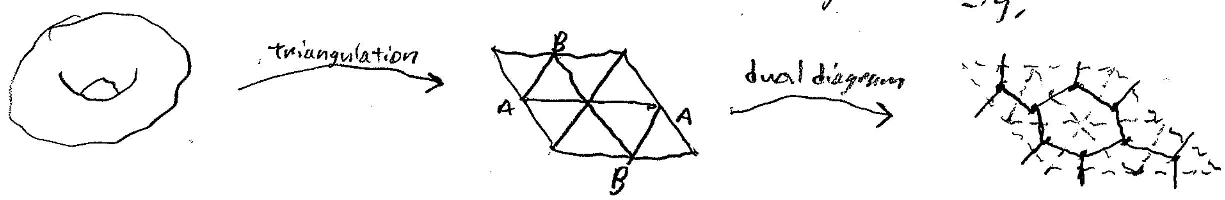
Invariants of 2-manifolds via Triangulation

can describe closed 2-mnfds by a triangulation. However, this description + very combinatorial b/c equivalence isn't nicely local. The solution to this problem is to pass to the dual diagram. E.g.

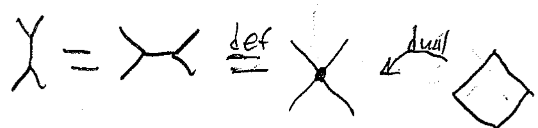


To recover the triangulation we also need a cyclic ordering of edges at each vertex. In the oriented case this can stay implicit. We'll deal with non-oriented later.

Now we have a category.

<u>Objects</u> disjoint unions of pts.

<u>Morphisms</u> 3-valent cyclically ordered graphs with oriented edges.

Since  we can instead have arbitrary graphs,

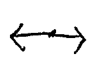
Bigons are , and one-gons are 

Similar argument to last week gives that $F(\bullet)$ is a <u>symmetric</u> (not necessarily commutative) Frobenius algebra, with $\varepsilon(1) = 1$.

<u>Ex</u> semi-simple algebra with $\varepsilon = \frac{\mathrm{Tr}}{\dim A}$

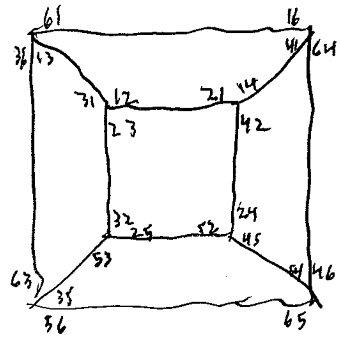
How to compute such an invariant? If our algebra A has a basis s.t. $V_i \cdot V_j = V_k$ for some k, then we can just count labellings. This is called a <u>state sum</u>

<u>Ex</u> Let $A = M_n(k)$, $\varepsilon = \frac{\mathrm{Tr}}{n}$,

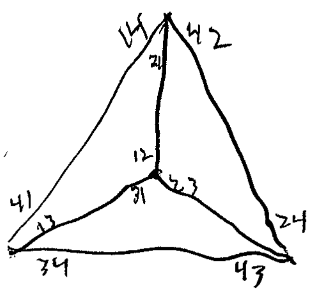
Choose the basis e_{ij} of matrix elements. Write dual diagram in terms of  and . Each  outputs $\sum_{i,j} \frac{e_{ij}}{\sqrt{n}} \otimes \frac{e_{ji}}{\sqrt{n}}$, Each  gives $\frac{n}{\sqrt{n}^3}$ if the cyclic product is 1 and 0 else.

So the value of the state sum is the number of ways of writing labels ij at each end of each edge such that edges look like ij ji vertices like , and then multiply this number by $\left(\frac{1}{\sqrt{n}}\right)^{\#V}$.

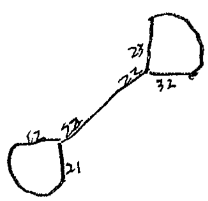
So:



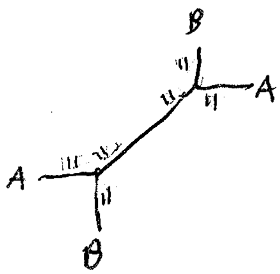
$$\longrightarrow n^6 \cdot \left(\frac{1}{\sqrt{n}}\right)^8 = n^2$$



$$\longrightarrow n^4 \cdot \left(\frac{1}{\sqrt{n}}\right)^4 = n^2$$



$$\longrightarrow n^3 \cdot \left(\frac{1}{\sqrt{n}}\right)^2 = n^2$$



$$\longrightarrow n \cdot \left(\frac{1}{\sqrt{n}}\right)^2 = 1$$

Observe all such labellings come from a labelling of the faces. When looking at a corner look left for j and right for i. Therefore,

$$F(M) = n^{\#F} \cdot \left(\frac{1}{\sqrt{n}}\right)^{\#V} = n^{\#F - \frac{1}{2}\#V} = n^{-\chi}$$ is an invariant.

Ex Let $A = \mathbb{C}[G]$ and $\varepsilon = \frac{1}{\#G} \mathrm{Tr}$.

Here let's think about the original triangulation instead. The state sum becomes label each oriented edge with some $g \in G$, Every oriented path around a Δ is 1, and take $\#\text{states} \cdot \left(\frac{1}{\sqrt{\#G}}\right)^{\#\text{Faces}}$.

To count the states pick a base vertex and a path to each other vertex. To label the remaining edges we need a map $\mathrm{Hom}(\pi_1, G)$,

so $F(M) = \#\mathrm{Hom}(\pi_1, G) \cdot \#G^{\#V-1} \cdot \left(\frac{1}{\sqrt{G}}\right)^{\#F} = \#\mathrm{Hom}(\pi_1, G)\, \#G^{-\chi - 1}$!

$\mathbb{C}[G] \cong \bigoplus_{V \text{ irrep}} \mathrm{End}(V)$ implies $\displaystyle\sum_V \left(\frac{\#G}{\dim V}\right)^{\chi} = \frac{\#\mathrm{Hom}(\pi_1, G)}{\#G}$

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non-orentable case

We throw in a "flipping bigon"  with long edges contracted. Denoted by  (with flavors  etc.) There are only two new relations!

 $= 1$ and  $=$ .

On the algebra side $a^{**} = a$, $(ab)^* = b^* a^*$.

Ex

For $\mathbb{R}[G]$ the natural $*$ is $g^* = g^{-1}$

For $M_n(\mathbb{C})$ the natural $*$ is $M \to \overline{M^T}$ not $\mathbb{C}$-linear.

So, we look over $\mathbb{R}$.

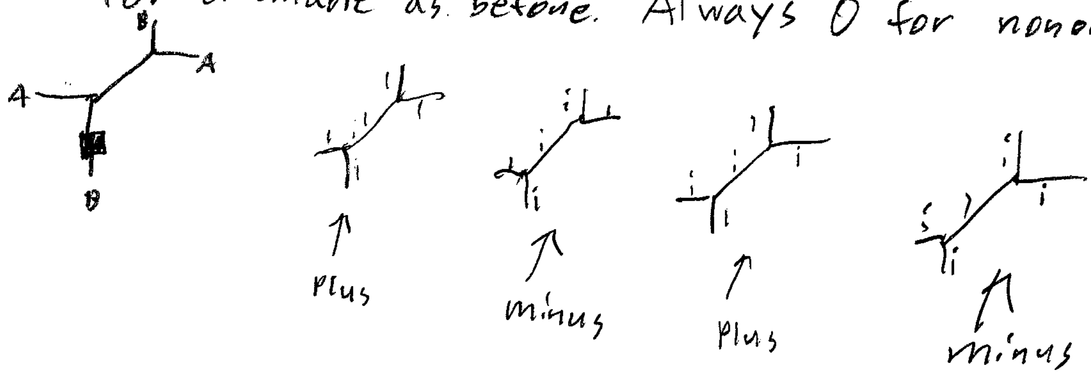
$$\mathbb{R}[G] \cong \left(\bigoplus_{V \text{real}} M_{\dim V}(\mathbb{R})\right) \oplus \left(\bigoplus_{V \text{complex}} M_{\frac{\dim V}{2}}(\mathbb{C})\right) \oplus \left(\bigoplus_{V \text{quat.}} M_{\frac{\dim V}{2}}(\mathbb{H})\right)$$

Now we need to compute the invariant for $M_n(\mathbb{R})$, $M_n(\mathbb{C})$, and $M_n(\mathbb{H})$

$\mathbb{R}$: $n^{-\chi}$ just as before

$\mathbb{C}$: $(2n)^{-\chi}$ for orientable as before. Always 0 for nonorientable.

e.g.



$\mathbb{H}$: $(4n)^{-\chi}$ for orientable, $(-2n)^{-\chi}$ for non-orientable.

So: $$\frac{\# \mathrm{Hom}(\pi_1, G)}{\# G} = \sum_{V \text{ real}} \left(\frac{\# G}{\dim V}\right)^{\chi} + \sum_{V \text{ quat.}} \left(-\frac{\# G}{\dim V}\right)^{\chi}$$