Toward splitting conditions for vector bundles over Lagrangian Grassmannians

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The Grassmannian and Lagrangian Grassmannian

- The Grassmannian is a generalization of projective space.

$$Gr(k,n) = \{k\text{-planes in } \mathbb{P}^n\}$$

- Note that $Gr(0,n) = \mathbb{P}^n$.
- Let $\mathbb{P}^n \simeq \mathbb{P}V$ with n-odd and $\omega \in \Lambda^2 V^*$ be non-degenerate.

$$LG(k,n) = \{\mathbb{P}E \in Gr(k,n) \mid \omega(v,w) = 0 \; \forall v,w \in E\}$$

- Note also that $LG(0,n) = \mathbb{P}^n$.
- Both $Gr(k,n)$ and $LG(k,n)$ are G/P for $G = SL(n+1)$ and $G = SP(n+1)$, respectively.

Horrock's Condition

A vector bundle $E \to X$ *splits* if it is isomorphic to a direct sum of line bundles.

For projective space, we have the following condition:

Theorem (Horrocks 1964)

A vector bundle $E \to \mathbb{P}^n$ splits if and only if

$$H^i\left(\mathbb{P}^n, E(t)\right) = 0 \qquad 0 < i < n, \;\; \forall \;\; t \in \mathbb{Z}$$

Ottaviani's Extension

Theorem (Ottaviani 1989)

Let E be a vector bundle on $Gr(k,n)$. Then E splits if and only if

$$H^i\left(Gr(k,n), \Lambda^{j_1}Q^* \otimes \cdots \otimes \Lambda^{j_s}Q^* \otimes E(t)\right) = 0 \qquad \forall i_1, \dots, i_s$$

such that $0 \le j_1, \dots, j_s \le n-k$, $s \le k$; $\forall t \in \mathbb{Z}$; $\forall i$ *such that*

$$0 < i < (k+1)(n-k) = \dim(Gr(k,n))$$

where Q is the quotient bundle on $Gr(k,n)$.

Main tools

Koszul complex, representation theory and Bott's theorem.

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Does a similar theorem hold for $LG(k,n)$?

Some Results

Proposition (Splitting criterion on $LG(1, n)$.)

Let $n \geq 3$ be an odd number and let E be a vector bundle $LG(1, n)$. Then E splits if and only if

$$H^i\left(LG(1,n), \Lambda^j Q^* \otimes E(t)\right) = 0,$$

for all $t \in \mathbb{Z}$ and all i, j such that $0 < i < \dim LG(1, n)$ and $0 \leq j < n - 1$.

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Key to the proof

$LG(1,n)$ is a hyperplane section of $Gr(1,n)$.

Naive conjecture

Let E be a vector bundle on $LG(k,n)$. Then E splits if and only if

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such that $0 \leq j_1, \ldots, j_s \leq n-k$, $s \leq k$; $\forall t \in \mathbb{Z}$; $\forall i$ such that

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Counterexample

The splitting conditions on the Grassmannian *do not* work on $LG(2,5)$:

$$H^1\left(LG(2,5), Q^* \otimes Q^*\right) \neq 0.$$

Modified Conditions

Proposition (Sharper sufficient splitting criterion.)

Let $n \geq 3$ be an odd number and let E be a vector bundle on $LG(k,n)$ such that

$$H^i\left(LG(k,n), \Lambda^{j_1}Q^* \otimes \cdots \otimes \Lambda^{j_k}Q^* \otimes E(t)\right) = 0,$$

for all $t \in \mathbb{Z}$ and all $i, j_1, \dots, j_k$ such that $i > 0$, $0 \leq j_q \leq n-k-q+1$, and

$$\textstyle\sum_{q=1}^{k} j_q \leq i < \sum_{q=1}^{k} j_q + n - 2k.$$

Then E splits as a sum of line bundles.

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Conjecture

These conditions are also necessary.

Some Additional Machinery (Bott's Theorem)

- $\Lambda^{j_1}Q^* \otimes \cdots \otimes \Lambda^{j_s}Q^*(t)$ are homogenous vector bundles over a homogeneous variety.
- Bott's theorem gives a straightforward combinatorial algorithm to compute

$$H^i\left(\Lambda^{j_1}Q^* \otimes \cdots \otimes \Lambda^{j_s}Q^*(t)\right).$$

- This algorithm is implemented in the computer program LiE, which we have used to verify the conjecture in several examples.
- We hope to be able to do these calculations in general, and this remains work in progress.

First cases of $LG(k, 2k+1)$

$$
\begin{aligned}
&H^1\left(Q^*(t)\right) = 0,\\
&H^2\left(Q^* \otimes Q^*(t)\right) = H^2\left(\Lambda^2 Q^*(t)\right) = 0,\\
&H^3\left(Q^* \otimes Q^* \otimes Q^*(t)\right) = H^3\left(\Lambda^2 Q^* \otimes Q^*(t)\right) = H^3\left(\Lambda^3 Q^*(t)\right) = 0,\\
&H^4\left(\Lambda^2 Q^* \otimes Q^* \otimes Q^*(t)\right) = H^4\left(\Lambda^2 Q^* \otimes \Lambda^2 Q^*(t)\right) =\\
&\qquad H^4\left(\Lambda^3 Q^* \otimes Q^*(t)\right) = 0,\\
&H^5\left(\Lambda^2 Q^* \otimes \Lambda^2 Q^* \otimes Q^*(t)\right) = H^5\left(\Lambda^3 Q^* \otimes Q^* \otimes Q^*(t)\right) =\\
&\qquad H^5\left(\Lambda^3 Q^* \otimes \Lambda^2 Q^*(t)\right) = 0,\\
&\forall t \in \mathbb{Z}.
\end{aligned}
$$

First cases of $LG(k, 2k+1)$

Proposition

The conjecture is true for $LG(k, 2k+1)$, for $1 \leq k \leq 4$.

Future Directions

To study the cases

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$$LG(k, n), \text{ for general } n$$

$\nearrow$

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Compact Hermitian symmetric spaces

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$LG(k,2k+1)$ $\nearrow$ $LG(k,n)$ $\searrow$ G/P

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