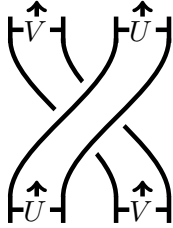


# Noah Snyder: Research Statement

## INTRODUCTION

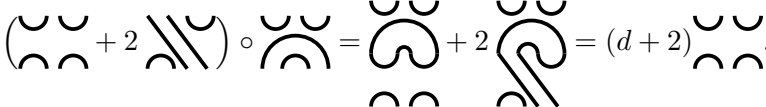
My research involves relationships between algebra (particularly representation theory) and low-dimensional topology (particularly knot theory). This connection comes up in two ways. On the one hand, one can interpret pictures as maps of representations. On the other hand, one can study algebras whose elements are formal sums of pictures.

A typical example of the first type is the  $R$ -matrix for the quantum group  $U_q(\mathfrak{sl}_2)$  which gives the following translation between a crossing in a framed link and a map of representations:



$$\leftrightarrow \left( u \otimes v \mapsto e^{\frac{H \otimes H}{2}} \sum_n q^{\frac{n(n+1)}{2}} \frac{(1-q^2)^n}{[n]_q!} F^n(v) \otimes E^n(u) \right).$$

A typical example of the second type is the Temperley-Lieb algebra  $TL_d$ . The following computation illustrates the definition of  $TL_d$  and its multiplication rule,



$$\left( \text{two crossings} + 2 \text{ one crossing} \right) \circ \text{pair of arcs} = \text{two crossings} + 2 \text{ one crossing} = (d+2) \text{ pair of arcs}.$$

My research fits into three main programs. First, I plan to investigate invariants of hyperbolic knots and hyperbolic 3-manifolds coming from the De Concini-Kac specialization of quantum  $U_q(\mathfrak{sl}_2)$  at roots of unity [4, 6] in the spirit of the work of Kashaev and Reshetikhin [19, 20, 39]. Second, I plan to study and classify fusion categories via Jones's theory of planar algebras [17]. Finally, I plan to investigate commutators coming from half-twists [13, 18, 40] and possible applications to the theory of finite type knot invariants and invariants of knots on non-orientable surfaces modulo stabilization.

## RESEARCH OBJECTIVES AND METHODS

### **Project 1:** *Invariants of hyperbolic knots and quantum groups at roots of unity*

**Background.** Two major themes in modern low-dimensional topology are the geometrization program of W. Thurston, and quantum topology in the style of Jones, Witten, Reshetikhin, Turaev, etc. An important current goal in quantum topology is to find a connection between these two programs. In particular, since a typical prime knot has a unique finite volume hyperbolic structure, one would like to find a connection between certain quantum invariants and this hyperbolic structure. A big breakthrough in this direction is the

hyperbolic volume conjecture of Kashaev [21] and Murakami-Murakami [29]. This says that the behavior as  $m$  goes to infinity of the  $m$ th colored Jones polynomial (corresponding to the  $m$ -dimensional representation of  $U_q(\mathfrak{sl}_2)$ ) at an  $m$ th root of unity is determined by the hyperbolic volume of the knot.

A different direction where a connection has been found between hyperbolic structures and quantum invariants is through quantum-style invariants which use the hyperbolic structure as a direct input. A hyperbolic structure on a 3-manifold  $M$  is given by a map  $\text{Hom}(\pi_1(M), \text{SL}_2(\mathbb{C}))$  modulo the conjugation action of  $\text{SL}_2(\mathbb{C})$ . Such a map is closely related to a choice of coloring of a knot with representations in the De Concini-Kac form of the quantum group at a root of unity [4,6]. In analogy with the Reshetikhin-Turaev [37] construction of knot invariants from ordinary quantum groups, Kashaev and Reshetikhin [19,20,39] constructed invariants of knots together with a choice of hyperbolic structure on the complement. There is a related construction due to Baseilhac and Benedetti [1,2] which is analogous to the Ocneanu-Turaev-Viro invariants [31,42].

**Research in progress.** In current work with Scott Morrison [28], we clarify several details from the papers of Kashaev and Reshetikhin to give a fully rigorous construction of their new knot invariants. This invariant is a function on the space  $\text{Hom}(G_m(K), \text{SL}_2(\mathbb{C}))/\text{SL}_2(\mathbb{C})$  where  $G_m(K)$  is the  $m$ th generalized knot group (a slight modification of  $\pi_1(S^3 - K)$  defined in [22,43]), and the action of  $\text{SL}_2$  is by conjugation. We prove several basic results, for example, that the value of the function on the trivial point is  $|J_m(e^{\frac{2\pi i}{m}})|^2$  (the same expression that occurs in the hyperbolic volume conjecture). Furthermore, we give the first computations of this invariant for nontrivial knots. For several small twist knots we compute explicitly the entire Kashaev-Reshetikhin knot invariant at a third root of unity and at a fifth root of unity. For a larger collection of twist knots we compute the value of the knot invariant at the finite volume hyperbolic point.

In related work in progress with Nicolai Reshetikhin [38], we extend some of the results of Kashaev and Reshetikhin from  $\mathfrak{sl}_2$  to a general simple quantum group. In [39] they show that the R-matrix induces an outer automorphism in a formal neighborhood of the trivial central character. For  $\mathfrak{sl}_2$  they showed that this outer automorphism is given by rational functions and thus extends to a Zariski open set. For more general quantum groups we show that this outer automorphism is not given by rational functions, but instead by compositions of rational functions and square roots. In particular, we show that the outer automorphism extends to an explicit analytic (rather than just formal) neighborhood of the trivial central character. Thus we get invariants of knots together with certain maps in  $\text{Hom}(\pi_1(S^3 - K), G)/G$  for arbitrary Lie group  $G$ .

### Proposed research.

- (1) In the above constructions, for technical reasons we are forced to use the simple bimodule  $\text{End}(V)$  everywhere that we would like to use the irreducible representation  $V$ . Removing this obstacle would clarify the general theory and dramatically speed up computations. Such a speed-up might allow us to find numerical evidence relating these invariants to the volume conjecture.

- (2) Clarify the square-root ambiguity in the construction for a general Lie group.
- (3) Give a complete description of the tensor category of representations of  $U_q(\mathfrak{sl}_2)$  relating the description at generic points in [5] with the description at the special points in [36]. Use this description to define a surgery invariant of hyperbolic 3-manifolds.
- (4) Implement computations of Beseilhac-Benedetti invariants and compare them to Kashaev-Reshetikhin invariants. Find a relationship between these invariants and try to prove it, probably via Turaev's shadow world.
- (5) Look at the Habiro half divided powers integral form of  $U_q(\mathfrak{sl}_2)$  [12] specialized to roots of unity. This Hopf algebra is the Drinfel'd double of the upper Borel, so it has a more natural  $R$ -matrix than the other specializations. Use this integral form to clarify the above theory.

## Project 2: *Fusion categories and planar algebras*

**Background.** A planar algebra is a combinatorial model for a pivotal fusion category (or relatedly, a subfactor) introduced by Jones [17] (closely related to Kuperberg's spiders [24]). A typical example of a pivotal fusion category is the semi-simplified category of representations of a quantum group at a root of unity. Ordinary algebra takes place on a line: you can multiply on the left or on the right, but you can't multiply above or beneath. In a planar algebra there are many more "multiplications." For example, you can multiply three elements by gluing them together in the following way:



The relationship between planar algebras and fusion categories is that a planar algebra describes the Hom spaces between tensor powers of a chosen fundamental object in a fusion category. However, the theory of planar algebras can be built up on its own and gives an independent elementary perspective on quantum groups.

This program has two main goals: to describe known fusion categories or subfactors by explicit generators and relations, and to find new fusion categories and subfactors by generator and relation constructions. The first of these goals has already been accomplished in many instances, starting with rank 2 quantum groups [24], and continuing with work of Westbury, Morrison, and Peters (on  $U_q(\mathfrak{so}(7))$  [44],  $U_q(\mathfrak{sl}_n)$  [25], and Haagerup's exceptional subfactor [35], respectively). The second of these goals is still in its infancy, but recent work of Jones [15] suggests optimism.

Planar algebras are best for studying fusion categories with a tensor generator which is not too complicated [3, 14, 15, 41]. In particular, they are good for studying fusion categories with a tensor generator of small dimension. Most previous work in classifying fusion categories has focused on categories of small rank [33, 34] or categories of small

integer global dimension [8–10], so one can expect planar algebra techniques to give new results.

**Research in Progress.** In a preprint with Emily Peters and Scott Morrison [27], we give a presentation of the subfactor planar algebra  $D_{2n}$  (corresponding to a “quantum subgroup” of  $U_q(\mathfrak{sl}_2)$  in the sense of the quantum MacKay correspondence [23, 30]) by generators and relations. We use this presentation to give a direct construction of the  $D_{2n}$  planar algebra and to prove all of its basic properties. In work in progress [26], we also use this planar algebra to prove several new identities between classical knot polynomials. The simplest of these identities says that if  $K$  is a knot (not a link!), then the colored Jones polynomial coming from a 3-dimensional representation at a 24th root of unity is always 2. After deriving these knot polynomial identities, we prove that they all come from coincidences of small tensor categories, and we prove these identities using three types of unexpected symmetries among finite quantum groups (an exceptional case of level-rank duality, coincidences of small Dynkin diagrams, and generalized Kirby-Melvin symmetry). One can think of these coincidences as quantum analogues of coincidences like  $A_5 \cong \mathrm{PSL}_2(\mathbb{F}_5)$ .

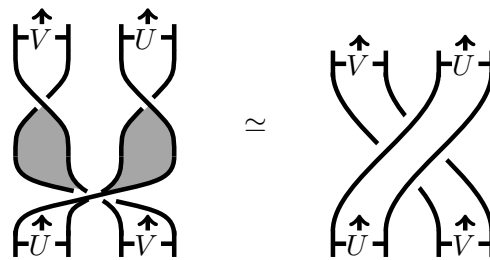
**Proposed research.**

- (1) Modify fundamental results in the theory of planar algebras [14–17] to the case of pivotal fusion categories by using oriented unshaded planar diagrams in place of the unoriented shaded diagrams relevant to subfactor theory.
- (2) Classify all oriented unitary planar algebras with  $d < 2$ . (In the unoriented case this is the *ADE* classification of [11, 23, 30].)
- (3) Discover and prove new coincidences between categories coming from quantum groups that are not predicted by level-rank duality. For example, we discovered coincidences between a modular category coming from  $\mathfrak{g}_2$  and a modular category coming from  $\mathfrak{so}_3$  [11].
- (4) Generalize our description of the  $D_{2n}$  planar algebra to quantum subgroups of other quantum groups (see [32]). Use the techniques of [14] combined with [24], to classify and describe all quantum subgroups of rank 2 quantum groups ( $\mathfrak{sl}_3$ ,  $\mathfrak{so}_5$ , and  $\mathfrak{g}_2$ ).
- (5) Characterize the  $U_q(\mathfrak{g}_2)$  as a braided tensor category. In particular, classify all braided tensor categories with a tensor generator  $X$  such that  $X \otimes X \cong 1 \oplus X \oplus A \oplus B$  with  $X$ ,  $A$ , and  $B$  irreducible.
- (6) Automate the search for subfactors of small index and small rank and for unitary fusion categories of small rank with a tensor generator of small dimension. This project is joint work with Scott Morrison and Emily Peters, and preliminary progress can be found at [tqft.net/wiki/Atlas\\_of\\_subfactors](http://tqft.net/wiki/Atlas_of_subfactors). Eventually we plan to make an online atlas summarizing this work much like the Knot Atlas [katlas.math.toronto.edu](http://katlas.math.toronto.edu).

**Project 3:** *Commutors coming from half-twists and cactus theory*

**Background.** Using an observation of Arkady Berenstein, Henriques and Kamnitzer [13] show the following. Let  $H$  be a Hopf algebra and  $t \in H$  be an invertible element such that conjugation by  $t$  is a coalgebra automorphism, then  $\text{swap} \circ (t^{-1} \otimes t^{-1})\Delta(t)$  gives a family of natural isomorphisms  $V \otimes W \rightarrow W \otimes V$ . Furthermore, they prove that if  $t$  satisfies some additional conditions then these commutors give an action of the Cactus group, an analogue the braid group. In [18], Kamnitzer and Tingley show that the  $R$ -matrix for a quantum group is given by a similar formula.

**Research in progress.** In [40], Peter Tingley and I rigorously interpret the following relation between pictures in the category of representations of a quantum group:



We introduce the notion of a half-ribbon Hopf algebra, which is a Hopf algebra  $H$  along with a distinguished element  $t$  in  $H$  such that  $(H, (t^{-1} \otimes t^{-1})\Delta(t), t^{-2})$  is a ribbon Hopf algebra. The element  $t$  is closely related to the topological ‘half-twist’, which twists a ribbon by 180-degrees. We show that  $U_q(\mathfrak{g})$  is a (topological) half-ribbon Hopf algebra, but that  $t^{-2}$  is not the standard ribbon element.

Using techniques from [40] we can show that for any Henriques-Kamnitzer commutator there is a consistent theory of quantum traces. This shows that quantum groups give invariants of cactus knots (which are not a type of knot, but rather the cactus analogue of a knot), which are described by generators and relations analogous to the Reidemeister moves.

**Proposed research.**

- (1) Using old work of Drinfel’d [7], I will show that any finite type invariant of knots also extends to an invariant of cactus knots. This extension uses an explicit formula for the  $n$ -strand cactus crossing as a power series in singular knots modulo the Vassiliev relations.
- (2) Cactus tangles are a planar subalgebra of the planar algebra of formal power series in singular tangles modulo the Vassiliev relations. I plan to look for other planar subalgebras, which would give other combinatorial analogues of knots which finite type invariants extend to.
- (3) According to an observation of Andre Henriques elements of the cactus group can be described by paths on non-orientable surfaces modulo cobordism. Similarly, cactus knots can be interpreted as circles embedded in a nonorientable surface

modulo cobordism. However, not all isotopies in the surface are allowed from this perspective because they don't respect the "direction of time" in the cactus group. I plan to implement a computer search for finite type invariants of knots which satisfy these additional isotopies.

- (4) Generalize our results classifying which ribbon elements extend to half-ribbon elements from  $U_q(\mathfrak{sl}_2)$  [40] to an arbitrary quantum group.
- (5) Investigate under what circumstances a quantum double has a half-ribbon structure.
- (6) Use the fact that an orientable surface connect sum a mobius strip is non-orientable to relate these invariants of knots drawn on non-oriented surfaces modulo stabilization to virtual knots.
- (7) Explore which properties of a finite type invariant of a knot are reflected in its values on cactus knots. In particular, investigate when two invariants can agree on all cactus knots.

#### REFERENCES

- [1] Stéphane Baseilhac and Riccardo Benedetti. Quantum hyperbolic invariants of 3-manifolds with  $\mathrm{PSL}(2, \mathbb{C})$ -characters. *Topology*, 43(6):1373–1423, 2004. arXiv:math/0306280 MR2081430.
- [2] Stéphane Baseilhac and Riccardo Benedetti. Classical and quantum dilogarithmic invariants of flat  $\mathrm{PSL}(2, \mathbb{C})$ -bundles over 3-manifolds. *Geom. Topol.*, 9:493–569 (electronic), 2005. arXiv:math/0409282 MR2140989.
- [3] Dietmar Bisch and Vaughan Jones. Singly generated planar algebras of small dimension. *Duke Math. J.*, 101(1):41–75, 2000. MR1733737.
- [4] C. De Concini, V. G. Kac, and C. Procesi. Quantum coadjoint action. *J. Amer. Math. Soc.*, 5(1):151–189, 1992. MR1124981.
- [5] C. De Concini, C. Procesi, N. Reshetikhin, and M. Rosso. Hopf algebras with trace and representations. *Invent. Math.*, 161(1):1–44, 2005. arXiv:math/0304313 MR2178656.
- [6] Corrado De Concini and Victor G. Kac. Representations of quantum groups at roots of 1. In *Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989)*, volume 92 of *Progr. Math.*, pages 471–506. Birkhäuser Boston, Boston, MA, 1990. MR1103601.
- [7] V. G. Drinfel'd. Quasi-Hopf algebras. *Algebra i Analiz*, 1(6):114–148, 1989. MR1047964.
- [8] Pavel Etingof, Shlomo Gelaki, and Viktor Ostrik. Classification of fusion categories of dimension  $pq$ . *Int. Math. Res. Not.*, (57):3041–3056, 2004. MR2098028 arXiv:math.QA/0304194.
- [9] Pavel Etingof, Dmitri Nikshych, and Victor Ostrik. Weakly group-theoretical and solvable fusion categories, 2008. arXiv:0809.3031.
- [10] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik. On fusion categories. *Ann. of Math. (2)*, 162(2):581–642, 2005. arXiv:math/0203060 MR2183279.
- [11] Frederick M. Goodman, Pierre de la Harpe, and Vaughan F. R. Jones. *Coxeter graphs and towers of algebras*, volume 14 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York, 1989. MR999799.
- [12] Kazuo Habiro. An integral form of the quantized enveloping algebra of  $\mathfrak{sl}_2$  and its completions. *J. Pure Appl. Algebra*, 211(1):265–292, 2007. arXiv:math/0605313 MR2333771.
- [13] André Henriques and Joel Kamnitzer. Crystals and coboundary categories. *Duke Math. J.*, 132(2):191–216, 2006. arXiv:math/0406478 MR2219257.
- [14] Vaughan Jones. Annular structure of subfactors, 2003. pre-print available at <http://math.berkeley.edu/~vfr>.

- [15] Vaughan Jones. Quadratic tangles in planar algebras, 2003. “pre”-pre-print available at <http://math.berkeley.edu/~vfr>.
- [16] Vaughan Jones, Dimitri Shlyakhtenko, and Kevin Walker. An orthogonal approach to the subfactor of a planar algebra, 2008. [arXiv:0807.4146](https://arxiv.org/abs/0807.4146).
- [17] Vaughan F. R. Jones. Planar algebras, I. [arXiv:math.QA/9909027](https://arxiv.org/abs/math/9909027).
- [18] Joel Kamnitzer and Peter Tingley. The crystal commutor and drinfeld’s unitarized r-matrix, 2007. [arXiv:0707.2248](https://arxiv.org/abs/0707.2248).
- [19] R. Kashaev and N. Reshetikhin. Invariants of links with flat connections in their complements.ii. holonomy r-matrices related to quantized universal enveloping algebras at roots of 1, 2002. [arXiv:math/0202212](https://arxiv.org/abs/math/0202212).
- [20] R. Kashaev and N. Reshetikhin. Invariants of tangles with flat connections in their complements.i. invariants and holonomy r-matrices, 2002. [arXiv:math/0202211](https://arxiv.org/abs/math/0202211).
- [21] R. M. Kashaev. The hyperbolic volume of knots from the quantum dilogarithm. *Lett. Math. Phys.*, 39(3):269–275, 1997. [MR1434238 arXiv:q-alg/9601025](https://arxiv.org/abs/math/9601025).
- [22] A. J. Kelly. Groups from link diagrams, ph. d. thesis, u. warwick, 1990.
- [23] Alexander Kirillov, Jr. and Viktor Ostrik. On a  $q$ -analogue of the McKay correspondence and the ADE classification of  $\mathfrak{sl}_2$  conformal field theories. *Adv. Math.*, 171(2):183–227, 2002. [MR1936496 arXiv:math.QA/0101219](https://arxiv.org/abs/math/0101219).
- [24] Greg Kuperberg. Spiders for rank 2 Lie algebras. *Comm. Math. Phys.*, 180(1):109–151, 1996. [MR1403861 arXiv:q-alg/9712003](https://arxiv.org/abs/q-alg/9712003).
- [25] Scott Morrison. A diagrammatic category for the representation theory of  $U_q(\mathfrak{sl}_n)$ , 2007. [arXiv:0704.1503](https://arxiv.org/abs/0704.1503).
- [26] Scott Morrison, Emily Peters, and Noah Snyder. Knot polynomial identities and quantum group coincidences. forthcoming, will be available at <http://tqft.net/identities>.
- [27] Scott Morrison, Emily Peters, and Noah Snyder. Skein theory for the  $D_{2n}$  planar algebras. [arXiv:0808.0764](https://arxiv.org/abs/0808.0764).
- [28] Scott Morrison and Noah Snyder. Kashaev-reshetikhin invariants of knots. in progress.
- [29] Hitoshi Murakami and Jun Murakami. The colored Jones polynomials and the simplicial volume of a knot. *Acta Math.*, 186(1):85–104, 2001. [MR1828373, arXiv:math/9905075](https://arxiv.org/abs/math/9905075).
- [30] Adrian Ocneanu. Quantized groups, string algebras and Galois theory for algebras. In *Operator algebras and applications, Vol. 2*, volume 136 of *London Math. Soc. Lecture Note Ser.*, pages 119–172. Cambridge Univ. Press, Cambridge, 1988. [MR996454](https://arxiv.org/abs/math/9906454).
- [31] Adrian Ocneanu. Chirality for operator algebras. In *Subfactors (Kyuzeso, 1993)*, pages 39–63. World Sci. Publ., River Edge, NJ, 1994. [MR1317353](https://arxiv.org/abs/math/9407353).
- [32] Adrian Ocneanu. The classification of subgroups of quantum  $SU(N)$ . In *Quantum symmetries in theoretical physics and mathematics (Bariloche, 2000)*, volume 294 of *Contemp. Math.*, pages 133–159. Amer. Math. Soc., Providence, RI, 2002. [MR1907188](https://arxiv.org/abs/math/971188).
- [33] Victor Ostrik. Pre-modular categories of rank 3, 1995. [arXiv:math.QA/0503564](https://arxiv.org/abs/math/9503564).
- [34] Viktor Ostrik. Fusion categories of rank 2. *Math. Res. Lett.*, 10(2-3):177–183, 2003. [MR1981895 arXiv:math.QA/0203255](https://arxiv.org/abs/math/0203255).
- [35] Emily Peters. *Constructing the Haagerup subfactor using planar algebras*. PhD thesis, University of California, Berkeley, 2009. forthcoming.
- [36] N. Reshetikhin and V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.*, 103(3):547–597, 1991. [MR1091619](https://arxiv.org/abs/math/91091619).
- [37] N. Yu. Reshetikhin and V. G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.*, 127(1):1–26, 1990. [MR1036112](https://arxiv.org/abs/math/901036112).
- [38] Nicolai Reshetikhin and Noah Snyder. Braidings for  $U_q(\mathfrak{g})$  at roots of 1. in progress.
- [39] R.Kashaev and N. Reshetikhin. Braiding for the quantum  $\mathfrak{gl}_2$  at roots of unity, 2004. [arXiv:math/0410182](https://arxiv.org/abs/math/0410182).
- [40] Noah Snyder and Peter Tingley. The half-twist for  $U_q(\mathfrak{g})$  representations, 2008. [arXiv:0810.0084](https://arxiv.org/abs/0810.0084).

- [41] Dylan P. Thurston. From dominoes to hexagons, 2004. [arXiv:math/0405482](#).
- [42] V. G. Turaev and O. Ya. Viro. State sum invariants of 3-manifolds and quantum  $6j$ -symbols. *Topology*, 31(4):865–902, 1992. [MR1191386](#).
- [43] Masaaki Wada. Group invariants of links. *Topology*, 31(2):399–406, 1992. [MR1167178](#).
- [44] Bruce W. Westbury. Invariant tensors for the spin representation of  $\mathfrak{so}(7)$ . *Math. Proc. Cambridge Philos. Soc.*, 144(1):217–240, 2008. [arXiv:math/0601209](#) [MR2388243](#).