Lower Bounds for Tensor Rank: 2×2×2×2×2 Tensors of Rank 5

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Recall Peter Burgisser's overview lecture (Jan Draisma's SIAM News article).

- Big Goal: Bound the computational complexity of $\det_n$, perm_n, $\mathcal{M}_n$.
- Algorithmic Complexity is often governed by the rank of a tensor.
- Need: tools for bounding the rank (or border rank) of tensors.
- How can we find such lower bounds? Polynomials!
- Common theme: Exploit all available symmetry to aid in computations.

Take the $n \times n$ matrix multiplication tensor $\mathcal{M}_n \in \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2} \otimes \mathbb{C}^{n^2}$.
In 2013 Ikenmeyer, Hauenstein and Landsberg provided a new computational proof that the border rank of 2×2 matrix multiplication $\mathcal{M}_2$ is 7 by providing a degree 20 invariant F for border rank 7 that did not vanish on $\mathcal{M}_2$.
Driasma highlighted the dramatic power of symmetry:

> *Straightforward combinatorics shows that the space of degree-20 polynomials on the 64-dimensional space is* $C(63+20, 20) = 8,179,808,679,272,664,720$*-dimensional — it is striking how representation theory helps us to find* F *and evaluate it at* $\mathcal{M}_2$*!*

Our work echoes this theme.

First questions for tensors

Consider tensors of format $n_1 \times \cdots \times n_d$.

1. What notion of rank are you using?
   (tensor rank / border rank via secant varieties)
2. What is the expected (generic) tensor rank? (defectivity)
3. How can you find a minimal decomposition of a given tensor?
   (Find *effective* algorithms)
4. How can you detect the rank of a given tensor? (Provide certificates)
5. How many decompositions does a given tensor have? (identifiability)

Knowing equations of secant varieties can help with all of these questions, especially if they're determinantal.

Secant varieties and tensors

Let $V_1, \ldots, V_d$, be $\mathbb{C}$-vector spaces. The tensor product $V_1 \otimes \cdots \otimes V_d$ is the vector space with elements $(T_{i_1,\ldots,i_d})$ considered as hyper-matrices or tensors.

- Segre variety (rank 1 tensors): Defined by

$$\begin{array}{rcl} \mathrm{Seg} : \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_d & \longrightarrow & \mathbb{P}\big(V_1 \otimes \cdots \otimes V_d\big) \\ ([v_1], \ldots, [v_d]) & \longmapsto & [v_1 \otimes \cdots \otimes v_d]. \end{array}$$

  In coordinates: $T_{i_1,\ldots,i_d} = v_{1,i_1} \cdot v_{2,i_2} \cdots v_{d,i_d}$.
- The r^{th} secant variety of a variety $X \subset \mathbb{P}^N$:

$$\sigma_r(X) := \overline{\bigcup_{x_1,\ldots,x_r \in X} \mathbb{P}(\mathrm{span}\{x_1, \ldots, x_r\})} \subset \mathbb{P}^N.$$

General points of $\sigma_r(\mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_d))$ have *rank* r:

$$\left[\sum_{s=1}^{r} v_1^s \otimes v_2^s \otimes \ldots \otimes v_d^s\right],$$

or in coordinates: $T_{i_1,\ldots,i_n} = \sum_{s=1}^r v^s_{1,i_1} \cdot v^s_{2,i_2} \cdots v^s_{d,i_d}$.

A first case: matrices

- Suppose $k \leq m \leq n$. If $M \in Mat_{m\times n}(\mathbb{C})$ has rank k then $\exists$ (full rank) $A \in Mat_{m\times k}(\mathbb{C})$ and $\exists$ (full rank) $B \in Mat_{k\times n}(\mathbb{C})$ such that

$$M = AB \qquad \text{but also} \qquad M = (AU)(U^{-1}B),$$

  for any (full rank) $U \in Mat_{k\times k}(\mathbb{C})$.

  So $\dim(\sigma_k(\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}))$ has dimension $m \cdot k + n \cdot k - k^2 - 1$.

- Expected dimension: $ExpDim(\sigma_k(\mathbb{P}^{m-1} \times \mathbb{P}^{n-1})) = \min\{n \cdot m - 1, k \cdot (n + m - 1) - 1\}$.

- So the defect is $k \cdot (n + m - 1) - 1 - (m \cdot k + n \cdot k - k^2 - 1) = k^2 - k$.

- Rank k matrices are defective (and thus not identifiable) for $k \neq 0, 1$.

First examples of equations of secant varieties: Flattenings

Given $\mathcal{T} \in V_1 \otimes \cdots \otimes V_d$, take index sets $I \sqcup J = [d]$.
Tensor product is associative, so $V_1 \otimes \cdots \otimes V_d = (\bigotimes_{i \in I} V_i) \otimes (\bigotimes_{j \in J} V_j)$,
And we can view $\mathcal{T}$ as a (flattening) matrix:

$$F_I(\mathcal{T}) \colon V_I^* \to V_J$$

Facts:

- If $\operatorname{Rank}(\mathcal{T}) = 1$, then rank $F_I(\mathcal{T}) = 1$.
- $F(\mathcal{T} + \mathcal{T}') = F(\mathcal{T}) + F(\mathcal{T}')$.
- Sub-additivity of matrix rank implies if $\operatorname{Rank} \mathcal{T} = r$ then $\operatorname{Rank} F((T)) \leq r$.
- So, $(r+1) \times (r+1)$ minors of flattenings (if non-trivial) are equations for tensors of rank $\leq r$.

$2 \times 2 \times 2 \times 2$ tensors

Suppose $V_i = \mathbb{C}^2$ for $1 \leq i \leq 4$.
Four different 1-flattenings (up to transpose):

$$F_1(\mathcal{T})\colon V_1^* \to V_2 \otimes V_3 \otimes V_4$$

$$F_2(\mathcal{T})\colon V_2^* \to V_1 \otimes V_3 \otimes V_4$$

$$F_3(\mathcal{T})\colon V_3^* \to V_1 \otimes V_2 \otimes V_4$$

$$F_4(\mathcal{T})\colon V_4^* \to V_1 \otimes V_2 \otimes V_3$$

All 8×2 matrices, max rank 2. If all have rank 1, then the tensor $\mathcal{T}$ actually has rank 1 (and vice versa).
Three different 2-flattenings (up to transpose)

$$F_{1,2}(\mathcal{T})\colon (V_1 \otimes V_2)^* \to V_3 \otimes V_4$$

$$F_{1,3}(\mathcal{T})\colon (V_1 \otimes V_3)^* \to V_2 \otimes V_3$$

$$F_{1,4}(\mathcal{T})\colon (V_1 \otimes V_4)^* \to V_2 \otimes V_3$$

All 4×4. So determinants vanish for tensors of rank 3.

A defective secant variety

Suppose $V_i = \mathbb{C}^2$ for $1 \leq i \leq 4$. Counting parameters, expect $\sigma_3(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3 \times \mathbb{P}V_4)$ to be $3 \cdot (1+1+1+1) + 2 = 14$, so expect codimension 1 in $\mathbb{P}^{15} = \mathbb{P}(V_1 \otimes V_2 \otimes V_3 \otimes V_4)$.
On the other hand,

$$F_{1,2}(\mathcal{T})\colon (V_1 \otimes V_2)^* \to V_3 \otimes V_4$$
$$F_{1,3}(\mathcal{T})\colon (V_1 \otimes V_3)^* \to V_2 \otimes V_3$$
$$F_{1,4}(\mathcal{T})\colon (V_1 \otimes V_4)^* \to V_2 \otimes V_3$$

Check: any two of $\det(F_{1,2}(\mathcal{T}))$, $\det(F_{1,3}(\mathcal{T}))$, $\det(F_{1,4}(\mathcal{T}))$ are algebraically independent, so

$$\dim zeros(\det(F_{1,2}(\mathcal{T})), \det(F_{1,3}(\mathcal{T}))) = 13$$

In fact, $\sigma_3(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3 \times \mathbb{P}V_4) = zeros(\det(F_{1,2}(\mathcal{T})), \det(F_{1,3}(\mathcal{T})))$
The expected dimension is not the actual dimension, so the variety is defective, and thus rank 3 tensors of format $2 \times 2 \times 2 \times 2$ are not identifiable.

Theorem (Catalisano-Geramita-Gimigliano)

Suppose $V_i \cong \mathbb{C}^2$ for $1 \leq i \leq d$. For all $d \geq 5$ and all k, $\sigma_k(\mathbb{P}V_1 \times \cdots \times V_d)$ has the expected dimension (non-defective).

Identifiability for Binary Tensors

$\dim \sigma_3(\mathbb{P}V_1 \times \mathbb{P}V_2 \times \mathbb{P}V_3 \times \mathbb{P}V_4) = 13 < 14$ also implies that the generic tensor of rank 3 and format $2 \times 2 \times 2 \times 2$ has infinitely many decompositions.

Definition

If the general tensor of format $n_1 \times \cdots \times n_d$ and rank k has finitely many decompositions the variety is not k-defective.

Definition

A tensor format $n_1 \times n_2 \times \ldots n_d$ is called k-identifiable if the generic tensor of that format and rank k has a unique (up to trivial re-ordering) decomposition as the sum of k rank-1 tensors.

Theorem (Bocci-Chiantini 2014)

$2 \times 2 \times 2 \times 2 \times 2$ tensors are not identifiable in rank 5, but the generic tensor of that format has exactly 2 decompositions.

Theorem (Bocci-Chiantini-Ottaviani 2014)

For ≥ 6 factors, the Segre is almost always k-identifiable.

Identifiability for Perfect Tensors

A tensor of format $n_1 \times \cdots \times n_d$ is of *perfect* format if $[\prod_{i=1} n_i]/[1 + \sum_i (n_i + 1)]$ is an integer (generically have finitely many decompositions).

Theorem (Hauenstein-Oeding-Ottaviani-Sommese '14)

- *The general* $3 \times 4 \times 5$ *tensor has a unique decomposition of rank* 6*.*
- *The general* $2 \times 2 \times 2 \times 3$ *tensor has a unique decomposition of rank* 4*.*

Conjecture (Hauenstein-Oeding-Ottaviani-Sommese '14)

The only perfect formats $(n_1, \ldots, n_d)$ where a general tensor has a unique decomposition are:

1. $(2, k, k)$ for some k — matrix pencils, classical Kronecker normal form,
2. (3, 4, 5), and
3. (2, 2, 2, 3).

(see arXiv:1501.00090)

Out of Bernd Sturmfels's *Algebraic Fitness Session*

Find the equations of $\sigma_5(\mathrm{Seg}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1))$:

Theorem⋆ (Oeding-Sam [*Exp. Math 2015*])

The affine cone of $\sigma_5(\mathrm{Seg}(\mathbb{P}^{1^{\times 5}}))$ *is a complete intersection of two equations: one of degree* 6*, and one of degree* 16*.*

The star refers to the careful numerical, sometimes probablistic computations used in our proofs, which took around *two weeks of human/computer time.*

Note this result implies that: $\sigma_5(\mathrm{Seg}(\mathbb{P}^{1^{\times 5}}))$ is arithmetically Cohen-Macaulay.

Find equations in the ideal

Consider $\sigma_5(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^{31}$. Look for equations in $R = \mathbb{C}[x_{00000}, \ldots, x_{11111}]$ of low degree.
The Hilbert function of R starts like this:

d	=	0	1	2	3	4	5	6	7	...
$HF_R(d)$	=	1	32	528	5984	52360	376992	2324784	12620256	...
$HF_{R/I}(d)$	=	1	32	528	5984	52360	376992	2324783	?	...
$\dim(I_d)$	=	0	0	0	0	0	0	1	?	...

Naively, to find the space of sextics in the ideal: compute 2324784 points on the variety, evaluate them on the 2324784 monomials of degree 6 and compute the kernel of the resulting 2324784×2324784 matrix.

The space of degree 16 equations has dimension 511 738 760 544, so it seems hopeless to work here.

The equation f_6

Choose a basis e_0, e_1 for V_i so that we can identify the coordinates of $\mathbb{P}(\mathbb{V})$ with x_I where $I \in \{0,1\}^5$. Given a monomial in the x_I, define its skew-symmetrization to be $c^{-1}\sum_{\sigma\in\Sigma_5}\operatorname{sgn}(\sigma)x_{\sigma(I)}$ where c is the coefficient of x_I in the sum. The polynomial f_6 has 864 monomials and is the sum of the skew-symmetrizations of the following 15 monomials:

$$\begin{aligned}
&-x_{00000}x_{01010}x_{01101}x_{10011}x_{10100}x_{11111}, && x_{00000}x_{01100}x_{01111}x_{10010}x_{10111}x_{11001},\\
&-x_{00000}x_{01100}x_{01111}x_{10011}x_{10110}x_{11001}, && x_{00000}x_{01101}x_{01110}x_{10011}x_{10110}x_{11001},\\
&-x_{00110}x_{01000}x_{01101}x_{10000}x_{10011}x_{11111}, && x_{00100}x_{01010}x_{01111}x_{10000}x_{10111}x_{11001},\\
&x_{00100}x_{01000}x_{01111}x_{10011}x_{10110}x_{11001}, && x_{00110}x_{01000}x_{01101}x_{10001}x_{10010}x_{11111},\\
&-x_{00100}x_{01010}x_{01111}x_{10001}x_{10111}x_{11000}, && x_{00100}x_{01010}x_{01111}x_{10011}x_{10101}x_{11000},\\
&-x_{00101}x_{01010}x_{01111}x_{10000}x_{10110}x_{11001}, && x_{00100}x_{01011}x_{01110}x_{10011}x_{10101}x_{11000},\\
&-x_{00110}x_{01001}x_{01100}x_{10001}x_{10010}x_{11111}, && x_{00110}x_{01001}x_{01111}x_{10011}x_{10100}x_{11000},\\
& && x_{00111}x_{01010}x_{01101}x_{10011}x_{10100}x_{11000}.
\end{aligned}$$

Alternative description in terms of Young symmetrizers, see [Bates-Oeding'10].
The Young Symmetrizer algorithm takes as input a set of fillings of five Young diagrams, performs a series of skew-symmetrizations and symmetrizations, and produces as output a polynomial in the associated Schur module.

There are 5 standard tableaux of shape $(3,3)$ and content $\{1,2,\ldots,6\}$.

The following Schur module, which uses one of each of the 5 standard fillings, realizes the non-trivial copy of $\bigotimes_{i=1}^{5}(S_{3,3}V_i)$ inside of $\mathrm{Sym}^6(V_1 \otimes V_2 \otimes V_3 \otimes V_4 \otimes V_5)$

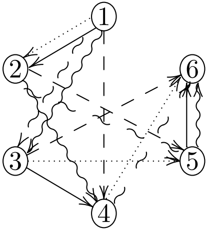
$$S_{\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array}}V_1 \otimes S_{\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array}}V_2 \otimes S_{\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array}}V_3 \otimes S_{\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}}V_4 \otimes S_{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array}}V_5.$$

Can show that the image of the Young symmetrizer vanishes on an open subset of points of X.

From Ikenmeyer's intro lecture: Polynomials and graphs

Each tableau gets a color. Each column becomes a directed colored arrow.
where color 1 corresponds to $\longrightarrow$, color 2 to $\cdots\cdots\rightarrow$,
color 3 to $\rightsquigarrow$, color 4 to $\sim\ \rightarrow$. color 5 to $\dashrightarrow$,
The degree 6 invariant is described by

$$\mathbf{S}_{\begin{smallmatrix}1&3&5\\2&4&6\end{smallmatrix}}V_1 \otimes \mathbf{S}_{\begin{smallmatrix}1&3&4\\2&5&6\end{smallmatrix}}V_2 \otimes \mathbf{S}_{\begin{smallmatrix}1&2&5\\3&4&6\end{smallmatrix}}V_3 \otimes \mathbf{S}_{\begin{smallmatrix}1&2&4\\3&5&6\end{smallmatrix}}V_4 \otimes \mathbf{S}_{\begin{smallmatrix}1&2&3\\4&5&6\end{smallmatrix}}V_5.$$



Combinatorics of these graphs played a strong role in the resolution of the *Garcia-Stillman-Sturmfels Conjecture* on ideals of secant line varieties [Raicu'12], and the resolution of the *Landsberg-Weyman Conjecture* on ideals of tangential varieties [Oeding-Raicu'13].

The Young symmetrizer algorithm for evaluations

1. **input**:
   - Point: $Z \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$,
   - Multi-partition: $((3,3),(3,3),(3,3),(3,3),(3,3))$
   - Filling: $F = \begin{array}{|c|c|c|}\hline a & c & e \\ \hline b & d & f \\ \hline\end{array} \otimes \begin{array}{|c|c|c|}\hline a & c & d \\ \hline b & e & f \\ \hline\end{array} \otimes \begin{array}{|c|c|c|}\hline a & b & e \\ \hline c & d & f \\ \hline\end{array} \otimes \begin{array}{|c|c|c|}\hline a & b & d \\ \hline c & e & f \\ \hline\end{array} \otimes \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline\end{array}$
2. Construct a product of determinants:

$$p = \begin{vmatrix} a_1^1 & a_2^1 \\ b_1^1 & b_2^1 \end{vmatrix} \begin{vmatrix} c_1^1 & c_2^1 \\ d_1^1 & d_2^1 \end{vmatrix} \begin{vmatrix} e_1^1 & e_2^1 \\ f_1^1 & f_2^1 \end{vmatrix} \begin{vmatrix} a_1^2 & a_2^2 \\ b_1^2 & b_2^2 \end{vmatrix} \begin{vmatrix} c_1^2 & c_2^2 \\ e_1^2 & e_2^2 \end{vmatrix} \begin{vmatrix} d_1^2 & d_2^2 \\ f_1^2 & f_2^2 \end{vmatrix} \begin{vmatrix} a_1^3 & a_2^3 \\ c_1^3 & c_2^3 \end{vmatrix} \begin{vmatrix} b_1^3 & b_2^3 \\ d_1^3 & d_2^3 \end{vmatrix} \begin{vmatrix} e_1^3 & e_2^3 \\ f_1^3 & f_2^3 \end{vmatrix}$$

$$\times \begin{vmatrix} a_1^4 & a_2^4 \\ c_1^4 & c_2^4 \end{vmatrix} \begin{vmatrix} b_1^4 & b_2^4 \\ e_1^4 & e_2^4 \end{vmatrix} \begin{vmatrix} d_1^4 & d_2^4 \\ f_1^4 & f_2^4 \end{vmatrix} \begin{vmatrix} a_1^5 & a_2^5 \\ d_1^5 & d_2^5 \end{vmatrix} \begin{vmatrix} b_1^5 & b_2^5 \\ e_1^5 & e_2^5 \end{vmatrix} \begin{vmatrix} c_1^5 & c_2^5 \\ f_1^5 & f_2^5 \end{vmatrix} = 2^{15} \text{ terms (don't expand!).}$$

3. Start with $p(a,b,c,d,e,f,x)$ of multi-degree $(5,5,5,5,5,5,0)$.
4. Substitutions: $a_i^1 a_j^2 a_k^3 a_l^4 a_m^5 a_n^6 \to x_{i,j,k,l,m,n}$ and $x_{i,j,k,l,m,n} \to Z_{i,j,k,l,m,n}$.
   - Produce a polynomial of multi-degree $(0,5,5,5,5,5,0)$.
5. Substitutions: $b_i^1 b_j^2 b_k^3 b_l^4 b_m^5 b_n^6 \to x_{i,j,k,l,m,n}$ and $x_{i,j,k,l,m,n} \to Z_{i,j,k,l,m,n}$.
   - Produce a polynomial of multi-degree $(0,0,5,5,5,5,0)$.
6. Repeat for c, d, e, f
7. **output**: the value of $p(Z)$.
   - Producing $p(Z)$ takes *much* less time and memory than $p(x)$.

Evaluate the highest-weight space of $S_\pi V \otimes M_\pi$ on X.

1. Use characters to compute $m := \dim M_{(d,d),(d,d),(d,d),(d,d),(d,d)}$.
2. Make m random points y_i of ambient space, and $m+5$ points x_i of X.
3. Repeat the following for $k = 1..m$:
   - 3.0 Start with linearly independent fillings $\mathcal{F} = \{F_1, \dots, F_{k-1}\}$.
   - 3.1 At step k take a random filling F_k of shape $(d,d)^{\times 5}$.
   - 3.2 Evaluate $p_k(y_1)$ using the Young symmetrizer algorithm for F_k.
     - ★ If $p_k(y_1)$ is non-zero, continue.
     - ★ Otherwise return to (3.1).
   - 3.3 Populate the $k \times k$ matrix (one processor core per entry)

$$Q_k(y) := (p_j(y_i))_{(i,j)}$$

   - 3.4 Compute $\mathrm{Rank}(Q_k(y))$
     - ★ If Q_k has rank k, add F_k to $\mathcal{F}$ increment k, and return to (3.0).
     - ★ Otherwise return to (3.1).
4. Take linearly independent fillings $\mathcal{F} = \{F_1, \dots, F_m\}$
5. Populate the $(m+5) \times m$ matrix (one processor core per entry)

$$Q_m(x) := (p_j(x_i))_{i,j}$$

6. The kernel of $Q_m(x)$ is the subspace of M_π vanishing on the x_i
7. The extra points increases likelihood that $\ker(Q_m(x))$ also vanishes on X.

From Jon Hauenstein's Intro Lectures: Numerical Algebraic Geometry & Bertini

input: An irreducible variety $\mathcal{H}$.
Output: $\deg \mathcal{H}$

1. Choose a random linear space $\mathcal{L}$ with $\dim \mathcal{L} = \operatorname{codim} \mathcal{H}$.
2. Generate a point $x \in \mathcal{H} \cap \mathcal{L}$. Initialize $\mathcal{W} := \{x\}$.
3. Perform a random monodromy loop starting at the points in $\mathcal{W}$:
   (a) Pick a random loop $\mathcal{M}(t)$ in the grassmannian of linear spaces so that $\mathcal{M}(0) = \mathcal{M}(1) = \mathcal{L}$.
   (b) Trace the curves $\mathcal{H} \cap \mathcal{M}(t)$ starting at the points in $\mathcal{W}$ at $t = 0$ to compute the endpoints $\mathcal{E}$ at $t = 1$. (Hence, $\mathcal{E} \subset \mathcal{H} \cap \mathcal{L}$).
   (c) Update $\mathcal{W} := \mathcal{W} \cup \mathcal{E}$.
4. Repeat (2) until $\#W$ stabilizes.
5. Use the trace test to verify that $\mathcal{W} = \mathcal{H} \cap \mathcal{L}$.
6. Return $\deg \mathcal{H} = \#\mathcal{H}(\cap \mathcal{L})$.

Proposition*

The degree of $\sigma_5\left((\mathbb{P}^1)^{\times 5}\right)$ is 96.

More Symmetry!

Let $\mathbb{V} := V_1 \otimes V_2 \otimes V_3 \otimes V_4 \otimes V_5$. Let $U_d = \mathrm{Sym}^d(\mathbb{V})^{\mathrm{SL}_2^{\times 5}}$ (the superscript denotes taking invariants).
This has an action of Σ_5.
Here are the dimensions of U_d, and the spaces of Σ_5-invariants and Σ_5 skew-invariants in U_d.

Degree d	$\dim U_d$	$\dim U_d^{\Sigma_5}$	$\dim U_d^{\Sigma_5,\mathrm{sgn}}$
2	0	0	0
4	5	1	0
6	1	0	1
8	36	4	0
10	15	0	2
12	228	12	2
14	231	2	9
16	1313	39	10

This follows from standard character theory calculations.

Use linear algebra to compute bases of each space of invariants, and compute the subspaces of invariants vanishing on rank 5 tensors.

Invariant Invariants

How to compute with $U_d^{\Sigma_5}$ and $U_d^{\Sigma_5, sgn}$:
Apply same Young symmetrizer method to detect non-zero fillings F.
Compute for y in ambient space,

$$\sum_{\sigma\in\Sigma_5} \sigma.p_F(y) \qquad \text{or} \qquad \sum_{\sigma\in\Sigma_5} \sigma.p_F(y)\cdot sgn(\sigma)$$

by evaluating $\sigma.p_F(y) = p_{\sigma.F}(y)$ on a different processor for each σ.
Then sum the results (with or without signs).
Repeat for new fillings and evaluating on y in ambient space until finding enough linearly independent fillings.
Evaluate again on $x_j \in X$ and compute the kernel of the associated matrix

$$Q(x) = \left(\sum_{\sigma\in\Sigma} p_{\sigma F_j}(x_i) \right).$$

Each evaluation took between 500 and 23,000 seconds and up to approximately 10GB of RAM on our servers:
24 cores 2.8 GHz Intel Xeon processors, 144 GB RAM
40 cores 2.8 GHz Intel Xeon processors, 256GB of RAM

Check invariants of degrees 8, 10, 12, 14, 16. In degree 16, $\dim U_{16} = 1313$.
Compute evaluations in parallel, and then sum results with / without signs.

- Try $\mathfrak{S}_5$-skew-invariants, $U_{16}^{\mathfrak{S}_5, sgn}$:
  - Find a basis of the 10-dimensional space $U_{16}^{\mathfrak{S}_5, sgn}$ of skew invariants.
  - Evaluate the basis on ≥ 10 random points of σ_5
  - Store results in a matrix and compute its rank
  - discover $U_{16}^{\mathfrak{S}_5, sgn} \cap I(X)$ is full-dimensional, so no new equations.
- Try $\mathfrak{S}_5$-invariants, $U_{16}^{\mathfrak{S}_5}$:
  - Find a basis of the 39-dimensional space $U_{16}^{\mathfrak{S}_5}$ of invariants.
  - Evaluate the basis on ≥ 10 random points of σ_5
  - Store results in a matrix and compute its rank
  - discover $U_{16}^{\mathfrak{S}_5} \cap I(X)$ has dimension 36 (random). $f_6 \cdot U_{10}^{\mathfrak{S}_5, sgn}$ is 2-dimensional, so there is ≥ 1 minimal generator of degree 16 in $I(X)$.

Theorem* (Oeding-Sam 2015)

The affine cone of $\sigma_5(\mathrm{Seg}(\mathbb{P}^{1^{\times 5}}))$ *is a complete intersection of two equations: one of degree* 6*, and one of degree* 16*.*

- Use `Bertini` (with Hauenstein's help) to find $\deg \sigma_5(\mathrm{Seg}(\mathbb{P}^{1^{\times 5}})) = 96$
- Known codim 2, so we suspect complete intersection of two polynomials.
- Compute the only degree 6 invariant f_6, and show that it vanishes on an open subset of X (and thus on all of X).
- Check invariants of degree 8,10,12,14, 16. In degree 16, discover one new generator, f_{16} vanishes on any number of random points of X.
- $Y = V(f_6, f_{16})$, a complete intersection. Also $X \subseteq Y$ and $\deg X \geq \deg Y = 96$. Since X is irreducible of codimension 2, and Y is equidimensional, Y is also irreducible (otherwise the degree inequality would be violated). So X is the reduced subscheme of Y.
- Also, this implies that $\deg(X) = \deg(Y)$, so Y is generically reduced. Since Y is Cohen–Macaulay, generically reduced is equivalent to reduced. Hence $X = Y$ is a complete intersection.

Consequences of main result

Consider $\widetilde{X} := \sigma_5(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_5)$ for any V_i. Let $G = \prod_{i=1}^5 \mathrm{GL}(V_i)$.

- Functoriality implies that the equations $\langle G.f_6\rangle$ and $\langle G.f_{16}\rangle$ vanish on $\widetilde{X}$, and are the only minimal generators of $\mathcal{I}(\widetilde{X})$ coming from modules where all partitions have at most 2 parts. (Use Sam and Snowden's Δ-module and twisted commutative algebra theory.)
- These are new equations from secant varieties, which, as far as we know, don't come from flattenings.
- A new example of a secant variety of a segre product that is a complete intersection and hence arithmetically Cohen-Macaulay.

The End