Homotopy Techniques for Tensor Decomposition and Perfect Identifiability

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Tensors, Rank, and Identifiability

- A tensor of format $(n_1, n_2, \ldots, n_d)$ is a hypermatrix $\mathcal{T} = (\mathcal{T}_{i_1,i_2,\ldots,i_d})$ (assume entries in $\mathbb{C}$), with $1 \leq i_j \leq n_j$ for all j.
- Since $\mathcal{T}$ has $\prod_{j=1}^{d} n_j$ entries, it represents a huge set of data.
- Basic question: Find a sparse representation of $\mathcal{T}$.
- A rank-one tensor (*a point on a Segre variety*) is a tensor $\mathcal{T}$ such that $\mathcal{T}_{i_1,\ldots,i_d} = v_{1,j_1} \cdot v_{2,j_2} \cdots v_{d,j_d}$ for some vectors $\vec{v}_j$ of lengths n_j.
- Rank-one tensors only have essentially $\sum_{j=1}^{d} n_j$ pieces of information.
- A rank-r tensor (*a general point on the r-th secant variety of the Segre variety*) is the sum of r rank-one tensors.
- A rank-r tensor only contains essentially $r \cdot \sum_{j=1}^{d} n_j$ pieces of information, which is potentially much smaller than the full dimension $\prod_{j=1}^{d} n_j$.
- So a low-rank representation of $\mathcal{T}$ is a *sparse* presentation.

Some Applications of Secant Varieties

- Classical Algebraic Geometry: When can a given projective variety $X \subset \mathbb{P}^n$ be isomorphically projected into $\mathbb{P}^{n-1}$?

  Determined by the dimension of the secant variety $\sigma_2(X)$.

- Algebraic Complexity Theory: Bound the border rank of algorithms via equations of secant varieties. Berkeley-Simons program Fall'14

- Algebraic Statistics and Phylogenetics:
  Given contingency tables for DNA of several species, determine the correct statistical model for their evolution.

  Find invariants (equations) of mixture models (secant varieties).

  For star trees / bifurcating trees this is the salmon conjecture.

- Signal Processing: Blind identification of under-determined mixtures, analogous to CDMA technology for cell phones.

  A given signal is the sum of many signals, one for each user.

  Decompose the signal uniquely to recover each user's signal.

- Computer Vision, Neuroscience, Quantum Information Theory, Chemistry...

First algebraic / geometric questions for tensors

Let $X \subset \mathbb{P}\mathbb{C}^N$, with $N = n_1 \times \cdots \times n_d$, denote the set of rank-one tensors, and let $\sigma_r(X)$ denote the Zariski closure of the set of rank-r tensors.

1. [Dimensions] What is the dimension of $\sigma_r(X)$?
   – When does $\sigma_r(X)$ fill the ambient $\mathbb{P}\mathbb{C}^N$? (defectivity)
2. [Equations] What are the polynomial defining equations of $\sigma_r(X)$?
3. [Decomposition] For my favorite $\mathcal{T} \in \mathbb{C}^N$, can you find an expression of $\mathcal{T}$ as a sum of points from X?
4. [Specific Identifiability] For a given $\mathcal{T} \in \mathbb{C}^N$, does $\mathcal{T}$ have a unique decomposition (ignoring trivialities)?
5. [Generic Identifiability] For *generic* $\mathcal{T} \in \mathbb{C}^N$, does $\mathcal{T}$ have a unique decomposition (ignoring trivialities)?

Today: Focus on Generic Identifiability

Geometric version of identifiability

Let $X \subset \mathbb{C}^N$, with $N = n_1 \times \cdots \times n_d$, denote the set of rank-one tensors, and let $\sigma_r(X)$ denote the Zariski closure of the set of rank-r tensors.
Construct the *incidence variety*

$$\mathcal{I} := \{([\mathcal{T}], [\mathcal{T}^1] \ldots, [\mathcal{T}^r]) \mid [\mathcal{T}^i] \in X, \mathcal{T} \in \langle \mathcal{T}^1, \ldots, \mathcal{T}^r \rangle\}$$

$$\begin{array}{ccccccccc} \mathcal{I} & \subset & \mathbb{P}^N & \times & X & \times & \ldots & \times & X \\ & & \downarrow \pi & & & & & & \\ & & \sigma_r(X) & & & & & & \end{array}$$

- Projection onto the first factor: $\pi(\mathcal{I}) = \sigma_r(X)$.
- Note $\dim(\mathcal{I}) = r \cdot \dim(\widehat{X}) - 1$. If the fiber $\pi^{-1}([\mathcal{T}])$ over a generic $[\mathcal{T}] \in \sigma_r(X)$ is finite, then $\dim(\sigma_r(X)) = \dim(\mathcal{I})$.
- If r is the smallest such that $\sigma_r(X) = \mathbb{P}^N$, say that r is the generic rank.
- Moreover, $\#\pi^{-1}([\mathcal{T}])$ is the number of decompositions of $\mathcal{T}$.

Definition

If $[\mathcal{T}] \in \sigma_r(X)$ is such that $\#\pi^{-1}([\mathcal{T}]) = r!$, then we say that $\mathcal{T}$ is identifiable and that the decomposition of $\mathcal{T}$ is essentially unique.

Perfect identifiability for tensors

For $\mathcal{T} \in \mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_d}$, based on dimension count, the generic rank is at least

$$R(n_1, \ldots, n_d) := \frac{\prod_{i=1}^d n_i}{\sum_{i=1}^d (n_i - 1) + 1} = \frac{\prod_{i=1}^d n_i}{\left(\sum_{i=1}^d n_i\right) + 1 - d}$$

- The value $\lceil R(n_1, \ldots, n_d) \rceil$ is called the expected generic rank.
- A necessary condition for generically finitely many decompositions is for $R(n_1, \ldots, n_d)$ to be an integer, a.k.a perfect format.
- When the generic tensor of perfect format has an essentially unique decomposition, we say that perfect identifiability holds.

Known Results for "unbalanced formats"

Assume $d \geq 3$ and $2 \leq n_1 \leq n_2 \leq \ldots \leq n_d$. If $n_d \geq \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1}(n_i - 1)$, we say that the format $(n_1, \ldots, n_d)$ is unbalanced.

Theorem (Catalisano-Geramita-Gimigliano'02, Abo-Ottaviani-Peterson'09, Bocci-Chiantini-Ottaviani'13)

For formats $(n_1, \ldots, n_d)$, suppose that $n_d \geq \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1}(n_i - 1)$.

1. *The generic rank is* $\min\left(n_d,\ \prod_{i=1}^{d-1} n_i\right)$.
2. *A general tensor of rank r has a unique decomposition if* $r < \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1}(n_i - 1)$.
3. *A general tensor of rank* $r = \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1}(n_i - 1)$ *has exactly* $\binom{D}{r}$ *different decompositions where* $D = \frac{\left(\sum_{i=1}^{d-1}(n_i-1)\right)!}{(n_1-1)!\cdots(n_{d-1}-1)!}$.
   This value of r coincides with the generic rank in the perfect case: $r = n_d$.
4. *If* $n_d > \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1}(n_i - 1)$, *a general tensor of rank* $r > \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1}(n_i - 1)$ *has infinitely many decompositions.*

Monodromy for Tensor Identifiability (using `Bertini`)

The problem:

`Given` $\mathcal{T}$ `find rank-one tensors` $\mathcal{T}^i$ `so that` $\mathcal{T} = \sum_{i=1}^r \mathcal{T}^i$.

Asks to solve a straightforward system of polynomial equations. In general, this can be a very difficult problem.

- One method to numerically solve large systems of polynomials is to use homotopy continuation, in a software package like `Bertini`.
- The idea is to start with a similar system G whose solutions you know (like roots of unity). Then perform a homotopy to your system F:

  $$t \cdot G + (1-t) \cdot F \qquad t \in [0,1]$$

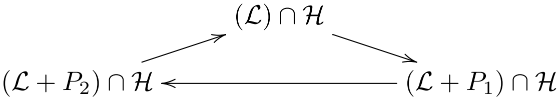
  and numerically track the paths traced out by the solutions of G. The paths should end in solutions of the F.
- *generically* can construct paths that avoid singularities and end points are non-singular (real 1-dimensional path, singular locus has complex codim 1, so real codim 2.)

Numerical Algebraic Geometry & Bertini

`input:` An affine variety $\mathcal{H}$.
`Output:` $\deg \mathcal{H}$

1. Choose a random linear space $\mathcal{L}$ with $\dim \mathcal{L} = \operatorname{codim} \mathcal{H}$.
2. Generate a point $x \in \mathcal{H} \cap \mathcal{L}$. Initialize $\mathcal{W} := \{x\}$.
3. Perform a random monodromy loop starting at the points in $\mathcal{W}$:
   - (a) Pick a random loop $\mathcal{M}(t)$ in the grassmannian of linear spaces so that $\mathcal{M}(0) = \mathcal{M}(1) = \mathcal{L}$.
   - (b) Trace the curves $\mathcal{H} \cap \mathcal{M}(t)$ starting at the points in $\mathcal{W}$ at $t = 0$ to compute the endpoints $\mathcal{E}$ at $t = 1$. (Hence, $\mathcal{E} \subset \mathcal{H} \cap \mathcal{L}$).
   - (c) Update $\mathcal{W} := \mathcal{W} \cup \mathcal{E}$, sort $\mathcal{W}$, remove repeats and symmetric copies.
4. Repeat (2) until $\#W$ stabilizes.
5. Use the trace test to verify that $\mathcal{W} = \mathcal{H} \cap \mathcal{L}$.
6. Return $\deg \mathcal{H} = \#\mathcal{H}(\cap\mathcal{L})$.

A triangular monodromy loop for random points P_1 and P_2 in $\mathbb{C}^N$:



Monodromy for Tensor Decomposition (using `Bertini`)

Start: A general tensor $\mathcal{T}$ of format $(n_1, \ldots, n_d)$ with known minimal decomposition, $\mathcal{T} = \sum_{i=1}^{r}(v_1^i \otimes \ldots \otimes v_d^i)$.
(dehomogenize): Set $(v_j^i)_1 = 1$ for $i = 1, \ldots, r$ and $j = 1, \ldots, d-1$.

- Input system:
$$F_{\mathcal{T}}(v_1^1, \ldots, v_d^r) = \begin{bmatrix} \mathcal{T} - \sum_{i=1}^{r}(v_1^i \otimes \ldots \otimes v_d^i) \\ (v_j^i)_1 - 1 \quad \text{for } i = 1, \ldots, r \text{ and } j = 1, \ldots, d-1 \end{bmatrix} = 0$$
- The system $F_{\mathcal{T}}$ consists of $\prod_{j=1}^{d} n_j + r(d-1)$ polynomials in $r \cdot \sum_{j=1}^{d} n_j$ variables. Balanced format $\Rightarrow$ square system.
- Let $\mathcal{W} \subset (\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_d})^r$ be the known decompositions of T.
- Homotopy: For a loop $\tau : [0,1] \to \mathbb{C}^{n_1 \cdots n_d}$ with $\tau(0) = \tau(1) = \mathcal{T}$, consider the homotopy
$$H(v_1^1, \ldots, v_d^r, s) = F_{\tau(s)}(v_1^1, \ldots, v_d^r) = 0.$$
- Endpoints are decompositions of $\mathcal{T}$. If new, add results to $\mathcal{W}$.
- Repeat until $|\mathcal{W}|$ stabilizes (at least 20 additional randomly selected loops failed to yield any new decompositions), and possibly use `AlphaCertify`.

Theorem (CGG'02, AOP'09, BCC'13)

For formats $(n_1, \ldots, n_d)$, *suppose that* $n_d \geq \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1}(n_i - 1)$.

1. *The generic rank is* $\min\left(n_d,\ \prod_{i=1}^{d-1} n_i\right)$.

2. *A general tensor of rank* r *has a unique decomposition if* $r < \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1}(n_i - 1)$.

3. *A general tensor of rank* $r = \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1}(n_i - 1)$ *has exactly* $\binom{D}{r}$ *different decompositions where*

$$D = \frac{\left(\sum_{i=1}^{d-1}(n_i - 1)\right)!}{(n_1 - 1)! \cdots (n_{d-1} - 1)!}.$$

   This value of r *coincides with the generic rank in the perfect case: when* $r = n_d$.

4. *If* $n_d > \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1}(n_i - 1)$, *a general tensor of rank* $r > \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1}(n_i - 1)$, *e.g., a general tensor of format* $(n_1, \ldots, n_d)$, *has infinitely many decompositions.*

Computational results: Unbalanced cases

Some known perfect cases and the number of decompositions.

$(n_1, \ldots, n_d)$	gen. rank	# of decomp. of general tensor
$(2, n, n)$	n	(Weierstrass-Kronecker) 1
$(3, 3, 5)$	5	6
$(3, 4, 7)$	7	120
$(3, 5, 9)$	9	5005
$(3, 6, 11)$	11	352716
$(4, 4, 10)$	10	184756
$(2, 2, 2, 5)$	5	6
$(2, 2, 3, 8)$	8	495

Computational results: Perfect cases, 3 factors

All perfect, balanced tensor formats of 3-tensors with $\prod_{i=1}^{3} n_i \leq 150$.

(n_1, n_2, n_3)	gen. rank	# of decomp. of general tensor
$(3, 4, 5)$	6	**1**
$(3, 6, 7)$	9	38
$(4, 4, 6)$	8	62
$(4, 5, 7)$	10	$\geq$ 222,556

After the numerical results, we were motivated to prove the following:

Theorem (HOOS 2015)

The general tensor of format $(3, 4, 5)$ *has a unique decomposition as a sum of* 6 *decomposable summands.*

Our proof relies on algebraic geometry, vector bundles and intersection theory, and relies on a notion of *non-abelian polarity*.

Computational results: Perfect cases, 4 factors

All perfect, balanced tensor formats with $d \geq 4$ and $\prod_{i=1}^{d} n_i \leq 100$.

$(n_1, \ldots, n_d)$	gen. rank	# of decomp. of general tensor
$(2,2,2,3)$	4	**1**
$(2,2,3,4)$	6	4
$(2,2,4,5)$	8	68
$(2,3,3,4)$	8	471
$(2,3,3,5)$	9	7225
$(3,3,3,3)$	9	20,596
$(2,2,2,2,4)$	8	447
$(2,2,2,3,3)$	9	18,854
$(2,2,2,2,2,3)$	12	$\geq$ 238,879

Again, motivated by the numerical evidence we were able to prove:

Theorem (HOOS 2015)

The general tensor of format $(2,2,2,3)$ has a unique decomposition as a sum of 4 decomposable summands.

A similar proof to the $(3,4,5)$-case also works here.

A conjecture

Conjecture (HOOS 2015)

The only perfect formats $(n_1, \ldots, n_d)$ where a general tensor has a unique decomposition are

1. $(2, k, k)$ *for some* k *– matrix pencils, known classically by Kronecker normal form,*
2. $(3, 4, 5)$, *and*
3. $(2, 2, 2, 3)$.

The generic rank is known to be equal to the expected one for the cubic format (n, n, n) [Lickteig'85], which is not perfect for $n \geq 3$, and in the binary case $(2, \ldots, 2)$ for at least $k \geq 5$ factors [CGG'11], which is perfect if $k + 1$ is a power of 2. A numerical check for $k = 7$ shows it is not identifiable.

Methods: Koszul Flattenings

The Koszul complex: linear maps $K_p\colon \bigwedge^p V \to \bigwedge^{p+1} V$ depending linearly on V.

$$K_p(v)(\varphi) = \varphi \wedge v \text{ for } p \geq 0, \qquad K_p(v)(\varphi) = \varphi(v) \text{ for } p < 0.$$

Set $V_I = \bigwedge^{i_1} V_1 \otimes \bigwedge^{i_2} V_2 \otimes \cdots \otimes \bigwedge^{i_d} V_d$, and form a tensor product of Koszul maps:

$$K_I\colon V_I \to V_{I+1^d},$$

that depend linearly on $V_{(1,\ldots,1)} = V_1 \otimes \cdots \otimes V_d$.

Lemma (Koszul Flattening)

Suppose $T \in V_{1,\ldots,1}$ has tensor rank r. Let $i_j \geq 0$ for $j = 1, \ldots, h$, $i_j < 0$ for $j = h+1, \ldots, d$. The Koszul flattening $K_I(T)\colon V_I \to V_{I+1^d}$ has rank at most

$$r_I := r \cdot \prod_{j=1}^{h} \binom{n_j - 1}{i_j} \cdot \prod_{j=h+1}^{d} \binom{n_j - 1}{-i_j - 1}.$$

In particular, the $(r_I + 1) \times (r_I + 1)$ minors of $K_I(T)$ vanish.
Meaningful if $r_I < \min\{\dim V_I, \dim V_{I+1^d}\}$.

Basic idea: A Koszul flattening of $\mathcal{T}$ is a matrix constructed from the entries of $\mathcal{T}$ that has rank at most a multiple of the rank of $\mathcal{T}$: detect $\mathrm{Rank}(\mathcal{T})$.

The $3 \times 4 \times 5$ case

Let us denote the three factors as $A = \mathbb{C}^3$, $B = \mathbb{C}^4$, $C = \mathbb{C}^5$. The following are all possible non-trivial, non-redundant Koszul flattenings (up to transpose).

- usual flattenings:

$$K_{(0,-1,-1)} \colon (B \otimes C)^* \to A \,,$$
$$K_{(-1,0,-1)} \colon (A \otimes C)^* \to B \,,$$
$$K_{(-1,-1,0)} \colon (A \otimes B)^* \to C \,,$$

- Koszul flattenings:

$$K_{(1,-1,0)} \colon B^* \otimes A \to C \otimes \bigwedge^2 A \,, \qquad K_{(1,0,-1)} \colon C^* \otimes A \to B \otimes \bigwedge^2 A \,,$$
$$K_{(0,1,-1)} \colon C^* \otimes B \to A \otimes \bigwedge^2 B \,, \qquad K_{(-1,1,0)} \colon A^* \otimes B \to C \otimes \bigwedge^2 B \,,$$
$$K_{(-1,0,1)} \colon A^* \otimes C \to B \otimes \bigwedge^2 C \,, \qquad K_{(0,-1,1)} \colon B^* \otimes C \to A \otimes \bigwedge^2 C \,,$$
$$K_{(-1,0,2)} \colon A^* \otimes \bigwedge^2 C \to B \otimes \bigwedge^3 C \,, \qquad K_{(0,-1,2)} \colon B^* \otimes \bigwedge^2 C \to A \otimes \bigwedge^3 C \,.$$

An example Koszul flattening

$$K_{(0,1,-1)} \colon C^* \otimes B \to A \otimes \bigwedge^2 B$$

$K_{0,1,-1}(a \otimes b \otimes c)$ has image

$$(\bigwedge^0 A \wedge a) \otimes (\bigwedge^1 B \wedge b) \otimes (C^*(c)) \subset \bigwedge^1 A \otimes \bigwedge^2 B \otimes \bigwedge^0 C.$$

The factor $C^*(c)$ is just a scalar that is obtained by contracting c with C^*.

We are left with $(\bigwedge^0 A \wedge a) = \langle a \rangle$ tensored with $(\bigwedge^1 B \wedge b) \subset \bigwedge^2 B$,

but $(\bigwedge^1 B \wedge b) \cong (B/b) \otimes \langle b \rangle$, which is 3 dimensional.

So $K_{0,1,-1}(\mathcal{T})$ has rank that is at most 3 times the rank of $\mathcal{T}$. And since it is 18×20, it has a chance to detect up to rank 6 tensors.

map	size	mult-factor	max tensor rank detected
$K_{(0,-1,-1)}$	3×20	1	3
$K_{(-1,0,-1)}$	4×15	1	4
$K_{(-1,-1,0)}$	5×12	1	5
$K_{(1,-1,0)}$	15×12	2	6
$K_{(1,0,-1)}$	12×15	2	6
$K_{(0,1,-1)}$	18×20	3	6
$K_{(-1,1,0)}$	12×30	3	4
$K_{(-1,0,1)}$	40×15	4	4
$K_{(0,-1,1)}$	30×20	4	5
$K_{(-1,0,2)}$	40×30	6	5
$K_{(0,-1,2)}$	30×40	6	5
$K_{(0,-1,2)}$	30×40	6	5

We see that the only maps that distinguish between tensor rank 5 and 6 are $K_{(1,-1,0)}$, $K_{(1,0,-1)}$, and $K_{(0,1,-1)}$. Since $\bigwedge^2 A \cong A^*$, the first two maps are transposes of each other:

$$K_{(1,-1,0)} = (K_{(1,0,-1)})^t.$$

Thus, we proceed by considering $K_{(1,0,-1)}$ and $K_{(0,1,-1)}$.

Methods: Apolarity

The definition of the Koszul Flattening implies

$$T = v_1 \otimes \ldots \otimes v_d \in \ker K_I(T) \iff \bigotimes_{j=1}^{h} (\varphi_j \wedge v_j) \otimes \bigotimes_{j=h+1}^{d} (\varphi_j(v_j)) = 0$$

for all basis elements $\varphi \in V_I$.
Think of elements of the kernel of $K_I(T)$ as linear mappings.
Let $N \sqcup P = \{1, \ldots, d\}$ be the set partition such that $-I_N \in \mathbb{Z}^d_{>0}$, $I_P \in \mathbb{Z}^d_{\geq 0}$.

Lemma (Non-abelian Apolarity Lemma [Landsberg-Ottaviani'13:])

Suppose $T = \sum_{s=1}^r v_1^s \otimes \ldots \otimes v_d^s$. *The kernel* $\ker K_I(T)$ *contains all maps* $\psi \in \mathrm{Hom}(V_{-I_N}, V_{I_P})$ *such that*

$$\psi\Big(V_{-I_N + 1_N} \wedge \bigotimes_{j \in N} v_j^s\Big) \wedge \Big(\bigotimes_{j \in P} v_j^s\Big) = 0$$

for $s = 1, \ldots, r$.

Basic idea: the kernel of a flattening of $\mathcal{T}$ can be used to gain information about the decomposition of $\mathcal{T}$.

In our case the Apolarity Lemma says that

$$\ker K_{1,0,-1}(\textstyle\sum_{i=1}^{s} a_i b_i c_i) \supset \{\varphi \in Hom(C, A) | \varphi(c_i) \wedge a_i = 0 \text{ for } i = 1, \ldots, s\}. \tag{1}$$

and

$$\ker K_{0,1,-1}(\textstyle\sum_{i=1}^{s} a_i b_i c_i) \supset \{\varphi \in Hom(C, B) | \varphi(c_i) \wedge b_i = 0 \text{ for } i = 1, \ldots, s\}.$$

Equality should hold for honest decompositions.

Basic result from Oeding-Ottaviani [OO'13] and Landsberg-Ottaviani [LO'11]:

The set of eigenvectors of a general element in $\ker(K_I(\mathcal{T}))$ (interpreted as the common base locus of general sections of a certain vector bundle) contains the set of (pieces of) rank-one summands in a decomposition of $\mathcal{T}$.

Proof of Theorem 3-4-5: Vector Bundles

For general $\mathcal{T} \in A \otimes B \otimes C$, $K_{1,0,-1}(f)$ is surjective and $\ker K_{1,0,-1}(\mathcal{T})$ has dimension $\dim Hom(C, A) - \dim \wedge^2 A \otimes B = 15 - 12 = 3$.

Interpret $K_{1,0,-1}(\mathcal{T})$ as a map between sections of vector bundles.

Let $X = \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$ with (pull-back) line bundle. $\mathcal{O}(\alpha, \beta, \gamma)$

Let Q_A be the pullback of the quotient bundle on $\mathbb{P}(A)$.

Let $E = Q_A \otimes \mathcal{O}(0, 0, 1)$ (a rank 2 bundle on X) and $L = \mathcal{O}(1, 1, 1)$.

As in [OO'13], [LO'13], the map $K_{1,0,-1}(\mathcal{T})$ can be identified with contraction $K_{1,0,-1}(\mathcal{T})\colon H^0(E) \longrightarrow H^0(E^* \otimes L)^*$ depending linearly on $\mathcal{T} \in H^0(L)^*$.

Apolarity: compute the common base locus of the sections of the vector bundle $\ker(K_{0,1,-2}(\mathcal{T}))$ to find the decomposition of $\mathcal{T}$.

Proof of Theorem 3-4-5: Intersection Theory I

- Have $K_{1,0,-1}(\mathcal{T})\colon H^0(E) \longrightarrow H^0(E^* \otimes L)^*$ depending linearly on $\mathcal{T} \in H^0(L)^*$.

- The general element in $H^0(E)$ vanishes on a codimension two subvariety of X which has homology class $c_2(E) \in H^*(X, \mathbb{Z})$.

- The ring $H^*(X, \mathbb{Z})$ can be identified with $\mathbb{Z}[t_A, t_B, t_C]/(t_A^3, t_B^4, t_C^5)$.

- The Chern polynomial of Q_A is $\frac{1}{1+t_A}$, so $c_2(E) = t_A^2 + t_A t_C + t_C^2$.

- Three general sections of $H^0(E)$ have common base locus given by $c_2(E)^3 = \left(t_A^2 + t_A t_C + t_C^2\right)^3 = 6t_A^2 t_C^4$.

- This coefficient 6 coincides with the generic rank and it is the key to the computation.

Proof of Theorem 3-4-5: Intersection Theory II

- A `Macaulay2` test (`M2` file on arXiv) performed on a random tensor $\mathcal{T}$ gives that the common base locus of $\ker K_{1,0,-1}(\mathcal{T})$ is given by 6 points (a_i, c_i) for $i = 1, \ldots, 6$ on the 2-factor Segre variety $\mathbb{P}(A) \times \mathbb{P}(C)$.
- By semicontinuity, the common base locus of $\ker K_{1,0,-1}(\mathcal{T})$ is given by 6 points for general tensor $\mathcal{T}$. Hence, for the general tensor $\mathcal{T}$, equality holds in the Apolarity Lemma.
- In particular, the decomposition $\mathcal{T} = \sum_{i=1}^{6} a_i \otimes b_i \otimes c_i$ has a unique solution (up to scalar) for a_i, c_i. It follows that also the remaining vectors b_i can be recovered uniquely, by solving a linear system.

Thanks!

The $2 \times 2 \times 2 \times 3$ case

For this part, let $A \cong B \cong C \cong \mathbb{C}^2$ and $D \cong \mathbb{C}^3$. The only interesting Koszul flattenings for tensors in $A \otimes B \otimes C \otimes D$ are the following maps, which depend linearly on $A \otimes B \otimes C \otimes D$.

The 1-flattenings (and their transposes):

$$K_{-1,0,0,0} \colon A^* \to B \otimes C \otimes D, \quad K_{0,-1,0,0} \colon B^* \to A \otimes C \otimes D,$$

$$K_{0,0,-1,0} \colon C^* \to A \otimes B \otimes D, \quad K_{0,0,0,-1} \colon D^* \to A \otimes B \otimes C,$$

which detect a maximum of rank 2 in the first 3 cases and a maximum of rank 3 in the last.

The 2-flattenings (and their transposes):

$$K_{0,0,-1,-1} \colon C^* \otimes D^* \to A \otimes B, \quad K_{0,-1,0,-1} \colon B^* \otimes D^* \to A \otimes C,$$

$$K_{-1,0,0,-1} \colon A^* \otimes D^* \to B \otimes C.$$

The maps are all 4×6 and detect a maximum of tensor rank 4.

The higher Koszul flattenings:

$$K_{-1,0,0,1} \colon A^* \otimes D \to B \otimes C \otimes \bigwedge^2 D, \quad K_{0,-1,0,1} \colon B^* \otimes C \to A \otimes C \otimes \bigwedge^2 D,$$

$$K_{0,0,-1,1} \colon C^* \otimes D \to A \otimes B \otimes \bigwedge^2 D$$

These maps are all 12×6, and detect a maximum of rank 3.

Proof of Theorem 2-2-2-3

Suppose $T \in A \otimes B \otimes C \otimes D$. Consider $K_{0,0,-1,-1} \colon C^* \otimes D^* \to A \otimes B$. If T is general of rank 4, then $\operatorname{Rank} K_{0,0,-1,-1}(T) = 4$ and $\dim \ker K_{0,0,-1,-1}(T) = 2$. Apolarity says that the points $\{c^s \otimes d^s\}$ are in the common base locus of the elements in the kernel of $K_{0,0,-1,-1}(T)$.

Consider line bundles $E = \mathcal{O}(0,0,1,1)$, $L = \mathcal{O}(1,1,1,1)$ over $\mathrm{Seg}(\mathbb{P}C^* \times \mathbb{P}D^*)$. Two general sections of E have common base locus given by a cubic curve, denoted $\mathcal{C}_{C,D}$ of bi-degree (1,2) on $\mathrm{Seg}(\mathbb{P}C \times \mathbb{P}D)$. The projection to $\mathbb{P}D$ is a conic, which we denote $\mathcal{Q}_C$.

Repeat the process for the next 2-flattening, $K_{0,-1,0,-1} \colon B^* \otimes D^* \to A \otimes C$,, changing the roles of C and B, we obtain another conic $\mathcal{Q}_B$ in $\mathbb{P}D^*$.

Finally, if $\mathcal{Q}_C$ and $\mathcal{Q}_B$ are general, Bézout's theorem implies that they intersect in 4 points in $\mathbb{P}D$, $\{[d^1], [d^2], [d^3], [d^4]\}$.

Pull back the $\{d_i\}$ to the curve $\mathcal{C}_{C,D}$ in $\mathrm{Seg}(\mathbb{P}C^* \times \mathbb{P}D^*)$ and project to $\mathbb{P}C$ to obtain 4 points $\{c_i\}$ on $\mathbb{P}C$.

Reverse the roles of B and C and repeat to find 4 points $\{b_i\}$ on $\mathbb{P}B$.

Reverse the roles of A and B and repeat to find 4 points $\{a_i\}$ on $\mathbb{P}A$.

The tensor products $a^i \otimes b^i \otimes c^i \otimes d^i$ obtained in this way are, up to scale, the indecomposable tensors in the decomposition of the original tensor T.

Finally we solve an easy linear system to determine the coefficients λ_i in the expression $T = \sum_{i=1}^4 \lambda_i a^i \otimes b^i \otimes c^i \otimes d^i$.