Possible additional topics

- 2-categories
- Vassiliev finite type invariants
- 4 dimensions

  etc.

2-dims via pants

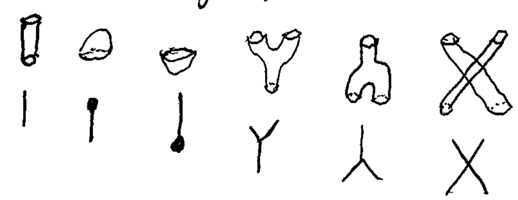
Let's mimic the approach from last week for 2-dims.

$\mathcal{C}_{top}$: Objects are disjoint unions of circles.
Morphisms are cobordisms

$\mathcal{C}_{univ}$: Generated by the object $\circ$

morphisms:

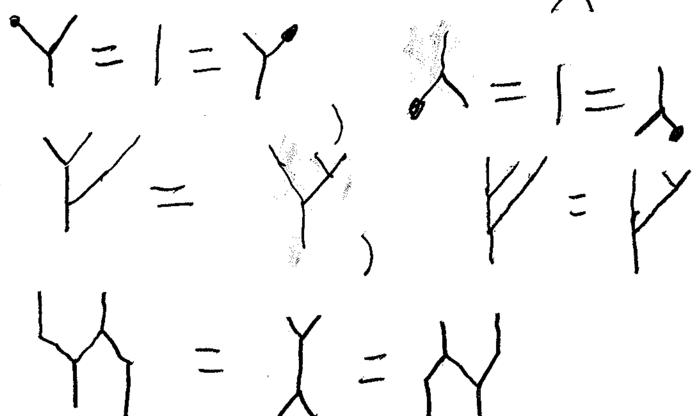
(shorthand)



monoidal

symmetric

Relations:



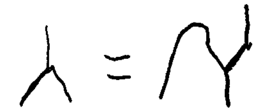
Now we consider symmetric monoidal functors $F: \mathcal{C}_{univ} \to \underline{Vect}$.
Let $V = F(\circ)$. The structure that carries over from $\mathcal{C}$ is:

• a multiplication $F(Y)$
• a trace $F(\text{!})$
• a unit $F(\text{¡})$

Satisfying

• mult. + unit mak V a commutative ring
• $\langle x \mid y \rangle = \varepsilon(xy)$ is non-denerate, symm.

Why? Let $\cap = \lambda$ and $\cup = Y$. We see $\text{Ч} = | = \text{Ν}$. Thus, as last week we see that $\cup$ determines $\cap$.

Now we can show that • and ε determine $F(\lambda)$: 

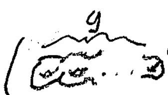
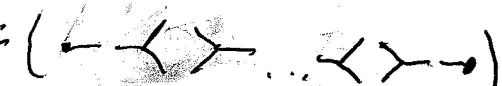
Such an algebra is called a <u>commutative Frobenius algebra</u>.
Notice that any Frobenius algebra has a <u>coalgebra</u> structure given by $F(\lambda)$.

> **!** If H is a Hopf algebra then (H, ε) is a Frobenius algebra but, the Hopf algebra Δ is <u>not</u> the Frob. algebra coproduct

<u>Ex.</u> V = algebra of functions on G with ε pulling out the x_1-coeff.

So any commutative Frob. alg. gives an invariant of 2-manifolds.
If (A_1, ε_1) and (A_2, ε_2) are two comm. Frob. algebras then $(A_1 \times A_2, \varepsilon_1 + \varepsilon_2)$ is a Frob. algebra, and the corresponding functor is the direct product.

Suppose A is semisimple. Then $A = \mathbb{C} \times \dots \times \mathbb{C}$ and $\varepsilon = \sum \varepsilon_i$. So enough to look at the case $(\mathbb{C}, \varepsilon)$. Here $\varepsilon: \mathbb{C} \to \mathbb{C}$ is just multiplication by some constant λ.

So: $F($$) = F($$) = 1 \mapsto 1 \mapsto \lambda^{-1} \otimes 1 \mapsto \lambda^{-1} \mapsto \dots \mapsto \lambda^{-g+1} \mapsto \lambda^{-g} \otimes 1 \mapsto \lambda^{-g} \mapsto \lambda$
$= \lambda^{1-g}$.

One non-semisimple example. $A = \mathbb{C}[x]/x^n$, $\varepsilon\left(\sum_{i<n} a_i x^i\right) = a_{n-1}$.
Here pairing is $\omega\left(\sum b_{ij}\, x^i \otimes x^j\right) = \varepsilon\left(\sum b_{ij}\, x^{i+j}\right) = \sum_{i+j=n-1} b_{ij}$.
So isom $A \to A^*$ sends $x^i \mapsto (x^{n-1-i})^*$. So copairing $\mathbb{C} \to A \otimes A$ sends $1 \mapsto \sum x^i \otimes x^{n-1-i}$
So comultiplication sends $x^m \mapsto \sum_{i<n} x^i \otimes x^{n-1-i+m}$.
Thus $F(\Diamond)$ is $x^m \mapsto \sum_{i<n} x^{n-1+m} = n\, \delta_{m0}\, x^{n-1}$.
So $F(M) = \varepsilon\left((n x^{n-1})^g\right) = n \cdot \delta_{g1}$, unless $n=1$ when $F(M) = n^g$.
So it detects whether M has exactly genus 1.

<u>Exercises</u>

1. $F(\Diamond)$ invertable iff A is semisimple.
*2. What about non-orientable