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Spiders for $U_q \mathfrak{sl}_n$

(week 5, Topological Invariants from Quantum Algebra)

Noah has imparted to us the prejudices appropriate to a quantum topologist —

"If you're looking for invariants of widgets, instead think about widgets with boundary, and make a category out of them. Work out what sort of category you've got, and invent functors from this category to 'pre-existing' categories of the same type. Now to each widget we can associate a self map of the trivial object in the target category".

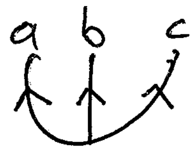
Last week Noah told us about how doing this for knots results in the category of tangles, which we identified as a <u>braided category with duals</u>, and more specifically a <u>planar category</u>.

Today, I'll show you a combinatorial model for some of the 'best' planar categories, namely representations of quantum $\mathfrak{sl}_n$'s, but without mentioning quantum $\mathfrak{sl}_n$.

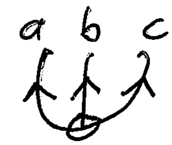
② To begin, let's define the 'free spider' category, FS_n. It has —

<u>Objects</u> $1, \dots, n-1$, and a 'tensor identity' $0 = n$, and tensor products. There's a duality functor
$$k^* = n-k.$$

<u>Morphisms</u> 'Planar generators' of two types:



and



$0 \longrightarrow a \otimes b \otimes c$
with $a+b+c = 2n$

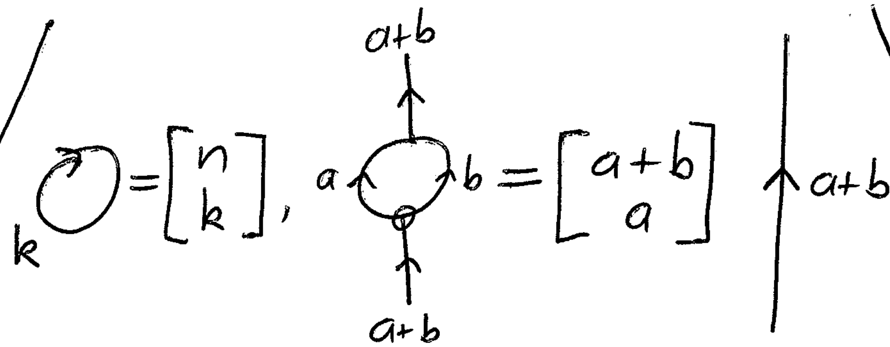
$0 \longrightarrow a \otimes b \otimes c$
with $a+b+c = n$.

What is a 'planar generator'? We want FS_n to be the monoidal category generated (by composition and tensor product) by the trivalent vertices, subject only to the axioms required for a planar category.

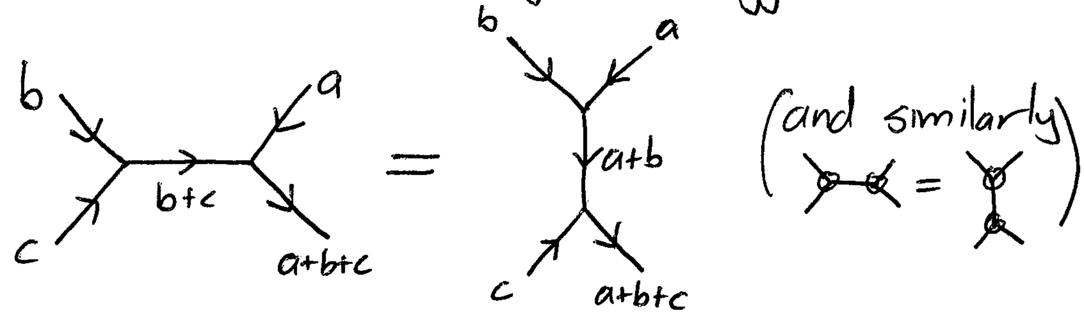
Equivalently, FS_n is 'the category of trivalent graphs, with oriented-labelled edges $1, \dots, n-1$, so the outgoing label sum at each vertex is either n or $2n$'.

③ Let's add some relations, to make things more interesting. First, the 'intermediate spider' category

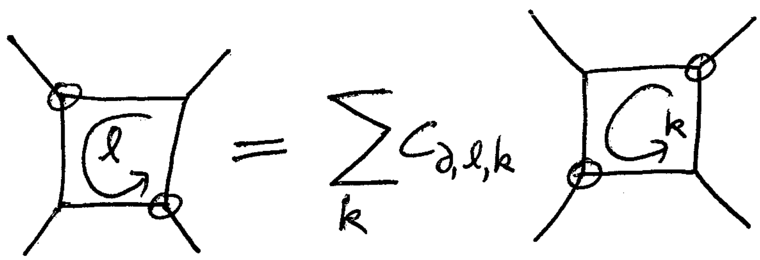
$IS_n = FS_n \big/$



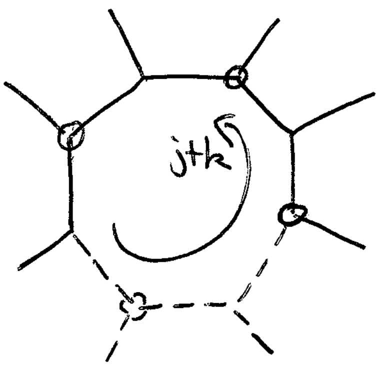
"deloop" and "let bigons be bygones"



the "I=H" relation



④ Let's add even more relations, to get the (conjectural) "representation spider":

$$RS_n = IS_n \Big/ \Big\langle \sum_k d_{\partial,j,k}$$ 

$$= 0 \Big\rangle$$

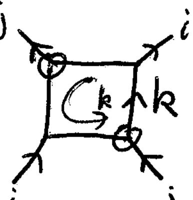
the "Kekulé" relation
(for each $j = n-1+\Sigma a, \ldots, \Sigma b$)

(with $\partial = (\bar{a}, \bar{b})$,
$d = (-1)^{k+j} \begin{bmatrix} n-k-j-\max(a) \\ n-1-j-\Sigma a \end{bmatrix} \begin{bmatrix} k+j+\min(b) \\ j+\Sigma b \end{bmatrix}$)

(this relation actually includes "delooping" and "bigons" as special cases, and for squares is sometimes redundant with the "square change" relation.)

In a moment we'll look at these for small values of n, but first —

<u>Theorem 1</u> (MOY) IS_n isn't just planar — it's braided too!

$$= \sum_k b_k$$ 

<u>"Theorem" 2</u> (Scott) RS_n is equivalent to $\mathrm{FundRep}\, U_q\mathfrak{sl}_n$

(the subcategory of $\mathrm{Rep}\, U_q\mathfrak{sl}_n$ where the objects are just tensor products of fundamental representations)
(you can add idempotents as new objects, and get all of Rep)

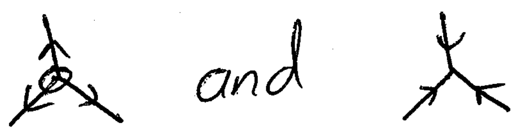
⑤ Okay.... $n=2$.

Now there are no vertices — $a+b+c=2$ has no solutions. Thus the only relation is $\bigcirc = [2]$, and we recover the Temperley-Lieb category.

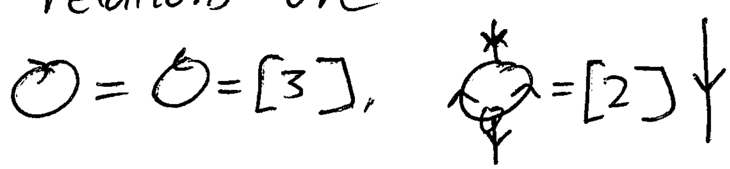
Here it's easy to see equivalence with $\mathrm{Rep}\, U_q \mathfrak{sl}_2$

- construct a surjective functor $RS_2 \twoheadrightarrow \mathrm{FundRep}\, U_q \mathfrak{sl}_2$ (actually works for all n)
- count dimensions — Catalan numbers on both sides.

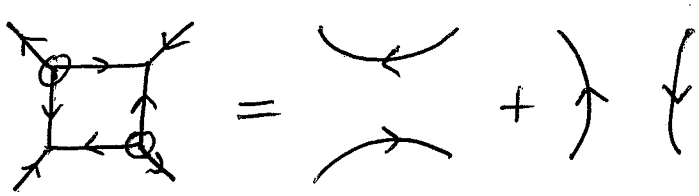
At $n=3$, there's one of each type of vertex:



The relations are —



and

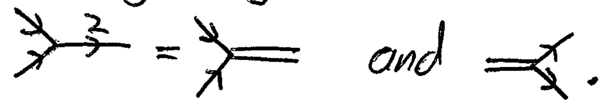


You can think of this last relation as either a "square-change" or "polygon relation" — $IS_3 = RS_3$.

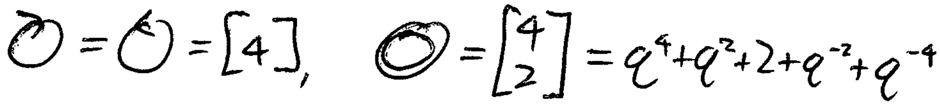
<u>Kuperberg</u> proved the 'equinumeration' theorem for $n=3$.

Even better, RS_3 has an obvious basis (diagrams with no loops, bigons or squares) which is <u>not quite the same</u> as the (dual?) canonical basis.

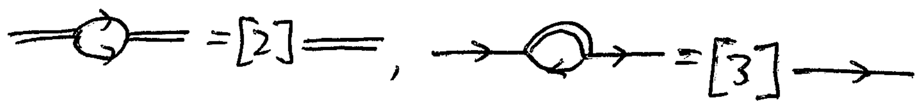
⑥ At $n=4$, we see everything!

The vertices are  and .

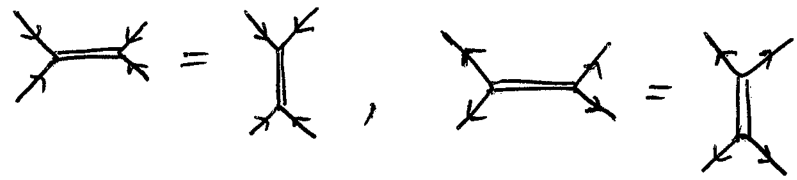
The relations are (correctly conjectured by D. Kim)



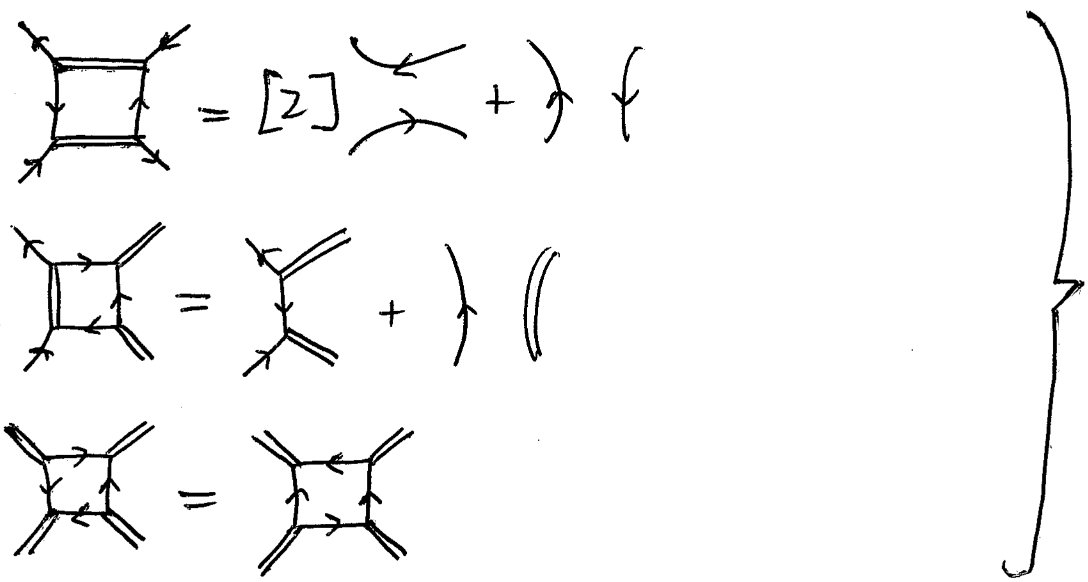
$= [4]$, $= \begin{bmatrix}4\\2\end{bmatrix} = q^4+q^2+2+q^{-2}+q^{-4}$ (loop)



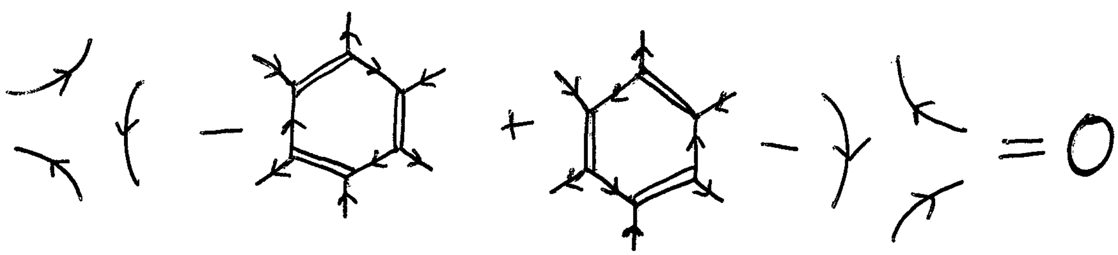
$=[2]$, $=[3]$ (bigon)



(I=H)

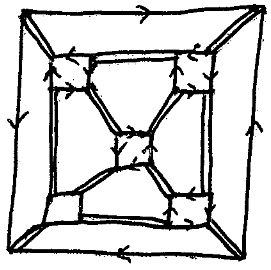


$= [2]$ (square-change)



$=0$ (Kekulé - hence the name...)

Now $RS_4 \neq IS_4$, and in fact End(ϕ) is infinite dimensional.

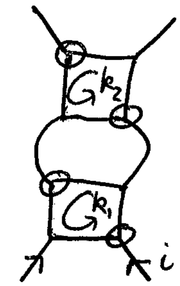
For example  is not evaluable in IS_4, but is in RS_4

(a truncated octahedron)

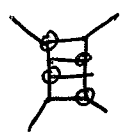
7

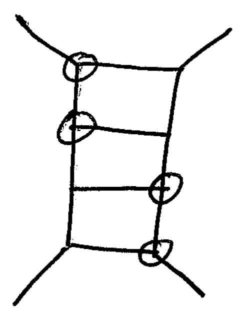
Let's understand the braiding. First we need to check  is the inverse of .

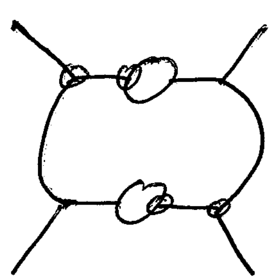
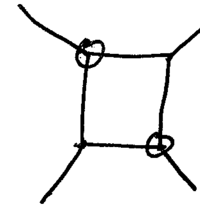
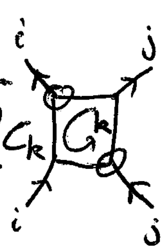
Calculating –

 $= \sum_{k_1,k_2} b_{k_1} b_{k_2}$ 

(We'll leave out all the coefficients, and indeed individual terms from here on, and just see the schematics of the calculation...

 means – some linear combination of diagrams with skeleton , and varying internal labels)

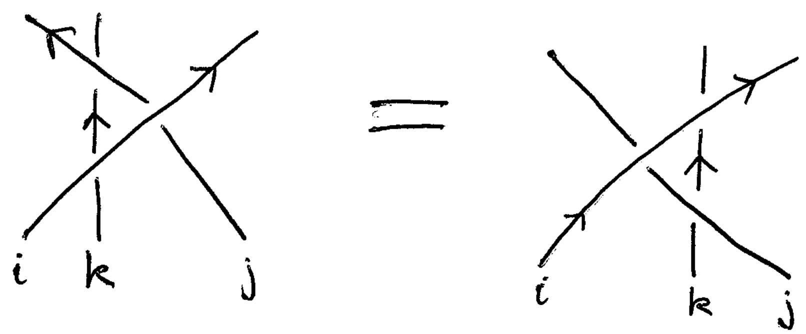
= (square-change in the middle) 

= (I=H relation)  = (bygones)  $= \sum_k c_k$ 

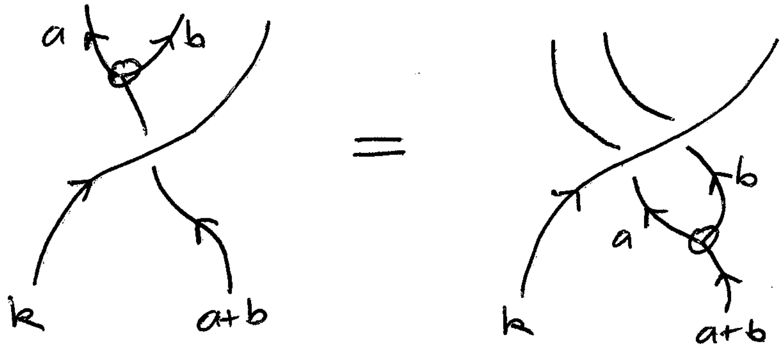
If you actually keep track of all the coefficients, just a single term survives, the $k=0$ term.

8 Next we want to check the Yang-Baxter equation. We'll do it at the same level of detail — trust me (!) that the magic of q-numbers makes all the coefficient come out right.

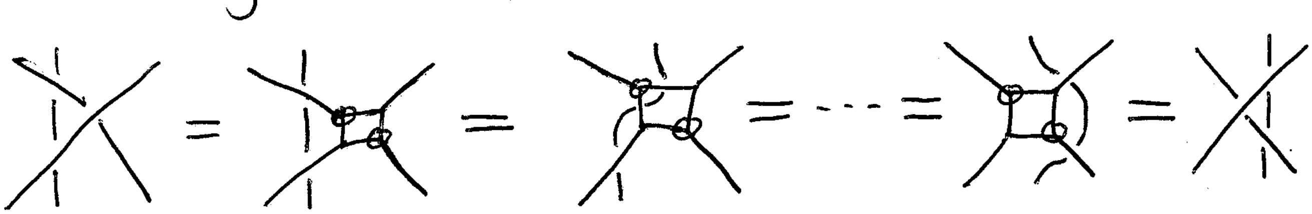
Instead of directly attacking the Yang-Baxter equation



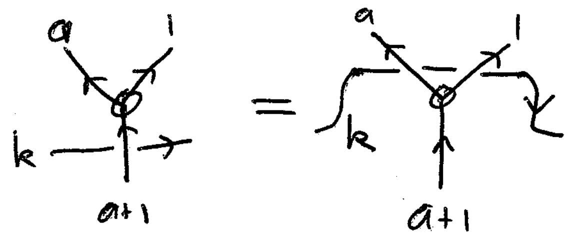
we'll follow Noah's approach of last week, and show the braiding is natural:



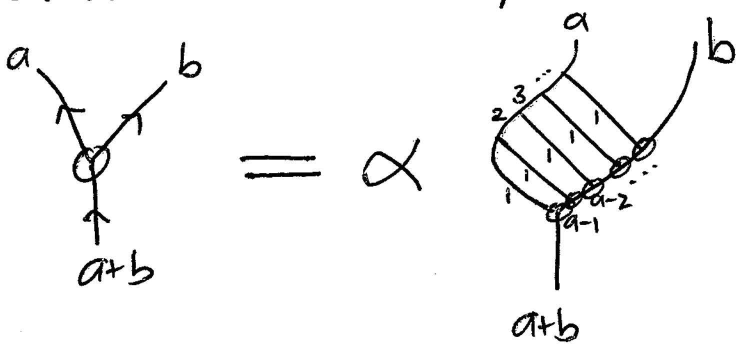
It's easy after that —



In fact, we only need to check naturality for

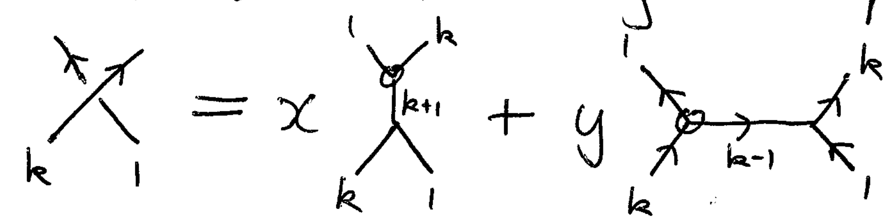


because of the decomposition

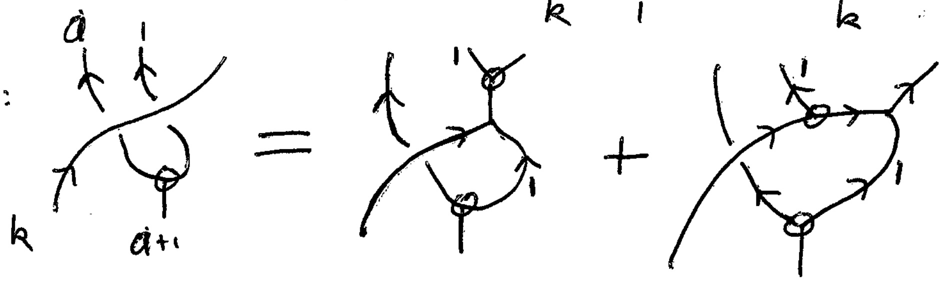


(using the bigon relation)

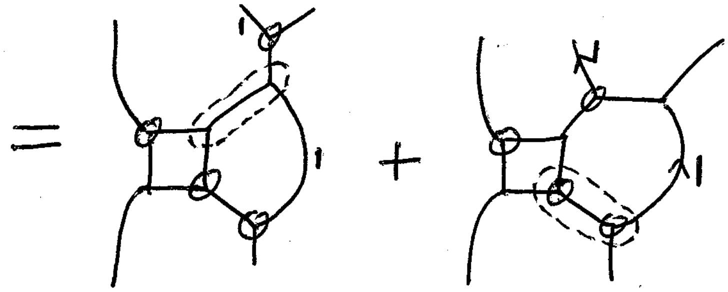
The braiding with a strand labelled by 1 is particularly simple



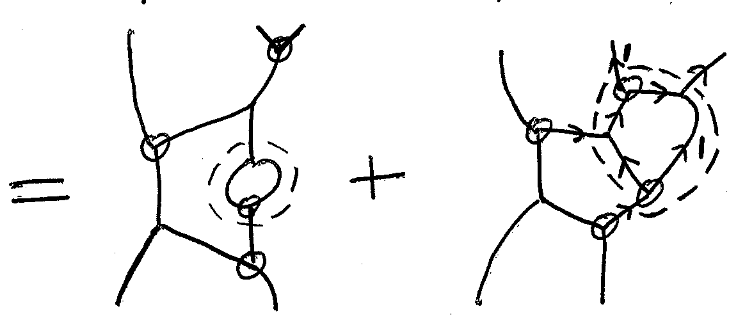
Thus:



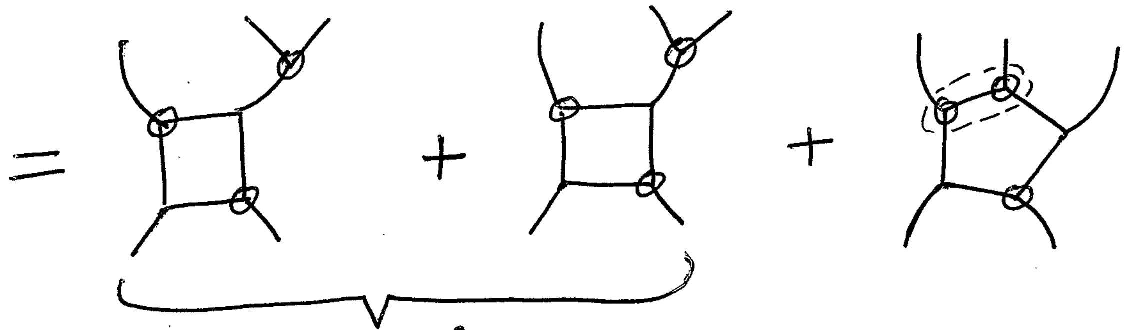
(expand one braiding)



(then the other)

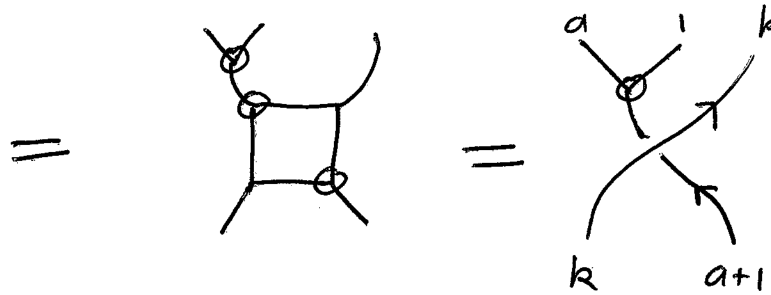


(I=H relations, in marked places)



these sets of terms cancel!

(let bigons be bygones, and change the marked square)



(more magic – the coefficients in the surviving terms are exactly right to be the braiding.)

Given this is page 10 already, perhaps I won't say anything about "Theorem 2" – the equivalence with the representation category. If you really care, you can see a sketch of the argument at

http://math.berkeley.edu/~scott/math/GeneratorsAndRelationsForRepUqsln_Slides.pdf