

Toward splitting conditions for vector bundles over Lagrangian Grassmannians

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The Grassmannian and Lagrangian Grassmannian

- The Grassmannian is a generalization of projective space.

$$Gr(k, n) = \{k\text{-planes in } \mathbb{P}^n\}$$

- Note that $Gr(0, n) = \mathbb{P}^n$.
- Let $\mathbb{P}^n \simeq \mathbb{P}V$ with n -odd and $\omega \in \Lambda^2 V^*$ be non-degenerate.

$$LG(k, n) = \{\mathbb{P}E \in Gr(k, n) \mid \omega(v, w) = 0 \forall v, w \in E\}$$

- Note also that $LG(0, n) = \mathbb{P}^n$.
- Both $Gr(k, n)$ and $LG(k, n)$ are G/P for $G = SL(n+1)$ and $G = SP(n+1)$, respectively.

Horrock's Condition

A vector bundle $E \rightarrow X$ *splits* if it is isomorphic to a direct sum of line bundles.

For projective space, we have the following condition:

Theorem (Horrocks 1964)

A vector bundle $E \rightarrow \mathbb{P}^n$ splits if and only if

$$H^i(\mathbb{P}^n, E(t)) = 0 \quad 0 < i < n, \quad \forall t \in \mathbb{Z}$$

Ottaviani's Extension

Theorem (Ottaviani 1989)

Let E be a vector bundle on $Gr(k, n)$. Then E splits if and only if

$$H^i \left(Gr(k, n), \mathcal{L}^{j_1} Q^* \otimes \dots \otimes \mathcal{L}^{j_s} Q^* \otimes E(t) \right) = 0 \quad \forall i_1, \dots, i_s$$

such that $0 \leq j_1, \dots, j_s \leq n - k$, $s \leq k$; $\forall t \in \mathbb{Z}$; $\forall i$ such that

$$0 < i < (k + 1)(n - k) = \dim(Gr(k, n))$$

where Q is the quotient bundle on $Gr(k, n)$.

Main tools

Koszul complex, representation theory and Bott's theorem.

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Does a similar theorem hold for $LG(k, n)$?

Proposition (Splitting criterion on $LG(1, n)$.)

Let $n \geq 3$ be an odd number and let E be a vector bundle $LG(1, n)$. Then E splits if and only if

$$H^i \left(LG(1, n), N^j Q^* \otimes E(t) \right) = 0,$$

for all $t \in \mathbb{Z}$ and all i, j such that $0 < i < \dim LG(1, n)$ and $0 \leq j < n - 1$.

Some Results

Proposition (Splitting criterion on $LG(1, n)$.)

Let $n \geq 3$ be an odd number and let E be a vector bundle $LG(1, n)$. Then E splits if and only if

$$H^i \left(LG(1, n), \mathcal{N}^j Q^* \otimes E(t) \right) = 0,$$

for all $t \in \mathbb{Z}$ and all i, j such that $0 < i < \dim LG(1, n)$ and $0 \leq j < n - 1$.

Key to the proof

$LG(1, n)$ is a hyperplane section of $Gr(1, n)$.

Naive conjecture

Let E be a vector bundle on $LG(k, n)$. Then E splits if and only if

$$H^i \left(LG(k, n), \Lambda^{j_1} Q^* \otimes \dots \otimes \Lambda^{j_s} Q^* \otimes E(t) \right) = 0 \quad \forall j_1, \dots, j_s$$

such that $0 \leq j_1, \dots, j_s \leq n - k$, $s \leq k$; $\forall t \in \mathbb{Z}$; $\forall i$ such that

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where Q is the quotient bundle on $LG(k, n)$.

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Counterexample

The splitting conditions on the Grassmannian *do not* work on $LG(2, 5)$:

$$H^1(LG(2, 5), Q^* \otimes Q^*) \neq 0.$$

Proposition (Sharper sufficient splitting criterion.)

Let $n \geq 3$ be an odd number and let E be a vector bundle on $LG(k, n)$ such that

$$H^i \left(LG(k, n), \Lambda^{j_1} Q^* \otimes \dots \otimes \Lambda^{j_k} Q^* \otimes E(t) \right) = 0,$$

for all $t \in \mathbb{Z}$ and all i, j_1, \dots, j_k such that $i > 0$, $0 \leq j_q \leq n - k - q + 1$, and

$$\sum_{q=1}^k j_q \leq i < \sum_{q=1}^k j_q + n - 2k.$$

Then E splits as a sum of line bundles.

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Conjecture

These conditions are also necessary.

Some Additional Machinery (Bott's Theorem)

- $\Lambda^{j_1} Q^* \otimes \dots \otimes \Lambda^{j_s} Q^*(t)$ are homogenous vector bundles over a homogeneous variety.
- Bott's theorem gives a straightforward combinatorial algorithm to compute

$$H^i \left(\Lambda^{j_1} Q^* \otimes \dots \otimes \Lambda^{j_s} Q^*(t) \right).$$

- This algorithm is implemented in the computer program LiE, which we have used to verify the conjecture in several examples.
- We hope to be able to do these calculations in general, and this remains work in progress.

First cases of $LG(k, 2k + 1)$

$$H^1(Q^*(t)) = 0,$$

$$H^2(Q^* \otimes Q^*(t)) = H^2(\Lambda^2 Q^*(t)) = 0,$$

$$H^3(Q^* \otimes Q^* \otimes Q^*(t)) = H^3(\Lambda^2 Q^* \otimes Q^*(t)) = H^3(\Lambda^3 Q^*(t)) = 0,$$

$$H^4(\Lambda^2 Q^* \otimes Q^* \otimes Q^*(t)) = H^4(\Lambda^2 Q^* \otimes \Lambda^2 Q^*(t)) =$$

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$$\forall t \in \mathbb{Z}.$$

First cases of $LG(k, 2k + 1)$

Proposition

The conjecture is true for $LG(k, 2k + 1)$, for $1 \leq k \leq 4$.

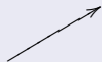
To study the cases

To study the cases

$$LG(k, 2k + 1)$$

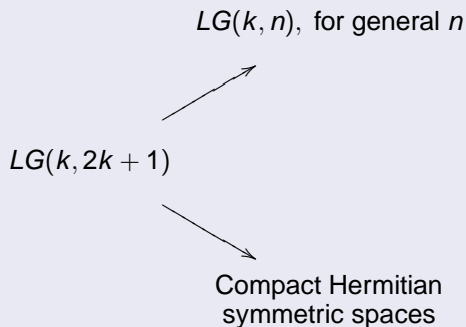
To study the cases

$LG(k, n)$, for general n



$LG(k, 2k + 1)$

To study the cases



To study the cases

