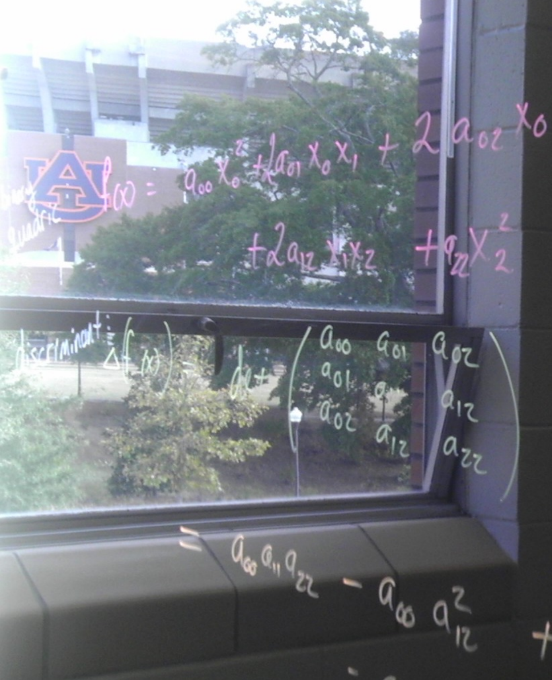


Symmetry and Large-Scale Computations for the Quadrifocal Variety

Luke Oeding
Auburn University, USA

Image reconstruction

Google “Duomo Florence”

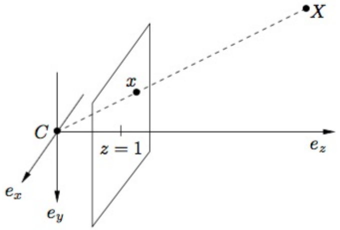
Try to reconstruct a 3D model of this beautiful structure from 2D images.

The Pinhole Camera

Model the 3D world as projective 3-space, $\mathbb{P}^3$.
Use homogeneous coordinates for points $[X] := [X_0 : X_1 : X_2 : X_3]$.

The 2D image is modeled by projective 2-space, $\mathbb{P}^2$.
Points $[x] := [x_0 : x_1 : x_2]$.

The standard pinhole camera is modeled by projection $\mathbb{P}^3 \to \mathbb{P}^2$.



The projection is induced from a linear map on affine spaces, represented by a 3×4 *camera matrix* A.

The projection is simply $[X] \mapsto [A.X] = [x]$

Assume A has full rank. Can choose coordinates so that $A = \left(I_3 | \vec{a}_4\right)$,
where $\vec{a}_4$ will be the coordinates of the image of the focal point of the camera.

Fundamental Matrices

For 2 camera matrices A_1, and A_2, point correspondences between two images are recorded by the 3×3 *fundamental matrix*, $\mathtt{F}$, which is defined by the following algebraic conditions:

$$\text{If } A_1 X = x \text{ and } A_2 X = x' \text{ (a point-point correspondence) then } x^\top \mathtt{F} x' = 0.$$

If the camera matrices are known, and the stacked camera matrix is (after change in coordinates)

$$M = (A_1^\top \mid A_2^\top) \cong \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{2,1} & x_{2,2} & x_{2,3} \end{array} \right).$$

The entries of the fundamental matrix are the 4×4 minors of A using 2 columns from each block:

$$F(M) = \begin{pmatrix} 0 & x_{1,3} - x_{2,3} & -x_{1,2} + x_{2,2} \\ -x_{1,3} + x_{2,3} & 0 & x_{1,1} - x_{2,1} \\ x_{1,2} - x_{2,2} & -x_{1,1} + x_{2,1} & 0 \end{pmatrix}.$$

Fundamental Matrices

Theorem (Hartley-Zisserman Thm. 9.10)

The fundamental matrix determines the camera matrices up to projective transformation. That is, if (P, P') and $(\tilde{P}, \tilde{P}')$ are two pairs of matrices with the same fundamental matrix $\mathtt{F}$, then there exists a nonsingular 4×4 matrix H such that $P = \tilde{P}H$ and $P' = \tilde{P}'H$.

If the camera matrices are not known, $\mathtt{F}$ has 9 homogeneous parameters (only defined up to scale) and must have rank 2, so it is determined by the linear conditions imposed by 7 point-point correspondences.

Once the two camera matrices are reconstructed, triangulation allows us to reconstruct the 3D world points associated to each point-point correspondence.

Typical method for camera matrix reconstruction: Random Sample Consensus (RANSAC) Algorithms.

Typical issue: sometimes difficult to determine 7 inliers in two different images.

Possible improvement: use more images to reduce the number of required inliers.

General multi-focal tensors

Stack camera matrices $A_i : \mathbb{C}^N \to \mathbb{C}^n$ to form the $N \times m \cdot n$ matrix

$$M = \left(\; A_1^\top \mid A_2^\top \mid \dots \mid A_m^\top \; \right).$$

M has maximal rank on an open set of camera configurations.

Fix a partition $\pi \vdash N$, with $\#\pi \leq m$ parts, $N \leq m \cdot n$, and $\pi_1 \leq n$:
Z_π is the variety of the maximal minors of M using π_i columns from $A_i^\top$.

- Z_π is an equivariant projection from the Grassmannian $Gr(N, mn)$.
- Z_π is invariant under the action of $\mathrm{GL}(n)^{\times \#\pi}$.
  - see also [Hartley-Zisserman], [Aholt-Sturmfels-Thomas], [Heyden], [Triggs], [Faugeras-Mourrain]...

We are most interested in the case $N = 4$, $m = 4$ $n = 3$:
$Z_{2,2}$ is the variety of *fundamental matrices* (skew-symmetric 3×3 matrices).
$Z_{2,1,1}$ is the variety of *trifocal tensors* (special $3 \times 3 \times 3$ tensors). (see [Aholt-O.'14]).
$Z_{1,1,1,1}$ is the variety of *quadrifocal tensors* (special $3 \times 3 \times 3 \times 3$ tensors).

Frank and Prank for trifocal tensors

A (rank 4) tensor in $A^* \otimes B^* \otimes C$:

$$T = a_1 \otimes b_2 \otimes c_1 + a_3 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_3 \otimes c_3.$$

3 contractions:

$$T(A) = \begin{pmatrix} a_3 & 0 & 0 \\ a_1 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix},$$

P-Rank$_A = 3$

$$T(B) = \begin{pmatrix} b_2 & 0 & 0 \\ 0 & b_2 & 0 \\ b_1 & 0 & b_3 \end{pmatrix},$$

P-Rank$_B = 3$

$$T(C) = \begin{pmatrix} 0 & c_1 & 0 \\ 0 & c_2 & 0 \\ c_1 & 0 & c_3 \end{pmatrix}$$

P-Rank$_C = 2$ $\Rightarrow$ 10 cubics

3 flattenings to 3×9 matries:

$$F(A) = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

F-Rank$_A = 3$

$$F(B) = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

F-Rank$_B = 3$

$$F(C) = \left(\begin{array}{ccc|ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

F-Rank$_C = 3$

Polarization: how to write down 10 special cubics

Think of a $3 \times 3 \times 3$ tensor as three 3×3 matrices stacked in the rows of a matrix $T = (T_1 \mid T_2 \mid T_3)$

$$= \left(\begin{array}{ccc|ccc|ccc} T000 & T100 & T200 & T001 & T101 & T201 & T002 & T102 & T202 \\ T010 & T110 & T210 & T011 & T111 & T211 & T012 & T112 & T212 \\ T020 & T120 & T220 & T021 & T121 & T221 & T022 & T122 & T222 \end{array}\right)$$

Use dummy variables $z1, z2, z3$ to form the generic contraction:

$$T(z) = z1\,T_1 + z2\,T_2 + z3\,T_3.$$

$$\begin{pmatrix} T000z0 + T001z1 + T002z2 & T100z0 + T101z1 + T102z2 & T200z0 + T201z1 + T202z2 \\ T010z0 + T011z1 + T012z2 & T110z0 + T111z1 + T112z2 & T210z0 + T211z1 + T212z2 \\ T020z0 + T021z1 + T022z2 & T120z0 + T121z1 + T122z2 & T220z0 + T221z1 + T222z2 \end{pmatrix}$$

If T is a trifocal tensor then $T(z)$ is a bifocal tensor for all z.

Recall F is a bifocal tensor if and only if it has rank 2.

Therefore $\det(T(z)) \equiv 0$. The coefficients in z are 10 cubic equations in the entries of T.

Gives a basis of the Schur Module $(S^3 V_1 \otimes \bigwedge^3 V_2 \otimes \bigwedge^3 V_3)^*$.

Polarization: how to write down 600 special cubics

Think of a $3 \times 3 \times 3 \times 3$ tensor as nine 3×3 matrices

$$Q(x,y) = \sum_{i,j=1}^{3} x_i y_j Q_{i,j}.$$

A bilinear function in (x_i, y_i).

If Q is a quadrifocal tensor then $Q(x,y)$ is a bifocal tensor for all x, y.

Therefore $\det(Q(x,y)) \equiv 0$. The coefficients in x, y are 100 cubic equations in the entries of Q.

Gives a basis of $S^3 S^3 \bigwedge^3 \bigwedge^3$ - the 6 permutations give the 600 cubic equations.

We would like to give a complete algebraic description of the quadrifocal variety Z by finding the generators of its defining ideal $I(Z)$ (the implicit defining equations of the model).

Naively, to find out if the variety Z lives in a linear subspace, put the coordinates of 81 points in the rows of a matrix $P = \begin{pmatrix} -Z_1- \\ \vdots \\ -Z_{81}- \end{pmatrix}$ The null space of P is the vector space of linear forms vanishing on the 81 points (and very likely all of Z).

If $\mathrm{Null}(P) = 0$, there are no linear forms in the ideal of Z.

In higher degree d we can Veronese re-embed the points and solve another linear algebra problem to find the space of degree d polynomials vanishing on Z.

But the dimensions grow quickly:
$(1 \quad 81 \quad 3321 \quad 91881 \quad 1929501 \quad 32801517 \quad 470155077 \quad 5843355957 \quad 64276915527 \quad \ldots)$

The ideal $I(Z)$ is a $G = \mathfrak{S}_4 \ltimes \mathrm{GL}(3)^{\times 4}$-submodule of $R = \mathbb{C}[\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3]$.
R has a G-isotypic decomposition:

$$R = \bigoplus_{d \geq 0} \bigoplus_{\pi \vdash_4 d} S_\pi V \otimes M_\pi .$$

where π is a multi-partition, the sum is over non-redundant permutations,

- Schur module: $S_\pi V = \bigoplus_{\sigma \in \mathfrak{S}_4/\sim} (S_{\pi_{\sigma.1}} V_1 \otimes S_{\pi_{\sigma.2}} V_2 \otimes S_{\pi_{\sigma.3}} V_3 \otimes S_{\pi_{\sigma.4}} V_4)$
- Multiplicity space (Specht Module): M_π.

Our tasks for small degree d are the following:

- Compute a basis of M_π for each π.
- Evaluate the highest-weight space of $S_\pi V \otimes M_\pi$ on points of Z.
- Obtain a list of G-modules (with multiplicity) in $I(Z)$.
- Determine which modules are among the minimal generators.
- Determine the maximal degree of minimal generators.

Invariant Theory and Young Symmetrizers

Multi-partition: (221, 221, 221,221), Filling: $F = \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \cline{1-1}\end{array}$

Auxiliary polynomial: $p = \begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ b_1^1 & b_2^1 & b_3^1 \\ c_1^1 & c_2^1 & c_3^1 \end{vmatrix} \begin{vmatrix} d_1^1 & d_2^1 \\ e_1^1 & e_2^1 \end{vmatrix} \cdot \begin{vmatrix} a_1^2 & a_2^2 & a_3^2 \\ b_1^2 & b_2^2 & b_3^2 \\ c_1^2 & c_2^2 & c_3^2 \end{vmatrix} \begin{vmatrix} d_1^2 & d_2^2 \\ e_1^2 & e_2^2 \end{vmatrix} \cdot \begin{vmatrix} a_1^3 & a_2^3 & a_3^3 \\ b_1^3 & b_2^3 & b_3^3 \\ c_1^3 & c_2^3 & c_3^3 \end{vmatrix} \begin{vmatrix} d_1^3 & d_2^3 \\ e_1^3 & e_2^3 \end{vmatrix} \cdot \begin{vmatrix} a_1^4 & a_2^4 & a_3^4 \\ b_1^4 & b_2^4 & b_3^4 \\ c_1^4 & c_2^4 & c_3^4 \end{vmatrix} \begin{vmatrix} d_1^4 & d_2^4 \\ e_1^4 & e_2^4 \end{vmatrix}$

1. Start with $p(a, b, c, d, e, x)$ of multi-degree $(4, 4, 4, 4, 4, 0)$.
2. Replace every $a_i^1 a_j^2 a_k^3 a_l^4$ with $x_{i,j,k,l}$
   - Produce a polynomial of multi-degree $(0, 4, 4, 4, 4, 1)$.
3. Replace every $b_i^1 b_j^2 b_k^3 b_l^4$ with $x_{i,j,k,l}$
   - Produce a polynomial of multi-degree $(0, 0, 4, 4, 4, 2)$.
4. Repeat for $c, d, e,$
   - Produce $P(x)$ of multi-degree $(0, 0, 0, 0, 0, 5)$ (possibly zero).
5. output: $P(x)$ highest weight vector of $S_{221}\mathbb{C}^3 \otimes S_{221}\mathbb{C}^3 \otimes S_{221}\mathbb{C}^3 \otimes S_{221}$.

Evaluation of Young Symmetrizers

Multi-partition: (221, 221, 221,221), Filling: $F = \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \hline \end{array}$

Auxiliary polynomial: $p = \begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ b_1^1 & b_2^1 & b_3^1 \\ c_1^1 & c_2^1 & c_3^1 \end{vmatrix} \begin{vmatrix} d_1^1 & d_2^1 \\ e_1^1 & e_2^1 \end{vmatrix} \cdot \begin{vmatrix} a_1^2 & a_2^2 & a_3^2 \\ b_1^2 & b_2^2 & b_3^2 \\ c_1^2 & c_2^2 & c_3^2 \end{vmatrix} \begin{vmatrix} d_1^2 & d_2^2 \\ e_1^2 & e_2^2 \end{vmatrix} \cdot \begin{vmatrix} a_1^3 & a_2^3 & a_3^3 \\ b_1^3 & b_2^3 & b_3^3 \\ c_1^3 & c_2^3 & c_3^3 \end{vmatrix} \begin{vmatrix} d_1^3 & d_2^3 \\ e_1^3 & e_2^3 \end{vmatrix} \cdot \begin{vmatrix} a_1^4 & a_2^4 & a_3^4 \\ b_1^4 & b_2^4 & b_3^4 \\ c_1^4 & c_2^4 & c_3^4 \end{vmatrix} \begin{vmatrix} d_1^4 & d_2^4 \\ e_1^4 & e_2^4 \end{vmatrix}$

Point: $Z \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

1. Start with $p(a, b, c, d, e, x)$ of multi-degree $(4, 4, 4, 4, 4, 0)$.
2. Replace every $a_i^1 a_j^2 a_k^3 a_l^4$ with $x_{i,j,k,l}$ and substitute $x_{i,j,k,l} \to Z_{i,j,k,l}$.
   - Produce a polynomial of multi-degree $(0, 4, 4, 4, 4, 0)$.
3. Replace every $b_i^1 b_j^2 b_k^3 b_l^4$ with $x_{i,j,k,l}$ and substitute $x_{i,j,k,l} \to Z_{i,j,k,l}$.
   - Produce a polynomial of multi-degree $(0, 0, 4, 4, 4, 0)$.
4. Repeat for c, d, e,
5. output: the value of $p(Z)$.
   - Producing $p(Z)$ takes *much* less time and memory than $p(x)$.

Compute a basis of M_π for each π.

The following fillings form a basis of $M_{(221),(221),(221),(221)}$:

$$F_1 = {\color{red}\begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \cline{1-1}\end{array}} \otimes {\color{red}\begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \cline{1-1}\end{array}} \otimes {\color{red}\begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \cline{1-1}\end{array}} \otimes {\color{red}\begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \cline{1-1}\end{array}}, \qquad F_2 = {\color{red}\begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \cline{1-1}\end{array}} \otimes {\color{red}\begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \cline{1-1}\end{array}} \otimes {\color{blue}\begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d & \\ \cline{1-1}\end{array}} \otimes {\color{blue}\begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d & \\ \cline{1-1}\end{array}}$$

$$F_3 = {\color{red}\begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \cline{1-1}\end{array}} \otimes {\color{red}\begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \cline{1-1}\end{array}} \otimes {\color{blue}\begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d & \\ \cline{1-1}\end{array}} \otimes \begin{array}{|c|c|}\hline a & c \\ \hline b & d \\ \hline e & \\ \cline{1-1}\end{array}, \qquad F_4 = {\color{red}\begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \cline{1-1}\end{array}} \otimes {\color{blue}\begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d & \\ \cline{1-1}\end{array}} \otimes {\color{red}\begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \cline{1-1}\end{array}} \otimes {\color{blue}\begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d & \\ \cline{1-1}\end{array}}$$

$$F_5 = {\color{red}\begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \cline{1-1}\end{array}} \otimes {\color{blue}\begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d & \\ \cline{1-1}\end{array}} \otimes {\color{red}\begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \cline{1-1}\end{array}} \otimes \begin{array}{|c|c|}\hline a & c \\ \hline b & d \\ \hline e & \\ \cline{1-1}\end{array}, \qquad F_6 = {\color{red}\begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \cline{1-1}\end{array}} \otimes {\color{blue}\begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d & \\ \cline{1-1}\end{array}} \otimes {\color{blue}\begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d & \\ \cline{1-1}\end{array}} \otimes {\color{red}\begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \cline{1-1}\end{array}}$$

Check rank$(p_i(Z_j))$ for 6 random points Z_j, $\Rightarrow$ independence.
A character computation $\Rightarrow$ spanning.

Evaluate the highest-weight space of $S_\pi V \otimes M_\pi$ on Z.

Using the basis of $M_{(221),(221),(221),(221)}$ and Young symmetrizers, populate the matrix (one processor core per entry)

$$(p_i(Z_j))$$

for 6 random points Z_j of Z.

Find null-space (kernel) is the span of

$$\begin{pmatrix} -11/12 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}^T$$

So $(S_{221}\mathbb{C}^3 \otimes S_{221}\mathbb{C}^3 \otimes S_{221}\mathbb{C}^3 \otimes S_{221}\mathbb{C}^3)$ has multiplicity 1 in $I(Z)$.

Quadrifocal Hilbert Function

Compute the Hilbert function for $I(Z) \subset R = \mathbb{C}[x_1, \dots, x_{81}]$ for as high a degree as possible.

Using Representation Theory and parallel computing we found:

d	$=$	0	1	2	3	4	5	6	7	8	
HF_R	$=$	(1	81	3321	91881	1929501	32801517	470155077	5843355957	64276915527	…
$\dim_I$	$=$	(0	0	0	600	48600	1993977	54890407	1140730128	18051062139	
mingens	$=$	(0	0	0	600	0	≥ 1377	37586	?0	?162000	

```
-- deg      # reps     max_mult
--------------------------------
-- 1 |           1           1          }
-- 2 |           3           1          } Use Symmetry
-- 3 |           9           3          }
-- 4 |          25           4          } Use Grid Computing
-- 5 |          59          13          }
-- 6 |         163          93          } Use Multi-threading
-- 7 |         288         301          }
-- 8 |         619         608          } Use a High Performance Cluster
-- 9 |        1205        2226          } (deg 9 seems slightly out of reach)
```

Obtain a list of G-modules (with multiplicity) in $I(Z)$.

Repeat the process for isotypic decompositions of $\mathbb{C}[(\mathbb{C}^3)^{\otimes 4}]_d$.

Obtain multiplicities of all modules in $I(Z)_d$ for small degree d.
For instance, $I(Z)_d = 0$ for $d = 1, 2$

Input the results into `SchurRings` (by Stillman and Raicu) in `Macaualy2`.

Modules are represented as polynomials, with coefficients the multiplicities.
For example $I(Z)_3$ is expressed as

$$\left(s_{(1,1,1)} t_{(1,1,1)} u_3 + \left(s_{(1,1,1)} t_3 + s_3 t_{(1,1,1)}\right) u_{(1,1,1)}\right) v_3 \\ + \left(\left(s_{(1,1,1)} t_3 + s_3 t_{(1,1,1)}\right) u_3 + s_3 t_3 u_{(1,1,1)}\right) v_{(1,1,1)}$$

or modding out by the $\mathfrak{S}_4$ action, $I(Z)_3 = s_3 t_3 u_{(1,1,1)} v_{(1,1,1)}$

which represents the module $\mathfrak{S}_4.(S_3\mathbb{C}^3 \otimes S_3\mathbb{C}^3 \otimes S_{1,1,1}\mathbb{C}^3 \otimes S_{1,1,1}\mathbb{C}^3)$.

Determine which modules are minimal generators.

Let $R = \mathbb{C}[(\mathbb{C}^3)^{\otimes 4}]$. Using the Young symmetrizer algorithm,
$I(Z)_4 = (s_4 t_4 + (s_4 + s_{(3,1)})t_{(3,1)})u_{(2,1,1)}v_{(2,1,1)}$

Using `SchurRings`, we find that
$I(Z)_3 \cdot R_1 = I(Z)_4$.
(all multiplicities are one, and every module in $I(Z)_4$ occurs in $I(Z)_3 \cdot R_1$)
So there are no minimal generators in degree 4.

We find two modules in $I(Z)_5$ that cannot occur in $I(Z)_3 \cdot R_2$:
$s_{(3,1,1)}t_{(3,1,1)}u_{(3,1,1)}v_{(3,1,1)} + s_{(2,2,1)}t_{(2,2,1)}u_{(2,2,1)}v_{(2,2,1)}$

We find the following modules occur in $I(Z)_6$ but cannot occur in $I(Z)_5 \cdot R_1$:
$(s_6 t_{(3,3)}u_{(3,3)} + ((2s_{(4,1,1)} + 2s_{(3,3)})t_{(3,3)} + s_{(3,2,1)}t_{(3,2,1)} + 2s_{(2,2,2)}t_{(2,2,2)})u_{(2,2,2)})v_{(2,2,2)}$
We find that all modules in $I(Z)_7$ can occur in $I(Z)_6 \cdot R_1$,
strong evidence that there are no minimal generators in degree 7.
In degree 8 we find a surprise: $S_{4,4}S_{4,4}S_{4,4}S_{4,2,2} \otimes \mathbb{C}^2$ must occur among the minimal generators.
In degree 9 we weren't able to compute all modules because of a lack of computing time, but the modules we were able to compute produced no new necessary minimal generators.

graded piece	dim I_d	necessary G modules of minimal generators	dimension of mingens
I_2	0	$\mathcal{M}_2 = 0$	0
I_3	600	$\mathcal{M}_3 = S_3 S_3 S_{1,1,1} S_{1,1,1}$	600
I_4	48,600	$\mathcal{M}_4 = 0$	0
I_5	1,993,977	$\mathcal{M}_5 = S_{3,1,1} S_{3,1,1} S_{3,1,1} S_{3,1,1}$ $\oplus S_{2,2,1} S_{2,2,1} S_{2,2,1} S_{2,2,1}$	1,377
I_6	54,890,407	$\mathcal{M}_6 = S_{4,1,1} S_{3,3} S_{2,2,2} S_{2,2,2} \otimes \mathbb{C}^2$ $\oplus S_{3,3} S_{3,3} S_{2,2,2} S_{2,2,2} \otimes \mathbb{C}^2$ $\oplus S_{3,2,1} S_{3,2,1} S_{2,2,2} S_{2,2,2}$ $\oplus S_{2,2,2} S_{2,2,2} S_{2,2,2} S_{2,2,2} \otimes \mathbb{C}^2$ $\oplus S_6 S_{3,3} S_{3,3} S_{2,2,2}$	37,586
I_7	1,140,730,128	$\mathcal{M}_7 = 0$	0
I_8	18,051,062,139	$\mathcal{M}_8 = S_{4,4} S_{4,4} S_{4,4} S_{4,2,2} \otimes \mathbb{C}^2$	162,000
I_9	$\geq$ 188,850,321,637	$\mathcal{M}_9 \geq S_{5,4} S_{5,4} S_{5,4} S_{4,3,2}$ $\oplus S_{5,4} S_{5,4} S_{5,4} S_{5,2,2}$	3,087,000

Table: The ideal of the quadrifocal variety up to degree 9. We used M2 to rule out many possible minimal generators and conjecture that these equations suffice to define the quadrifocal variety.