NOTES FOR GEOMETRIC ANALYSIS

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Contents

These notes were originated from the lecture notes for the graduate course Geometric Analysis at Zhejiang University in Spring 2022. This note only covers the contents from the first few weeks of the semester due to my laziness. Luckily, it is still largely self-contained if you are familiar with Riemannian geometry, especially the second fundamental form, first and second variations of arc length and Jacobi fields. The primary reference is [3].

1. Preliminaries

We recall some basic definitions to avoid ambiguity. The curvature tensor is defined as

$$R_{XY}(Z) := D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z.$$

The second fundamental form of a Riemannian submanifold $N \subset M$ is defined as

$$\bar{D}_X Y = D_X Y - II(X, Y),$$

where $D_X Y := \tan \bar{D}_X Y$ and $-II(X, Y) := \operatorname{nor} \bar{D}_X Y$.

2. Second fundamental form of Geodesic Spheres

We will develop a volume comparison theorem originally proved by Bishop. Let M^m be a complete Riemannian manifold. Take $p \in M$. In polar normal coordinates $(\exp_p; r, \theta)$, let $G(r, \theta) := \sqrt{\det g_{ij}}$ be the Jacobian of the exponential map, where $g_{ij} = \langle \partial_{\theta_i}, \partial_{\theta_j} \rangle$. With a slight abuse of notation, we have $\partial_r = d\exp_p(\partial_r)$ and $\partial_{\theta_i} = d\exp_p(\partial_{\theta_i})$. Then we can write the volume element as $dVol_{B_r} = G(r, \theta)\, d\theta\, dr$ and by Gauss Lemma, we have $dA_{\partial B_r} = G(r, \theta)\, d\theta$.

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Let $\vec{II}$ be the second fundamental form of $\partial B_p(r)$ at $x=(\theta,r)$ and $\vec{II}(X,Y)=II(X,Y)\partial_r$. Then

$$II_{ij}=-\langle D_{\partial_i}\partial_j,\partial_r\rangle=\langle\partial_j,D_{\partial_i}\partial_r\rangle=\langle\partial_j,D_{\partial_r}\partial_i\rangle=\frac{1}{2}\partial_r g_{ij},\tag{2.1}$$

where we abbreviate $\partial_i:=\partial_{\theta_i}$. We further compute

$$\partial_r II_{ij}=-\partial_r\langle D_{\partial_i}\partial_j,\partial_r\rangle=-\langle D_{\partial_r}D_{\partial_i}\partial_j,\partial_r\rangle=-\langle\partial_r,R(\partial_r,\partial_i)\partial_j\rangle-\langle\partial_r,D_{\partial_i}D_{\partial_r}\partial_j\rangle,\tag{2.2}$$

where the second equality follows from the fact $D_{\partial_r}\partial_r=0$ thanks to the equation of geodesics. The first term in (2.2) is equal to $-R(\partial_r,\partial_j,\partial_r,\partial_i)=-R_{mimj}$ and we compute the last term as follows.

$$\langle\partial_r,D_{\partial_i}D_{\partial_r}\partial_j\rangle=\langle\partial_r,D_{\partial_i}D_{\partial_j}\partial_r\rangle=-\langle D_{\partial_i}\partial_r,D_{\partial_j}\partial_r\rangle=-\langle II_{ik}g^{kl}\partial_l,II_{jr}g^{rs}\partial_s\rangle=-II_{ik}g^{kr}II_{rj},$$

where the second equality follows from $\langle\partial_r,\partial_r\rangle=1$ and the third follows from (2.1). We write

$$II^2_{ij}:=II_{ik}g^{kl}II_{lj}$$

and hence

$$\partial_r II_{ij}=-R_{mimj}+II_{ik}g^{kr}II_{rj}=-R_{mimj}+II^2_{ij}.\tag{2.3}$$

Furthermore,

$$(D_{\partial_r}II)(\partial_i,\partial_j)=\partial_r II_{ij}-II(D_{\partial_r}\partial_i,\partial_j)-II(\partial_i,D_{\partial_r}\partial_j)=-R_{mimj}-II^2_{ij}$$

and hence for all $X,Y\in T\partial B_p(r)$,

$$(D_{\partial_r}II)(X,Y)=-R(\partial_r,X,\partial_r,Y)-\sum_i II(X,e_i)II(e_i,Y)\tag{2.4}$$

where $e_1,\cdots,e_{m-1}\in T\partial B_p(r)$ are mutually orthonormal vector fields. Taking trace by metric, by (2.1), we have

$$\partial_r H=\partial_r(II_{ij}g^{ij})=\partial_r II_{ij}g^{ij}-2II^2_{ij}g^{ij}=-R_{mimj}g^{ij}-II^2_{ij}g^{ij},$$

thus

$$\partial_r H=-\mathrm{Ric}\,(\partial_r,\partial_r)-|II|^2.\tag{2.5}$$

3. ODE of area elements

We compute

$$\partial_r G=\frac{1}{2}g^{ij}\partial_r g_{ij}G=g^{ij}II_{ij}G=HG.\tag{3.1}$$

Differentiating again and use (2.5)

$$\partial_r^2 G=\left(-\mathrm{Ric}\,(\partial_r,\partial_r)-|II|^2+H^2\right)G.\tag{3.2}$$

Since

$$H^2=\left(\sum_{i=1}^{m-1}II(e_i,e_i)\right)^2\le(m-1)\sum_{i=1}^{m-1}II(e_i,e_i)II(e_i,e_i)\le(m-1)\sum_{i,j=1}^{m-1}II(e_i,e_j)II(e_j,e_i)\le(m-1)|II|^2,$$

we get

$$\partial_r^2 G\le\left(-\mathrm{Ric}\,(\partial_r,\partial_r)+\frac{m-2}{m-1}H^2\right)G=-\mathrm{Ric}\,(\partial_r)G+\frac{m-2}{m-1}(\partial_r G)^2/G.\tag{3.3}$$

Remark 3.1. Since any smooth metric on M^m is locally Euclidean, we have the initial conditions

$$G \sim r^{m-1}, \quad \partial_r G \sim (m-1) r^{m-2}$$

as $r \to 0$.

Remark 3.2. If M is a simply connected space form of constant sectional curvature K, then all the inequalities above become equalities, thus (2.5) and (3.3) become

$$\begin{aligned}\partial_r H &= -(m-1)K - \frac{1}{m-1} H^2, \\ \partial_r^2 G &= -(m-1)KG + \frac{m-2}{m-1}(\partial_r G)^2/G.\end{aligned}$$

4. Volume comparison

Theorem 4.1 (Bishop)**.** *Fix $p \in M$. Suppose $Ric(x) \geq (m-1)\mathcal{K}(d(x))g(x)$ for some function $\mathcal{K}$, where $d(x) = d_g(x,p)$. If $\bar{G}$ is a solution of*

$$\begin{cases} \frac{d^2}{dr^2}\bar{G} = -(m-1)\mathcal{K}(r)\bar{G} + \frac{m-2}{m-1}(\frac{d}{dr}\bar{G})^2/\bar{G}, \\ \lim_{r\to 0} \frac{\bar{G}}{r^{m-1}} = 1, \quad \lim_{r\to 0} \frac{d}{dr}\bar{G}/(m-1)r^{m-2} = 1. \end{cases} \tag{4.1}$$

Then within the cut locus of p:

(1) $G(\theta,r)/\bar{G}(r)$ is a nonincreasing function of r.

(2) $H(\theta,r) \leq \bar{H}(r) := \frac{d}{dr}\ln \bar{G}(r)$.

Proof. Set

$$f(r,\theta) = G^{\frac{1}{m-1}}(r,\theta) > 0, \quad \bar{f}(r) = \bar{G}^{\frac{1}{m-1}}(r).$$

It follows from (3.1) that

$$\partial_r f = \frac{1}{m-1} G^{\frac{1}{m-1}} H = \frac{1}{m-1} H f$$

and hence

$$\partial_r^2 f \leq \frac{1}{m-1}\left(\frac{1}{m-1}H^2 f + f\left(-\mathrm{Ric}\,(\partial_r) - \frac{1}{m-1}H^2\right)\right) = -\mathcal{K} f.$$

As a result,

$$\begin{cases} \partial_r^2 f \leq -\mathcal{K} f, \\ \lim_{r\to 0}\frac{f(\theta,r)}{r} = 1, \quad \partial_r f(\theta,0) = 1, \end{cases}$$

and one can verify that

$$\begin{cases} \frac{d^2}{dr^2}\bar{f} \leq -\mathcal{K}\bar{f}, \\ \lim_{r\to 0}\frac{\bar{f}(r)}{r} = 1, \quad \frac{d}{dr}\bar{f}(0) = 1. \end{cases}$$

Consider $F(\theta,r) = f(\theta,r)/\bar{f}(r) > 0$ if $r \in (0, \rho_0(\theta))$, where $(\theta, \rho_0(\theta))$ is a cut point. We have

$$\lim_{r\to 0} F(\theta,r) = \lim_{r\to 0}\frac{r+o(r)}{r+o(r)} = 1$$

and

$$\partial_r \ln F = \frac{1}{f}\partial_r f - \frac{1}{\bar{f}}\partial_r \bar{f} = \frac{1}{m-1}\left(H - \bar{H}\right).$$

and so

$$
\begin{aligned}
\partial_r^2 F =& F\left(\partial_r^2 \ln F + (\partial_r \ln F)^2\right)\\
\leq& \frac{1}{m-1}F\left(-\mathrm{Ric}\left(\partial_r\right) - \frac{1}{m-1}H^2 + (m-1)\mathcal{K} + \frac{1}{m-1}\bar{H}^2\right) + F\left(\frac{1}{m-1}\left(H-\bar{H}\right)\right)^2\\
\leq& \frac{2F}{(m-1)^2}\left(\bar{H}-H\right)\bar{H} = -\frac{2}{m-1}\bar{H}\partial_r F = -2\frac{\partial_r \bar{f}}{\bar{f}}\partial_r F.
\end{aligned}
$$

Thus,

$$\partial_r\left(\bar{f}^2\partial_r F\right) = 2\bar{f}\partial_r\bar{f}\partial_r F + \bar{f}^2\partial_r^2 F \leq 0,$$

which implies

$$\bar{f}(r)^2\partial_r F(\theta,r) \leq \bar{f}(\varepsilon)^2\partial_r F(\theta,\varepsilon) = \left(\bar{f}\partial_r f - f\partial_r\bar{f}\right)(\varepsilon) \to 0.$$

On the other hand, we get

$$\partial_r F(\theta,r) = \frac{F}{m-1}\left(H-\bar{H}\right) \leq 0,$$

which implies $H \leq \bar{H}$. $\square$

Corollary 4.2 (Myers Theorem). *Let M^m be a complete Riemannian manifold and Ric $\geq (m-1)K_0 g$ for some constant $K_0 > 0$. Then we have* $\operatorname{diam} M \leq \frac{\pi}{\sqrt{K_0}}$.

Proof. For any $p \in M$ fixed, one can check $\bar{G} := K_0^{-\frac{m-1}{2}}\sin^{m-1}(\sqrt{K_0}r)$ is a desired function satisfies (4.1) when $\mathcal{K} = K_0$ in our case. Then Bishop theorem implies

$$G(\theta,r) \leq \bar{G}(r)$$

and hence $G(\theta,\frac{\pi}{\sqrt{K_0}}) = 0$. Thus the conjugate point appears no later than $\frac{\pi}{\sqrt{K_0}}$ along any geodesic This completes the proof thanks to the Hopf-Rinow theorem. $\square$

Corollary 4.3. *Let M^m be a complete Riemannian manifold and Ric $\geq (m-1)K_0 g$ for some constant K_0. Let $\bar{M}_{K_0}$ be a space form of sectional curvature K_0.*

Fix a point $p \in M$ and let

$$A(r) = \mathit{area}\ \left(\partial B_r\right), \quad V(r) = \mathit{Vol}\ \left(B_r\right), \quad \bar{A}(r) = \mathit{area}\ \left(\partial\widetilde{B_r}\right), \quad \bar{V}(r) = \mathit{Vol}\ \left(\widetilde{B_r}\right),$$

where $B_r = B_p(r)$ and $\widetilde{B_r}$ are geodesic balls in M and $\bar{M}_{K_0}$, respectively. Then

(1) $A(r)/\bar{A}(r)$ is nonincreasing in r;
(2) $V(r) \leq \bar{V}(r)$.

Proof. Denote

$$C(r) := \left\{\theta \in S_p M : \exp_p(s\theta) \text{ is minimizing until } s = r\right\},$$

then one can observe that $C(r_2) \subset C(r_1)$ if $r_1 \leq r_2$. By Bishop theorem, we have

$$\frac{G(\theta,r_2)}{\bar{G}(r_2)} \leq \frac{G(\theta,r_1)}{\bar{G}(r_1)}, \quad \forall\theta \in C(r_2)$$

Integrating over $C(r_2)$ gives

$$\begin{aligned}\bar{G}(r_1)A(r_2) =&\bar{G}(r_1)\int_{C(r_2)} G(\theta, r_2)\, d\theta \le \bar{G}(r_2)\int_{C(r_2)} G(\theta, r_1)\, d\theta\\ \le&\bar{G}(r_2)\int_{C(r_1)} G(\theta, r_1)\, d\theta = \bar{G}(r_2)A(r_1),\end{aligned}$$

and hence $A(r_2)\bar{A}(r_1) \le \bar{A}(r_2)A(r_1)$.

Moreover,

$$\frac{d}{dr}\ln\frac{V}{\bar{V}} = \frac{A(r)}{V(r)} - \frac{\bar{A}(r)}{\bar{V}(r)} = \frac{1}{V(r)\bar{V}(r)}\left(\int_0^r A(r)\bar{A}(\tau) - \bar{A}(r)A(\tau)\, d\tau\right) \le 0,$$

which implies that $V(r) \le \bar{V}(r)$. $\square$

5. Laplacian comparison and Hessian comparison

Let M^m be a complete manifold. Suppose $p \in M$ is a fixed point and let us consider the distance function $\rho(x) = d_g(p, x)$ to p. The distance function in general is not smooth due to the presence of cut-points. However, it is a Lipschitz function with Lipschitz constant 1. In particular, we have $\operatorname{grad}\rho = \partial_r$ and $|\operatorname{grad}\rho|^2 = 1$ almost everywhere.

Lemma 5.1. *Suppose $x \in M$ is not a cut-point of p. Then the Hessian of ρ is $\mathcal{H}^\rho(e_i, e_j) = (D(D\rho))(e_i, e_j) = II(e_i, e_j)$ for all $e_i, e_j \perp \operatorname{grad}\rho(x)$ and $\Delta\rho(x) = H(x)$, where II is the second fundamental form of $\partial B_p(\rho(x))$.*

Proof. For all $e_i, e_j \perp \operatorname{grad}\rho(x)$,

$$\begin{aligned}\mathcal{H}^\rho(e_i, e_j) =&D(D\rho)(e_i, e_j) = D_{e_j}(d\rho)(e_i) = e_j\left(d\rho(e_i)\right) - d\rho(D_{e_j}e_i)\\ =&e_j e_i\rho - (D_{e_j}e_i)\rho = -\langle\operatorname{grad}\rho, D_{e_j}e_i\rangle = II_{ij}\end{aligned}$$

Taking traces gives $\Delta\rho = H$. $\square$

Theorem 5.2 (Laplacian comparison theorem)**.** *Let M^m be a complete Riemannian manifold and $Ric \ge (m-1)K_0 g$ for some constant K_0. Then in the sense of distributions, we have*

$$\Delta\rho(x) \le \begin{cases}(m-1)\sqrt{K_0}\cot(\sqrt{K_0}\rho), & K_0 > 0,\\ \frac{m-1}{\rho}, & K_0 = 0,\\ (m-1)\sqrt{-K_0}\coth(\sqrt{-K_0}\rho), & K_0 < 0.\end{cases} \tag{5.1}$$

Proof. If ρ is smooth at $x = (r, \theta)$, then by the lemma above and the Bishop theorem, we have $\Delta\rho(x) = H(\rho, \theta) \le \bar{H}(\rho)$, where $\bar{H}$ is the right hand side of (5.1).

Now we show that for all $\phi \in C_0^\infty(M), \phi \ge 0$, we have $\int_M \rho(\Delta\phi) \le \int_M \phi\bar{H}(\rho)$. In polar coordinates,

$$\begin{aligned}\int_M \phi\bar{H}(\rho) = \int_0^\infty\int_{C(r)} \phi\bar{H}(r)G(r,\theta)\, d\theta\, dr = \int_{S_pM}\int_0^{R(\theta)} \phi\bar{H}(r)G(r,\theta)\, dr\, d\theta\\ \ge \int_{S_pM}\int_0^{R(\theta)} \phi H(r)G(r,\theta)\, dr\, d\theta = \int_{S_pM}\int_0^{R(\theta)} \phi\partial_r G(r,\theta)\, dr\, d\theta\end{aligned}$$

where $R(\theta)$ is the cut-point on $\exp_p(s\theta)$, that is, the maximum value of $r > 0$ such that $\exp_p(s\theta)$ minimizes up to $s = r$. Furthermore, we integrate by parts and get

$$\begin{aligned}\int_M \phi \bar{H}(\rho) &= \int_{S_pM} (\phi G)(R(\theta), \theta) - \int_{S_pM} \int_0^{R(\theta)} \partial_r \phi \phi G(r, \theta)\, dr\, d\theta \\ &\geq - \int_M \partial_r \phi = - \int_M \langle \operatorname{grad} \phi, \operatorname{grad} \rho \rangle = \int_M \rho \Delta \phi,\end{aligned}$$

which completes the proof. □

Theorem 5.3 (Hessian comparison theorem)**.** *Let M^m be a complete Riemannian manifold such that for all $p \in M$, all $X, Y \in T_pM$, $K(\Pi_{X,Y})(p) \geq K_0$ for some constant K_0. Then*

$$\mathcal{H}^\rho(X, X) \leq \frac{1}{m-1} \bar{H} g(X, X), \quad \forall X \in T_pM$$

at all points $p = (\theta, r)$ where ρ is smooth.

Proof. If $X = f\partial_\rho$, then

$$\begin{aligned}\mathcal{H}^\rho(X, X) =& D_X(d\rho)(X) = X(X\rho) - (D_X X)\rho = Xf - \langle \operatorname{grad} \rho, D_X X \rangle \\ =& f\left(\partial_\rho f - f\langle \partial_\rho, D_{\partial_\rho}(f\partial_\rho) \rangle\right) = f\langle D_{\partial_\rho}\partial_\rho, f\partial_\rho \rangle = \frac{1}{2} f^2 \partial_\rho \langle \partial_\rho, \partial_\rho \rangle = 0,\end{aligned}$$

where we use the fact $\langle \partial_r, \partial_r \rangle = 1$.

Now we can assume with loss of generality that $X \perp \partial_\rho$ and $g(X, X) = 1$. By parallel transport, we assume X is defined on $\exp_p(s\theta)$. Recall from (2.4) that

$$(D_{\partial_r} II)(X, X) = -R(\partial_\rho, X, \partial_\rho, X) - \sum_i |II(X, e_i)|^2 \leq -K_0 - |II(X, X)|^2,$$

where the last inequality follows from $\langle X, X \rangle = 1$.

Let $g(r) = II(X, X)$ and $\bar{g}(r) = \bar{II}(X, X)$. Then by Lemma 5.1, it suffices to show $g(r) \leq \bar{g}(r)$.

Since X is obtained from parallel transport, $D_{\partial_r} X = 0$ and hence $\partial_r g = (D_{\partial_r} II)(X, X) \leq -K_0 - g^2$. Moreover, $\partial_r \bar{g} = -K_0 - \bar{g}^2$. We claim that $g(r) \sim \frac{1}{r} + O(r)$ and leave the vertification of the claim in the subsequent remark. Thus, we see from

$$\partial_r(g - \bar{g}) \leq -(g - \bar{g})(g + \bar{g})$$

that

$$(g - \bar{g})(r) \leq (g - \bar{g})(\varepsilon) \exp\left(\int_\varepsilon^r (g + \bar{g})\right) \to 0, \quad \text{as } \varepsilon \to 0,$$

which proves the theorem. □

Remark 5.4. Now we show the asymptotic behaviour of $II(X, X)$ with $X \perp \partial_r$ that was used above by studying its asymptotic behavior via Jacobi fields.

Fix $p \in M$. Let $v \in S_pM$ and $\{e_i\}$ be an orthonormal frame in $S_v(T_pM) = S_pM$. Since $\exp_p(t(v + se_i))$ is a variation of $\gamma(t) := \exp_p(tv)$, we know that $\partial_{\theta_i} = \partial_s|_{s=0} \exp_p(t(v + se_i))$ is a Jacobi field along γ. Hence it satisfies

$$D_{\dot{\gamma}} D_{\dot{\gamma}} \partial_{\theta_i} - R(\dot{\gamma}, \partial_{\theta_i})\dot{\gamma}$$

with initial data $\partial_{\theta_i}(0)=0$ and $D_{\dot\gamma}\partial_{\theta_i}=e_i$. By parallel transport, we obtain a orthonormal frame $\{E_i(t)\}$ along γ such that $E_i(0)=e_i$ for $i=1,\dots,m-1$ and $E_m(t)=\dot\gamma(t)$. We write $\partial_{\theta_i}=\sum_{j=1}^{m-1}\theta_i^j(t)E_j(t)$, then

$$\begin{cases}\ddot\theta_i^j(t)=-\sum\theta_i^k s_{kj},\\ \theta_i^j(0)=0,\quad \dot\theta_i^j(0)=\delta_i^j,\end{cases}$$

where $s_{kj}=R(\dot\gamma,E_k,\dot\gamma,E_j)$. Now

$$II_{ij}:=II(\partial_{\theta_i},\partial_{\theta_j})=\frac{1}{2}\partial_r g_{ij}=\sum_k\dot\theta_i^k\theta_j^k$$

and therefore

$$A_{ij}:=II(E_i,E_j)=\tilde\theta_i^k II_{kl}\tilde\theta_j^l=\sum_r\tilde\theta_i^k\dot\theta_k^r\theta_l^r\tilde\theta_j^l=\tilde\theta_i^k\dot\theta_k^j,$$

where $E_i=\tilde\theta_i^k\partial_{\theta_k}$, $\theta_l^k\tilde\theta_j^l=\delta_j^k$. It follows that

$$\begin{aligned}\theta_i^j&=\delta_i^j r+\frac{1}{6}s_{ij}(0)r^3+O(r^4),\\ \dot\theta_i^j&=\delta_i^j+\frac{1}{2}s_{ij}(0)r^2+O(r^3),\end{aligned}$$

and hence $A_{ij}=\frac{1}{r}\delta_{ij}+O(r)$.

6. Cheng's maximal diameter theorem

Proposition 6.1. *Let M^m be a complete Riemannian manifold and $Ric\geq(m-1)K_0g$ for some constant K_0. Let $\bar M_{K_0}$ be a space form of sectional curvature K_0. For $0\leq r_1\leq r_2,r_3\leq r_4<\infty$, there holds*

$$\frac{V(r_2)-V(r_1)}{\bar V(r_2)-\bar V(r_1)}\geq\frac{V(r_4)-V(r_3)}{\bar V(r_4)-\bar V(r_3)},$$

and equality holds if and only if
(1) $C(r_1)=C(r_4)$,
(2) and for all $r\in[0,r_4]$ and $\theta\in C(r_1)$, $G(\theta,r)=\bar G(r)$,
where $C(r)$ is as defined in Corollary 4.3.

Proof. Two cases are needed to consider:

Case A $r_1\leq r_2\leq r_3\leq r_4$: Integrating $A(t_1)\bar A(t_2)\geq A(t_2)\bar A(t_1)$ over $r_1\leq t_1\leq r_2$ and $r_3\leq t_2\leq r_4$, we find

$$(V(r_2)-V(r_1))\left(\bar V(r_4)-\bar V(r_3)\right)\geq(V(r_4)-V(r_3))\left(\bar V(r_2)-\bar V(r_1)\right)$$

Case B $r_1\leq r_3<r_2\leq r_4$: We have

$$\begin{aligned}&(V(r_2)-V(r_1))\left(\bar V(r_4)-\bar V(r_3)\right)\\ =&(V(r_2)-V(r_3)+V(r_3)-V(r_1))\left(\bar V(r_4)-\bar V(r_2)+\bar V(r_2)-\bar V(r_3)\right)\\ \geq&\left(\bar V(r_2)-\bar V(r_3)+\bar V(r_3)-\bar V(r_1)\right)(V(r_4)-V(r_2)+V(r_2)-V(r_3))\\ =&(V(r_4)-V(r_3))\left(\bar V(r_2)-\bar V(r_1)\right),\end{aligned}$$

where we use case A in the inequality. □

Theorem 6.2 (Cheng)**.** *Let M^m be a complete Riemannian manifold and $Ric \geq (m-1)K_0 g$ for some constant $K_0 > 0$. If the diameter of M is $\frac{\pi}{\sqrt{K_0}}$, then M is isometric to $S^m\left(\frac{1}{\sqrt{K_0}}\right)$.*

Proof. Let $p, q \in M$ such that $d_g(p,q) = \operatorname{diam} M := d$.

Step 1: By the volume comparison theorem (Theorem 4.1), we know that

$$\mathrm{Vol}\,(B_d(p)) \leq \frac{\bar{V}(d)}{\bar{V}(\frac{d}{2})}\mathrm{Vol}\,\left(B_{\frac{d}{2}}(p)\right) = 2\mathrm{Vol}\,\left(B_{\frac{d}{2}}(p)\right).$$

As a result,

$$\mathrm{Vol}\,(B_d(p)) + \mathrm{Vol}\,(B_d(q)) \leq 2\mathrm{Vol}\,\left(B_{\frac{d}{2}}(p)\right) + 2\mathrm{Vol}\,\left(B_{\frac{d}{2}}(q)\right).$$

Suppose $B_{\frac{d}{2}}(p) \cap B_{\frac{d}{2}}(q) \neq \varnothing$, we take $z \in B_{\frac{d}{2}}(p) \cap B_{\frac{d}{2}}(q)$, then $d_g(p,q) \leq d_g(p,z) + d_g(z,q) < \frac{d}{2} + \frac{d}{2} = d$, which is a contradiction. So the intersection must be empty and hence

$$\mathrm{Vol}\,\left(B_{\frac{d}{2}}(p)\right) + \mathrm{Vol}\,\left(B_{\frac{d}{2}}(q)\right) \leq \mathrm{Vol}\,(M).$$

On the other hand,

$$2\mathrm{Vol}\,(M) = \mathrm{Vol}\,(B_d(p)) + \mathrm{Vol}\,(B_d(q)).$$

By combining the estimates above, we get an equality, so

$$\mathrm{Vol}\,\left(B_{\frac{d}{2}}(p)\right) = \frac{1}{2}\mathrm{Vol}\,(B_d(p)).$$

We write $V(r) := \mathrm{Vol}\,(B_r(p))$, then

$$\frac{V(\frac{d}{2})}{\bar{V}(\frac{d}{2})} = \frac{V(d)}{\bar{V}(d)} = \frac{V(d) - V(\frac{d}{2})}{\bar{V}(d) - \bar{V}(\frac{d}{2})}.$$

By applying Proposition 6.1 with $r_1 = 0, r_2 = r_3 = \frac{d}{2}$ and $r_4 = d$, we know $C(d) = S^{m-1}$ and $G(\theta, r) = \bar{G}(r)$ for all $r \in [0,d]$ and all θ.

Step 2, Metrics: Since $\partial_r G = HG$, $\partial_r \bar{G} = \bar{H}\bar{G}$, and $G, \bar{G}$ have the same initial condition, we know from the conclusion of the first step that

$$H = \bar{H} = (m-1)\sqrt{K_0}\cot(\sqrt{K_0}r).$$

We then find

$$\partial_r g_{ij} = 2II_{ij} = 2\sqrt{K_0}\cot(\sqrt{K_0}r)g_{ij},$$

and hence

$$g_{ij} = \frac{\sin^2(\sqrt{K_0}r)}{K_0}\delta_{ij}.$$

This shows that $g = dr^2 + \frac{\sin^2(\sqrt{K_0}r)}{K_0}d\sigma^2$ holds in $B_d(p)$.

Step 3, Diffeomorphism: It follows from $C(d) = S^{m-1}$ that $B_d(p)$ is a normal neighbourhood of p and hence $\exp : B := B^{T_pM}(d) \subset T_pM \to B_d(p)$ is diffeomorphism. In particular, we know that $S^{m-1}(1/\sqrt{K_0})$ is homeomorphic to M if the cut-locus of p is $\{q\}$

since $S^{m-1}(1/\sqrt{K_0})$ is homeomorphic to $B/\partial B$. More specifically, we consider an isomorphism $I: T_pM \to T_{\tilde{p}}(S^m(\frac{1}{\sqrt{K_0}}))$ for some $\tilde{p} \in S^m(\frac{1}{\sqrt{K_0}})$. Then

$$\widetilde{\exp}_{\tilde{p}} \circ I \circ \exp_p^{-1} : M \setminus \{q\} = B_d(p) \to S^m(\frac{1}{\sqrt{K_0}}) \setminus \{\bar{\tilde{p}}\}$$

is a isometry, where $\bar{\tilde{p}}$ is the antipodal point of $\tilde{p}$. By continuity, M is isometric to $S^m(\frac{1}{\sqrt{K_0}})$. So now it suffices to prove $Cut(p) = \{q\}$.

We shall prove a stronger claim that for all $z \in M$, $d_g(p,z) + d_g(q,z) = d_g(p,q) = d$. This will implies that for any $z \in M \setminus \{p,q\}$, $d_g(p,z) < d$, which lies in $B_d(p)$ and shows that $Cut(p) = \{q\}$.

Suppose not, then $d_g(p,z) + d_g(q,z) > d$. Choosing $\delta_1 < d_g(p,z)$, $\delta_2 < d_g(q,z)$ such that $\delta_1 + \delta_2 = d$. We mimic the arguments in step one as follows. Note that $B_{\delta_1}(p)$ and $B_{\delta_2}(q)$ are disjoint, so

$$\text{Vol}\,(B_{\delta_1}(p)) + \text{Vol}\,(B_{\delta_2}(q)) \leq \text{Vol}\,(M).$$

On the other hand,

$$\bar{V}(\delta_1) + \bar{V}(\delta_2) = \text{Vol}\,(S^m(\frac{1}{\sqrt{K_0}})).$$

Moreover, from the volume comparison theorem, for $x \in \{p,q\}$, $\frac{\text{Vol}\,(B_\delta(x))}{\bar{V}(\delta)}$ is a nonincreasing function of δ within $0 \leq \delta < d$, so

$$\frac{\text{Vol}\,(B_\delta(x))}{\text{Vol(M)}} = \frac{\text{Vol}\,(B_\delta(x))}{\text{Vol}\,(B_d(x))} \geq \frac{\bar{V}(\delta)}{\bar{V}(d)} = \frac{\bar{V}(\delta)}{\text{Vol}\,(S^m(\frac{1}{\sqrt{K_0}}))}.$$

Combining the estimates above imply

$$1 \geq \frac{\text{Vol}\,(B_{\delta_1}(p)) + \text{Vol}\,(B_{\delta_2}(q))}{\text{Vol}\,(M)} \geq \frac{\bar{V}(\delta_1) + \bar{V}(\delta_2)}{\text{Vol}\,(S^m(\frac{1}{\sqrt{K_0}}))} = 1.$$

Thus the inequalities above are equalities, so in particular,

$$\text{Vol}\left(\overline{B_{\delta_1}(p)}\right) + \text{Vol}\left(\overline{B_{\delta_2}(q)}\right) \text{Vol}\,(B_{\delta_1}(p)) + \text{Vol}\,(B_{\delta_2}(q)) = \text{Vol}\,(M).$$

Hence, $d_g(p,z) + d_g(q,z) > d$ shows that $q \notin \overline{B_{\delta_1}(p)} \cup \overline{B_{\delta_2}(q)}$, and hence there is a neighborhood of q disjoint from this union. However, any such neighborhood would have positive volume, contradicting our assumption. Thus, the claim holds.

Hence, M is isometric to $S^m(\frac{1}{\sqrt{K_0}})$.

∎

Theorem 6.3 (Yau's linear growth of volume)**.** *Let M^m be a complete Riemannian manifold with nonnegative Ricci curvature. Then there exists $C_m > 0$ such that*

$$Vol(B_\rho(p)) \geq C_m\, Vol(B_1(p))\rho, \quad \forall \rho \in (2, \text{diam}\, M), \quad \forall p \in M.$$

Proof. Take $x \in \partial B_{1+\rho}(p)$, then by Proposition 6.1,

$$\frac{\text{Vol}\,(B_{2+\rho}(x)) - \text{Vol}\,(B_\rho(x))}{\bar{V}(2+\rho) - \bar{V}(\rho)} \leq \frac{\text{Vol}\,(B_\rho(x))}{\bar{V}(\rho)},$$

where $\bar{V}(r)$ denotes the Euclidean ball with radius r. Note that $B_1(p) \subset B_{2+\rho}(x) \setminus B_\rho(x)$, so $\mathrm{Vol}\,(B_{2+\rho}(x)) - \mathrm{Vol}\,(B_\rho(x)) \geq \mathrm{Vol}\,(B_1(p))$. Moreover, $B_\rho(x) \subset B_{2\rho+1}(p)$ implies $\mathrm{Vol}\,(B_\rho(x)) \leq \mathrm{Vol}\,(B_{2\rho+1}(p))$. Therefore,

$$\mathrm{Vol}\,(B_1(p)) \leq \frac{(2+\rho)^m - \rho^m}{\rho^m} \mathrm{Vol}\,(B_{2\rho+1}(p)) \leq C_m \rho^{-1} \mathrm{Vol}\,(B_{2\rho+1}(p)).$$

□

7. Cheeger-Gromoll splitting theorem

This theorem is originally due to [2].

Definition 7.1. *We say $\gamma : \mathbb{R} \to M$ is a line if it is a normal geodesic and any $\gamma|_{[a,b]}$ is minimal for all finite interval $[a,b] \subset \mathbb{R}$.*

Definition 7.2. *We say $\gamma_+ : \mathbb{R}_+ \to M$ is a ray if it is a normal minimizing geodesic.*

Definition 7.3. *For a ray γ_+ in M, its buseman function is defined as*

$$\beta^+(x) := \lim_{t \to \infty} \beta_t^+(x),$$

where $\beta_t^+(x) := t - d_g(\gamma_+(t), x)$.

Remark 7.4. The buseman function is well-defined due to the following two observations.

Observation 1: $\{\beta_t^+\}_t$ is uniformly bounded on compact Ω.

This is because

$$|\beta_t^+(x)| \leq d_g(x, \gamma_+(0)) \leq \max_{x \in \Omega} d_g(x, \gamma_+(0)).$$

Observation 2: For any fixed x, $\beta_t^+(x)$ is a nondecreasing function of t.

This is because

$$\beta_{t+\delta}^+(x) - \beta_t^+(x) = \delta - d_g(\gamma_+(t+\delta), x) + d_g(\gamma_+(t), x) \geq 0.$$

Hence, $\beta_t^+|_\Omega$ converges uniformly to a limit and hence β^+ is well-defined.

Theorem 7.5 (Cheeger-Gromoll)**.** *Let M^m be a complete manifold with nonnegative Ricci curvature. If there is a line in M, then M is isometric to $\mathbb{R} \times N$, where N is a $(m-1)$-dimensional Riemannian manifold with nonnegative Ricci curvature.*

Lemma 7.6. *$\beta^+(x)$ is a Lipschitz function with Lipschitz constant 1. In particular, $|\mathrm{grad}\,\beta^+| \leq 1$ if $\beta^+ \in C^1$.*

Proof. For $x, y \in M$, $t > 0$, we have

$$|\beta_t^+(x) - \beta_t^+(y)| = |d_g(\gamma_+(t), x) - d_g(\gamma_+(t), y)| \leq d_g(x, y).$$

Sending $t \to \infty$ gives the first assertion.

Now suppose $\beta^+ \in C^1$ and denote $\beta = \beta^+$ for simplicity. To prove $|\mathrm{grad}\,\beta(x)| \leq 1$, it suffices to prove $|\langle \mathrm{grad}\,\beta, v\rangle| \leq 1$ for all unit vector $v \in T_xM$. Choose γ such that $\gamma(0) = x$ and $\gamma'(0) = v$, then

$$1 \geq \frac{\beta(\gamma(t)) - \beta(x)}{t} \to \langle \mathrm{grad}\,\beta(x), v\rangle$$

as $t \to 0$, which completes the proof.

$\square$

Lemma 7.7. *Suppose γ is a line, then*

$$\begin{cases} \beta^+(x) + \beta^-(x) = 0, \forall x \in \gamma, \\ \beta^+(x) + \beta^-(x) \le 0, \forall x \in M, \end{cases}$$

where β^- is the buseman function of $\gamma(-t)$, that is, $\beta^-(x) = \lim_{t\to\infty} \beta_t^-(x)$, $\beta_t^-(x) = t - d_g(\gamma(-t), x)$.

Proof. The equality for $x \in \gamma$ is direct. On the other hand, the inequality for $x \in M$ follows from the computation

$$\beta_t^+(x) + \beta_t^-(x) = 2t - d_g(\gamma(t), x) - d_g(\gamma(-t), x) \le 2t - d_g(\gamma(t), \gamma(-t)) = 0.$$

$\square$

Lemma 7.8. *Let M^m be a complete manifold with nonnegative Ricci curvature. Suppose f is a Lipschitz function such that $f \ge 0$. If $\Delta f \le 0$ in the sense of distributions, then for all $x \in M$ and small $R > 0$,*

$$f(x) \ge \frac{n}{\omega_{n-1} R^n} \int_{B_R(x)} f, \tag{7.1}$$

where ω_{n-1} is the volume of the Euclidean unit $(n-1)$-sphere.

Proof. Let (r, θ) be the normal coordinates centered at x, where

$$ds^2 = dr^2 + r^2 g_{ij}(r, \theta)\, d\theta^i d\theta^j$$

with $\theta \in S^{n-1}$. We put $\widetilde{G}(r, \theta) = \sqrt{\det(g_{ij})}$. Note that we factor out r^2 in ds^2 since it will result in a nicer form in (7.2), and hence the volume element of ∂B_r is $r^{n-1}\widetilde{G}(r, \theta)\, d\theta$.

For a standard mollifier

$$\eta(x) = \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1}\right), & |x| < 1, \\ 0, & |x| \ge 1, \end{cases}$$

we have $f_\varepsilon(x) = (f * \eta_\varepsilon)(x) \ge 0$. Moreover, since $\eta_\varepsilon \in C_0^\infty(M)$, $\Delta f_\varepsilon = ((\Delta \eta_\varepsilon) * f)(x) \le 0$ since $\eta_\varepsilon(x - \cdot)$ is a test function. Hence, we can assume without loss of generality that $f \in C^\infty(M)$. Otherwise, we prove (7.1) for $f = f_\varepsilon$ and then letting $\varepsilon \to 0$ completes the proof.

Now we suppose $f \in C^\infty(M)$. Since $\Delta f \le 0$,

$$0 \ge \int_{B(t)} \Delta f = \int_{\partial B(t)} \frac{\partial f}{\partial r} t^{n-1} \widetilde{G}\, d\theta,$$

By a simple calculation,

$$\Delta r = \frac{n-1}{r} + \frac{\partial \ln \widetilde{G}}{\partial r}, \tag{7.2}$$

it follows from the Laplacian comparison theorem that $\frac{\partial \widetilde{G}}{\partial r} \le 0$. Therefore,

$$0 \ge \frac{1}{t^{n-1}} \int_{B(t)} \Delta f \ge \int_{\partial B(t)} \frac{\partial f}{\partial r} \widetilde{G} + f \frac{\partial \widetilde{G}}{\partial r}\, d\theta = \int_{\partial B(t)} \frac{\partial}{\partial r}(f\widetilde{G})\, d\theta = \frac{d}{dt} \int_{\partial B(t)} f\widetilde{G}\, d\theta.$$

The last equality holds trivially since the integral is taken over $\theta \in S^{n-1}$. Therefore, it follows from $\widetilde{G}(0) = 1$ that

$$\omega_{n-1} f(x) \geq \int_{\partial B(t)} f \widetilde{G} \, d\theta.$$

Thus,

$$\int_{B(r)} f = \int_0^r \int_{\partial B(t)} f t^{n-1} \widetilde{G} \, d\theta \, dt \leq \omega_{n-1} f(x) \int_0^r t^{n-1} \, dt \leq \frac{1}{n} \omega_{n-1} r^n f(x),$$

which completes the proof. □

Remark 7.9. There are different definitions about subharmonicity and the statements of the corresponding strong maximum principle are also different. Here we follow the proof in [6].

An alternative proof of the Cheeger-Gromoll splitting theorem can be found in [4, Section 7.3.2, Page 303], which adopts another definition of subharmonicity due to [1], relying on the idea of support functions. See [4, Section 7.1.3] for a reference of subharmonicity in the barrier sense.

A sketch of the proof of Theorem 7.5: By Laplacian comparison theorem, in the sense of distributions, $\Delta \beta_t^+(x) \geq \frac{m-1}{d_g(\gamma_+(t), x)}$. Since β_t^+ converges to $\beta^+(x)$ uniformly on compact sets, we see

$$\Delta \beta^+(x) \geq - \lim_{t \to \infty} \frac{m-1}{d_g(\gamma_+(t), x)} \geq - \lim_{t \to \infty} \frac{m-1}{t - d_g(\gamma_+(0), x)} = 0$$

in the distributional sense.

Similarly, β^- is subharmonic and hence $\Delta(\beta^+ + \beta^-) \geq 0$ in the sense of distributions. By Lemma 7.7, $\beta^+ + \beta^-$ attains its maximum in the interior of M. It then follows from the mean value inequality (Lemma 7.8) that

$$\beta^+(x) + \beta^-(x) \geq \frac{n}{\omega_{n-1} R^n} \int_{B_R(x)} \left(\beta^+(x) + \beta^-(x) \right) \, dx$$

for small R, which implies $\beta^+ + \beta^- \equiv 0$ on M by connectedness. (This can be viewed as the strong maximum principle for $\beta^+ + \beta^-$.)

As a result,

$$0 \leq \Delta \beta^+ \leq -\Delta \beta^- \leq 0,$$

and then $\Delta \beta^+ = \Delta \beta^- = 0$ in the sense of distribution. By regularity theory, therefore, β^+, β^- are harmonic, and in particular, they are smooth functions.

By Lemma 7.6, we have $|\mathrm{grad}\, \beta^+| \leq 1$. And in particular, $|\mathrm{grad}\, \beta^+| = 1$ on γ since

$$\begin{aligned} |\beta^+(\gamma_+(t_2)) - \beta^+(\gamma_+(t_1))| &= \left| \int_{t_1}^{t_2} \langle \mathrm{grad}\, \beta^+, \dot{\gamma}_+ \rangle \, ds \right| \\ &\leq \int_{t_1}^{t_2} |\mathrm{grad}\, \beta^+| |\dot{\gamma}_+| \, ds \leq \int_{t_1}^{t_2} |\dot{\gamma}_+| \, ds = d_g(\gamma_+(t_2), \gamma_+(t_1)) \end{aligned}$$

is in fact an equality. By the Bochner formula in polar coordinates, we have

$$\begin{aligned} \Delta |\mathrm{grad}\, \beta^+|^2 &= 2\Delta_i \left(D_i D_k \beta^+ D_k \beta^+ \right) = 2|D(D\beta^+)|^2 + 2D_k \beta^+ D_i D_k D_i \beta^+ \\ &= 2|D(D\beta^+)|^2 + 2D_k \beta^+ D_k D_i D_i \beta^+ + 2R^h_{iki} D_h \beta^+ = 2|D(D\beta^+)|^2 + 2\mathrm{Ric}\,(\mathrm{grad}\, \beta^+, \mathrm{grad}\, \beta^+) \geq 0 \end{aligned}$$

Hence, by the strong maximum principle, $|\mathrm{grad}\, \beta^+|^2 \equiv 1$ on M.

Thus, $D(D\beta^+) = 0$ and $\mathrm{Ric}(\mathrm{grad}\,\beta^+, \mathrm{grad}\,\beta^+) = 0$. In particular, $\mathrm{grad}\,\beta^+$ is a parallel vector field on M. Finally, we show M is isometric to the product manifold of an intergral curve of $\mathrm{grad}\,\beta^+$ and a level set of β^+. For more details, refer to [5, Chapter V, Lemma 3.10] or [6, Chapter I.2]. ☐

8. Rauch comparison theorem

Theorem 8.1 (Rauch comparison theorem)**.** *Let M_1^m, M_2^m be two Riemannian manifolds. Let J_k be Jacobi fields along normal geodesics γ_k with $J_1(0) = J_2(0) = 0$, $|\dot{J}_1(0)| = |\dot{J}_2(0)|$, and $\langle \dot{\gamma}_1, \dot{J}_1(0)\rangle = \langle \dot{\gamma}_2, \dot{J}_2(0)\rangle$.*

Suppose $K^+_{M_2}(t) \le K^-_{M_1}(t)$ for $t \in [0, r]$, where

$$K^+_{M_2}(t) := \max\{K(\Pi_{\gamma_2(t)}) : \Pi_{\gamma_2(t)} \ni \dot{\gamma}_2(t)\}, \quad K^-_{M_1}(t) := \min\{K(\Pi_{\gamma_1(t)}) : \Pi_{\gamma_1(t)} \ni \dot{\gamma}_1(t)\}.$$

If γ_1 does not have conjugate points, then $|J_1(t)| \le |J_2(t)|$ for $t \in [0, r]$.

Before we prove this theorem, we show some applications.

Corollary 8.2. *Let $\gamma_k : [0, r] \to M_k$ be normal geodesics, $k = 1, 2$. Denote $p_k = \gamma_k(0)$ for $k = 1, 2$. Suppose γ_1 has no conjugate points and $K^+_{M_2}(t) \le K^-_{M_1}(t)$ for $t \in [0, r]$. If $X_k \in T_{p_k}M_k$ satisfies $|X_1| = |X_2|$ and $\langle X_1, \dot{\gamma}_1(0)\rangle = \langle X_2, \dot{\gamma}_2(0)\rangle$, then*

$$\left|\left(d\exp_{p_1}\right)_{t\dot{\gamma}_1(0)}(X_1)\right| \le \left|\left(d\exp_{p_2}\right)_{t\dot{\gamma}_2(0)}(X_2)\right|.$$

Proof. Apply Rauch comparison theorem to the Jacobi fields

$$J_k(t) = \partial_s|_{s=0}\left(\exp_{p_k}(t(\dot{\gamma}_k(0) + sX_k))\right) = t\left(d\exp_{p_k}\right)_{t\dot{\gamma}_k(0)}(X_k)$$

☐

Corollary 8.3. *Let M be a complete manifold with non-positive sectional curvature. Then for all $p \in M$, $X \in T_pM$, $Y \in T_pM = T_XT_pM$,*

$$\left|\left(d\exp_p\right)_X(Y)\right| \ge |Y|.$$

In particular,

$$L(\gamma) \le L(\exp_p \circ \gamma), \quad \forall \gamma \subset T_pM.$$

Proof. Let $M_2 = (M, g)$ and $M_1 = (T_pM, g_p)$. Then applying Corollay 8.2 with $X_1 = X_2 = Y$, $p_1 = X, p_2 = p$, $\gamma_1(t) = X + tX$ and $\gamma_2(t) = \exp_p(tX)$ will proves this corollary. ☐

Corollary 8.4. *Let (M, g) be a simply connected complete manifold with non-positive sectional curvature. Consider the geodesic triangle on M whose side lengths are a, b, c with opposite angles A, B, C, respectively. Then*

(1) $a^2 + b^2 - 2ab\cos C \le c^2$;

(2) $A + B + C \le \pi$.

Proof. Denote the vertex at the angle C, A, B by p, q, r respectively. In the tangent space T_pM, draw a triangle $\triangle OQR$, where O is the origin of T_pM, so that $|OQ| = a$, $|OR| = b$ and the angle at O is equal to C. Then we know that $\exp_p(\vec{OQ}) = q$ and $\exp_p(\vec{OR}) = r$. Hence, suppose η is the pre-image of the geodesic c in T_pM, then the two endpoint of η are q, r. Therefore, $|PQ| \le L(\eta) \le c$, where the second inequality follows from Corollary 8.3. Hence, the first conclusion follows from the Euclidean cosine law. And the second conclusion follows from Euclidean sine law by considering the triangle in $\mathbb{R}^2$ whose side length are a, b, c. ☐

Proof of Theorem 8.1: We have

$$J_k = J_k^{\perp} + \tan J_k,$$

where $\tan J_k = f_k(t)\dot{\gamma}_k$ and $f_k(t) = \langle J_k(t), \dot{\gamma}_k(t)\rangle$. Since $f_1(0) = f_2(0) = 0$, $\dot{f}_1(0) = \langle \dot{J}_1(t), \dot{\gamma}_1(t)\rangle = \langle \dot{J}_2(t), \dot{\gamma}_2(t)\rangle = \dot{f}_2(0)$ and $\ddot{f}_1(t) = \ddot{f}_2(t)$, we have $f_1(t) \equiv f_2(t)$ for $t \in [0, r]$. Then it suffices to show $|J_1^{\perp}(t)| \leq |J_2^{\perp}(t)|$ for all $t \in [0, r]$.

We assume with loss of generality that $J_k \perp \dot{\gamma}_k$. We write $g_k(t) = |J_k(t)|^2$.

Step 1: Since there is no conjugate point along γ_1, if there is some $t_0 > 0$ such that $g_1(t_0) = 0$, then $g_1(t) \equiv 0$ for all t and we are done since $g_2(t) \geq 0 = g_1(t)$. Now we can assume that $g_1(t) > 0$ for all $t \in [0, r]$.

Step 2: Since $g_1(t) = 0$ only at $t = 0$, the ratio $\frac{g_2(t)}{g_1(t)}$ is well-defined except at $t = 0$. By L'Hôpital's rule,

$$\lim_{t\to 0} \frac{g_2(t)}{g_1(t)} = \lim_{t\to 0} \frac{\langle J_2(t), \dot{J}_2(t)\rangle}{\langle J_1(t), \dot{J}_1(t)\rangle} = \lim_{t\to 0} \frac{\langle \dot{J}_2(t), \dot{J}_2(t)\rangle + R(J_1(0), \dot{\gamma}_1(0), J_1(0), \dot{\gamma}_1(0))}{\langle \dot{J}_1(t), \dot{J}_1(t)\rangle + R(J_2(0), \dot{\gamma}_2(0), J_2(0), \dot{\gamma}_2(0))} = 1.$$

thus it suffices to show $\frac{d}{dt} \log \frac{g_2(t)}{g_1(t)} \geq 0$, which is equivalent to

$$\frac{\langle \dot{J}_2, J_2\rangle}{|J_2(t)|^2} \geq \frac{\langle \dot{J}_1, J_1\rangle}{|J_1(t)|^2}, \quad \forall t \in [0, r].$$

Step 3: Let $l > 0$ be such that $g_2(t) > 0$ for all $t \in (0, l)$ and $g_2(l) = 0$. (Since $\gamma_2(t)|_{[0,\delta]}$ has no conjugate points for $\delta > 0$ small enough, so we know $g_2(t) > 0$ for all $t \in (0, \delta)$. Otherwise, $g_2(t) \equiv 0$ and by $\lim_{t\to 0} \frac{g_2(t)}{g_1(t)} = 1$ and $g_1(t) > 0$ when $t > 0$, we derive a contradiction. So such $l > 0$ exists.)

Fix $t_0 \in (0, l)$, let $J_k^{t_0}(t) = \frac{1}{|J_k(t_0)|} J_k(t)$, We claim that

$$I_{\gamma_1}(J_1^{t_0}, J_1^{t_0}) \leq I_{\gamma_2}(J_2^{t_0}, J_2^{t_0}). \tag{8.1}$$

Recall that after integration by parts,

$$I_\gamma(X, X) = \langle \dot{X}, X\rangle|_0^{t_0}$$

if X is a Jacobi field along γ. Hence, if (8.1) holds, then

$$\frac{\langle \dot{J}_2, J_2\rangle(t_0)}{|J_2(t_0)|^2} = \langle \dot{J}_2^{t_0}(t_0), J_2^{t_0}(t_0)\rangle \geq \langle \dot{J}_1^{t_0}(t_0), J_1^{t_0}(t_0)\rangle = \frac{\langle \dot{J}_1, J_1\rangle(t_0)}{|J_1(t_0)|^2}.$$

Thus, $|J_1(t)| \leq |J_2(t)|$ for $t \in (0, l)$. If $l < r$, then $|J_2(l)| \geq |J_1(l)| > 0$, contradicting the choice of l. Hence, $l = r$, that is, $|J_1(t)| \leq |J_2(t)|$ for $t \in (0, r)$. This proves the theorem.

Step 4: Now we prove the claim (8.1). For reader's convenience, we reformulate the claim as follows. We shall show that if $J_k(t) \perp \dot{\gamma}_k(t)$ for $t \in [0, r]$ is a Jacobi field $J_k(0) = 0$, $|J_k(r)| = 1$. Then $I_{\gamma_1}(J_1, J_1) \leq I_{\gamma_2}(J_2, J_2)$.

We choose a parallel orthonormal frame $\{E_{j,k}(t)\}_{j=1}^m$ such that $E_{m,k}(t) = \dot{\gamma}_k(t)$ and $E_{1,k}(r) = J_k(r)$. We write $J_2(t) = \sum_{i=1}^{m-1} \eta^i E_{i,2}(t)$. Put $Y(t) = \sum_{i=1}^{m-1} \eta^i E_{i,1}(t)$. Since

$J_1(0) = 0 = Y(0)$, $J_1(r) = E_{1,1}(r) = Y(r)$, it follows from the basic index lemma that

$$\begin{aligned} I_{\gamma_1}(J_1, J_1) \leq & I_{\gamma_1}(Y, Y) = \int_0^r \sum (\dot{\eta}^i)^2 - R_{M_1}(\dot{\gamma}_1, E_{i,1}, \dot{\gamma}_1, E_{i,1})(\eta^i)^2 \\ & \leq \int_0^r \sum (\dot{\eta}^i)^2 - R_{M_2}(\dot{\gamma}_1, E_{i,1}, \dot{\gamma}_1, E_{i,1})(\eta^i)^2 = I_{\gamma_2}(J_2, J_2), \end{aligned}$$

which completes the proof. □

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