb{P}(Sym^d V) \cap S^k(Seg(\mathbb{P}V \times \ldots \mathbb{P}V)) =^? S^k(\mathbb{P}(Sym^d V) \cap Seg(\mathbb{P}V \times \ldots \mathbb{P}V))$

The salmon Prize

In 2007, E. Allman offered a prize of Alaskan salmon (!) to whomever finds the defining ideal of

$$S^3\left(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3\right).$$

This algebraic variety may be viewed as a statistical model for evolution.



Possible observed values for DNA: $\{A, C, G, T\}$. Assume observations at extant species are independent. Ancestor unknown. Mixture model of 4 independence models.

Secant varieties as statistical models

Let $A \otimes B \otimes C$ denote the tensor product of $\mathbb{C}$-vector spaces A, B, C.

- Segre variety (rank 1 tensors): (Independence model) Defined by

$$\begin{array}{rcl} Seg : \mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C & \longrightarrow & \mathbb{P}\big(A \otimes B \otimes C\big) \\ ([a], [b], [c]) & \longmapsto & [a \otimes b \otimes c]. \end{array}$$

- The *r-secant variety* of a variety $X \subset \mathbb{P}^n$: (Mixture model)

$$S^r\big(Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)\big)$$

(*Can also work over $\mathbb{R}$ or Δ-probability simplex, but not today.)

Motivation

Invariants of this statistical model $\leftrightarrow$ ideal of the algebraic variety.

The main motivation: Work of Allman-Rhodes implies that solving this problem would provide all invariants for the mixture model of *any* binary evolutionary tree with *any* number of states!

Plan: Use Geometry and Representation theory to find equations of secant varieties via this somewhat *fishy* example.

Symmetry of the salmon variety

- The symmetry group of the salmon variety

$$S^3\left(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C\right)$$

  is change of coordinates in each factor, *i.e.* $GL(A) \times GL(B) \times GL(C)$

  (or $GL(A) \times GL(B) \times GL(C) \rtimes \mathfrak{S}_3$ when $A \cong B \cong C$).

- Good news: A large group acts! Can use tools from representation theory!

- This symmetry is a powerful tool (kind of like a RADAR device for finding hidden fish) we *should* exploit it!

Ideals with Symmetry

Recall: projective varieties have homogeneous ideals. This symmetry induces grading by degree.

$$\begin{array}{ccc} \mathbb{C}[x_0,\ldots,x_N] & = & \bigoplus_d \mathbb{C}[x_0,\ldots,x_N]_d \\ \cup & & \cup \\ \mathcal{I}(X) & = & \bigoplus_d \mathcal{I}_d(X) \end{array}$$

When a larger group G acts on X (and on $\mathcal{I}(X)$), we get a finer decomposition of $\mathcal{I}(X)$ into G-modules using representation theory. E.g. can get a grading by multi-degree, and even more.

This is good because
Ideal Mantra: "Polynomials in G-modules are like musketeers - one for all and all for one!"

What is an irreducible module?

Think: A G-module is a vector space with a G-action

An *irreducible module* is one with no proper nontrivial submodules.
Example: The space of square matrices $V \otimes V$ is not an irreducible $GL(V)$ module since it splits as

$$V \otimes V = S^2 V \oplus \bigwedge^2 V$$

(You already knew this: every square matrix may be written as a sum of a symmetric and a skew symmetric matrix)

Representation Theory Notation

- Module notation: $S^d(A \otimes B \otimes C) = \mathbb{C}[p_{ijk}]_d$.
- Fact: $S^d(A \otimes B \otimes C)$ is a $GL(A) \times GL(B) \times GL(C)$-module.
- The irreducible submodules of $S^d(A \otimes B \otimes C)$ are isomorphic to Schur modules indexed by certain partitions π_1, π_2, π_3 of d:

$$S_{\pi_1} A \otimes S_{\pi_2} B \otimes S_{\pi_3} C,$$

  and usually occur with multiplicity - this makes us work harder.
- Given π_1, π_2, π_3 and the multiplicity, there is a combinatorial algorithm for constructing polynomials!

Using Representation Theory

- For the groups we encounter, irreducible G-modules M satisfy

$$\mathrm{span}\{G.f\} = M \text{ for } 0 \neq f \in M$$

- In practice, use a distinguished f called a "highest weight vector".
- Can test if an irreducible $M \subset \mathcal{I}(X)$ by testing if $f \in \mathcal{I}(X)$!
- If we have $f \in \mathcal{I}(X)$, can find entire modules in $\mathcal{I}(X)$!
- Fact: isotypic decomposition of $S^d(A \otimes B \otimes C)$:

$$S^d(A \otimes B \otimes C) = \bigoplus_{|\pi|=d} \left(S_{\pi_1}A \otimes S_{\pi_2}B \otimes S_{\pi_3}C\right)^{\oplus M_{\pi_1,\pi_2,\pi_3}},$$

where the multiplicity M_{π_1,π_2,π_3} can be computed via characters.

What is a flattening?

Think: $U \otimes V =$ space of matrices.
3 canonical ways to express a tensor $T = \sum_{i,j,k} p_{ijk} a_i \otimes b_j \otimes c_k \in A \otimes B \otimes C$

as a matrix:

$$T = \sum_i a_i \otimes (\sum_{j,k} p_{ijk} b_j \otimes c_k) \in A \otimes (B \otimes C) \text{ or}$$

$$T = \sum_j b_j \otimes (\sum_{i,k} p_{ijk} a_i \otimes c_k) \in B \otimes (A \otimes C) \text{ or}$$

$$T = \sum_k (\sum_{i,j} p_{ijk} a_i \otimes b_j) \otimes c_k \in (A \otimes B) \otimes C.$$

An example flattening of $T = [p_{ijk}] \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ to $\mathbb{C}^3 \otimes (\mathbb{C}^3 \otimes \mathbb{C}^3) \cong \mathbb{C}^3 \otimes \mathbb{C}^9$:

$$\left(\begin{array}{ccc|ccc|ccc} p_{111} & p_{121} & p_{131} & p_{112} & p_{122} & p_{132} & p_{113} & p_{123} & p_{133} \\ p_{211} & p_{221} & p_{231} & p_{212} & p_{222} & p_{232} & p_{213} & p_{223} & p_{233} \\ p_{311} & p_{321} & p_{331} & p_{312} & p_{322} & p_{332} & p_{313} & p_{323} & p_{333} \end{array}\right)$$

A familiar G-module

The $GL(A) \times GL(B) \times GL(C)$-module of 3×3 minors of the flattening $A \otimes B \otimes C \to A \otimes (B \otimes C)$ is

$$F := S_{(1,1,1)}A \otimes S_{(1,1,1)}(B \otimes C) = \bigwedge^3 A \otimes \bigwedge^3 (B \otimes C)$$

This module is not irreducible: $F = F_1 \oplus F_2 \oplus F_3 =$

$$\left(\bigwedge^3 A \otimes \bigwedge^3 B \otimes S^3 C\right) \oplus \left(\bigwedge^3 A \otimes S_{(2,1)} B \otimes S_{(2,1)} C\right) \oplus \left(\bigwedge^3 A \otimes S^3 B \otimes \bigwedge^3 C\right)$$

After choosing ordered (or weighted) bases of A, B, C, can define a highest weight. For example, the highest weight vector of $\bigwedge^3 A \otimes \bigwedge^3 B \otimes S^3 C$ is

$$(a_1 \wedge a_2 \wedge a_3) \otimes (b_1 \wedge b_2 \wedge b_3) \otimes (c_1^{\otimes 3}) = \det \begin{pmatrix} p_{111} & p_{121} & p_{131} \\ p_{211} & p_{221} & p_{231} \\ p_{311} & p_{321} & p_{331} \end{pmatrix}$$

Can do ideal membership test for each irreducible module by testing vanishing of its highest weight vector!

Symmetric Flattenings

Consider $\phi \in Sym^d V$ as a symmetric multilinear form: eats d vectors (symmetrically) and spits out a number.

If we only feed ϕ s vectors, it still wants to eat $d-s$ more. So we can construct a linear map

$$\begin{aligned} \phi_{s,d-s} : Sym^s(V^*) &\to Sym^{d-s} V \\ [v_1, \dots, v_s] &\mapsto \phi(v_1, \dots, v_s, \cdot, \dots, \cdot) \end{aligned}$$

Macaulay (1916) showed that the border rank of ϕ is at least as big as the rank of $\phi_{s,d-s}$ for all $1 \leq s \leq d$.
The minors of $\phi_{s,d-s}$ are called *minors of Catalecticant matrices* or *minors of symmetric flattenings*.
These give some equations for the secant varieties to Veronese varieties.

Flattenings Again

A flattening is the observation that

$$\begin{array}{ccc}
\mathbb{P}(A \otimes B \otimes C) & = & \mathbb{P}(A \otimes (B \otimes C)) \\
\cup & & \cup \\
Seg(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C) & \subset & Seg(\mathbb{P}A \times \mathbb{P}(B \otimes C))
\end{array}$$

A symmetric flattening is the observation that

$$\begin{array}{ccc}
\mathbb{P}(Sym^d V) & \subset & \mathbb{P}\left(Sym^s V \otimes Sym^{d-s} V\right) \\
\cup & & \cup \\
v_d(\mathbb{P}V) & \subset & Seg(v_s(\mathbb{P}V) \times v_{d-s}(\mathbb{P}V))
\end{array}$$

In both cases, minors of the matrices gotten by flattening give some equations for the secant varieties.

Highlights: Flattenings and subspace varieties

Definition: $$Sub_{p,q,r}(A \otimes B \otimes C) := \{[T] \in \mathbb{P}(A \otimes B \otimes C) \mid \exists \mathbb{C}^p \subseteq A, \mathbb{C}^q \subseteq B, \mathbb{C}^r \subseteq C, \text{ and } [T] \in \mathbb{P}(\mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^r)\}$$

i.e. Tensors that can be written using fewer variables.

Theorem (3.1, Landsberg–Weyman '07)

$Sub_{p,q,r}(A \otimes B \otimes C)$ is normal with rational singularities. Its ideal is generated by the minors of flattenings;

$$\left(\bigwedge^{p+1} A \otimes \bigwedge^{p+1}(B \otimes C)\right) \oplus \left(\bigwedge^{q+1} B \otimes \bigwedge^{q+1}(A \otimes C)\right) \oplus \left(\bigwedge^{r+1}(A \otimes B) \otimes \bigwedge^{r+1} C\right)$$

Fact: $Sub_{r,r,r}(A \otimes B \otimes C) \supseteq S^{r-1}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$

Key Point: The subspace varieties contain secant varieties, and therefore they give some of the equations of the secant varieties.

Flattenings and the Segre variety

Note, the ideal of $Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)$ is generated by all 2×2 minors of flattenings. Garcia, Stillman and Sturmfels conjectured that the ideal of the secant variety should be generated by the 3×3 minors of flattenings.

Theorem (Landsberg-Manivel)

$S(Seg(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n))$ is cut out (set theoretically for all n and ideal theoretically for $n = 3$) by the 3×3 minors of flattenings.

However, $S^3(Seg(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3))$ is an example of a secant variety which does not have any equations coming from flattenings since there are no 5×5 minors of 3×12 or 9×4 matrices.

Highlights: Inheritance via an example

Proposition (example of Proposition 4.4 Landsberg–Manivel , '04)

$$\tilde{M}_6 := S_{(2,2,2)}\mathbb{C}^4 \otimes S_{(2,2,2)}\mathbb{C}^4 \otimes S_{(3,1,1,1)}\mathbb{C}^4 \in \mathcal{I}\big(S^3(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)\big)$$

if and only if

$$M_6 := S_{(2,2,2)}\mathbb{C}^3 \otimes S_{(2,2,2)}\mathbb{C}^3 \otimes S_{(3,1,1,1)}\mathbb{C}^4 \in \mathcal{I}\big(S^3(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)\big).$$

Note: $\dim(\tilde{M}_6) = 10^3$ but $\dim(M_6) = 10$, and has basis of polynomials, each with 576 or 936 monomials.

The point: The number of parts of π_1, π_2, π_3 tell us which secant variety to look at. This is a significant dimension reduction.

For $S^3(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$ we only need to consider $S_{\pi_1}A \otimes S_{\pi_2}B \otimes S_{\pi_3}C$ where π_1, π_2, π_3 have 4 parts, and those equations we get from inheritance.

Highlights: Inheritance (full details)

Proposition (4.4 Landsberg–Manivel , '04)

If an irreducible module

$$S_{\pi_1}A \otimes S_{\pi_2}B \otimes S_{\pi_3}C \subset \mathcal{I}_d(S^{r-1}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$$

then for all vector spaces $A' \supseteq A, B' \supseteq B, C' \supseteq C$ *we have*

$$S_{\pi_1}A' \otimes S_{\pi_2}B' \otimes S_{\pi_3}C' \subset \mathcal{I}_d(S^{r-1}(\mathbb{P}A' \times \mathbb{P}B' \times \mathbb{P}C')).$$

Moreover, a module $S_{\pi_1}A' \otimes S_{\pi_2}B' \otimes S_{\pi_3}C'$ *where* $l(\pi_1) \leq a$, $l(\pi_2) \leq b$ $l(\pi_3) \leq c$ *is in* $\mathcal{I}_d(S^{r-1}(\mathbb{P}A' \times \mathbb{P}B' \times \mathbb{P}C'))$ *iff the corresponding module is in* $\mathcal{I}_d(S^{r-1}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$.

Landsberg and Manivel's Reduction

Theorem (Landsberg-Manivel '08 Corollary 5.6)

$S^3\left(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3\right)$ *is the zero set of:*

1. $$\begin{aligned} M_5 := &\left(S_{(3,1,1)}\mathbb{C}^4 \otimes S_{(2,1,1,1)}\mathbb{C}^4 \otimes S_{(2,1,1,1)}\mathbb{C}^4\right) \\ &\oplus\left(S_{(2,1,1,1)}\mathbb{C}^4 \otimes S_{(3,1,1)}\mathbb{C}^4 \otimes S_{(2,1,1,1)}\mathbb{C}^4\right) \\ &\oplus\left(S_{(2,1,1,1)}\mathbb{C}^4 \otimes S_{(2,1,1,1)}\mathbb{C}^4 \otimes S_{(3,1,1)}\mathbb{C}^4\right) \end{aligned}$$
2. *Equations inherited from* $S^3\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\right)$

- Key point: It remains to find the equations of $S^3(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$!
- In fact, $S^3(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$ is the *smallest* secant variety to a Segre product whose equations are unknown!
- Note: M_5 is a 1728 dimensional irreducible G-module, for $G = GL(4) \times GL(4) \times GL(4) \times \mathfrak{S}_3$ with a natural basis of polynomials with 180 or 360 or 540 monomials.

A result of Strassen

Theorem (Strassen 1988 (reinterpreted))

The ideal of the hypersurface $S^3(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^{26}$ is generated in degree 9 by a nonzero vector in the 1 dimensional module

$$S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^3$$

Since $S^3(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2) \subset S^3(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$, inheritance implies that $M_9 := S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^4 \subset \mathcal{I}(S^3(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3))$

Note: Strassen's polynomial *only* has $9,216$ monomials, and $\dim(M_9) = 20$, has natural basis of polynomials with $9,216$ or $25,488$ or $43,668$ monomials! It is a 56 Mb file of polynomials... :-(

What is known about $\mathcal{I}\left(S^3\left(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^3\right)\right)$?

- General theory: $\mathcal{I}_s(S^{k-1}(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C))=0$ for $s\leq k$.
- Computational tests: (Please download my Maple code and double check this work!)

$$\mathcal{I}_5\left(S^3\left(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^3\right)\right)=0$$

$$M_6:=\mathcal{I}_6\left(S^3\left(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^3\right)\right)=S_{(2,2,2)}\mathbb{C}^3\otimes S_{(2,2,2)}\mathbb{C}^3\otimes S_{(3,1,1,1)}\mathbb{C}^4 \qquad *$$

- Strassen:
  $M_9:=S_{(3,3,3)}\mathbb{C}^3\otimes S_{(3,3,3)}\mathbb{C}^3\otimes S_{(3,3,3)}\mathbb{C}^4\in\mathcal{I}(S^3(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^3))$

Do M_6 and M_9 suffice to cut out $S^3\left(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^3\right)$?

*(correction)

Status of the salmon conjecture

Known equations of $S^3\big(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\big)$:

$$M_6 = S_{(2,2,2)}\mathbb{C}^3 \otimes S_{(2,2,2)}\mathbb{C}^3 \otimes S_{(3,1,1,1)}\mathbb{C}^4$$
$$M_9 = S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^4$$

Shape of partitions implies that $\langle M_9 \rangle \not\subset \langle M_6 \rangle$.

It is known that $\mathcal{V}(M_6) = S^3\big(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\big) \cup Sub_{3,3,3} \cup X$, where X is the "left-over" part of the zero-set.

Also, we know that $x \in \mathcal{V}(M_9) \cap Sub_{3,3,3} \Rightarrow x$ is in some $S^3\big(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2\big) \subset S^3\big(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\big)$.

Does $\mathcal{V}(M_6 + M_9)$ eliminate the "left-overs" X? If so, this would resolve the salmon problem (at least set theoretically)... work in progress.

A template for finding equations of varieties coming from applications

The salmon variety has been studied via the following:

1. Input: statistical model, space of special tensors, etc.
2. Find the corresponding algebraic variety X.
3. Find the largest symmetry group G acting X.
4. Study $\mathcal{I}(X)$ as a G-module using representation theory.
5. Use computational tools to study modules potentially in $\mathcal{I}(X)$.
   (works well for low degree) - See me for Maple implementations.
   Note: Representation theory tells where to look for invariants as well as how to get new invariants from old.
6. Try to make geometric reductions to show that the known invariants suffice.

This template should be useful for studying other varieties coming from applications since they often have nice symmetry as we have observed in this example.