NOTES FOR ORDINARY DIFFERENTIAL EQUATIONS

TRANSCRIBED BY NING TANG INSTRUCTOR: PROFESSOR SUNG-JIN OH

Contents

Date: December 30, 2022.

Syllabus

The prerequisites for this course are:

- Real Analysis(Math 202A-B)
- Complex Analysis(Math 185)
- Fundamental theorem of ODE(Picard-Lindlöf, Cauchy-Lipschitz, Well-posedness of IVP)

There will be two major parts of this course. In each part, our plan is as follows:

- Part I. Linear ODE and spectral theory: We will study Sturm-Liouville type problem. To be more specific, set $\mathcal{L}f = \frac{1}{r}\left(-\frac{d}{dx}\left(p\frac{d}{dx}f\right)+q\right)$, where p, q, r are real-valued functions on I. Then we consider $\mathcal{L}f = \lambda f$ with certain boundary conditions. Here, $\mathcal{L}$ is formally symmetric with respect to $\langle\cdot,\cdot\rangle_r$ in the sense that for all $f, g \in C_c^\infty(I)$, $\langle f, \mathcal{L}g\rangle_r = \langle \mathcal{L}f, g\rangle_r$, where $\langle f, g\rangle_r = \int_I \bar{f} gr\,dx$.

  This part will be useful since many natural problems arises in physics and geometry can be reduced to this form and be fruitful since some can be explicitly solvable.
  - Regular Sturm-Liouville problems (Dirichlet, Neumann, Robin boundary condition)
  - Singular Sturm-Liouville problems (Many problems in the whole space $\mathbb{R}^d$ can be reduced to this kind of problem)
  - Special functions (This part may need some complex analysis)

  For instance, you will study something like $\Box\phi = \partial\phi\partial\phi := N(\phi)$ when ϕ is small, so you want to approximate solutions ϕ to the solution ψ to the wave equation $\Box\psi = 0$, where $\Box = -\partial_t^2 + \Delta$. Usually, solutions to the wave equation disperse. If I snap the fingers, then the energy disperses. But the solitons are something stable, which is very different from the linear phenomenon. Solitons Q shall satisfy $\Box Q = N(Q)$, $\partial_t Q = 0$, $Q \neq 0$. To analyze this, one can linearize the equation by writing $\phi = Q + \psi$ and you hope ψ is small. Now we get $\Box\psi - N(Q)\psi = \widetilde{N}(\psi)$ from $\Box(Q+\psi) = N(Q+\psi)$. One may expect the asymptotic behavior of a solution is like the sum of a solution solution and a radiation.
- Part II. Nonlinear ODE and dynamics: We will study ODEs in the form of $\frac{d}{dt}u = F(u)$, which is related to evolutionary problem.
  - Linearization and invariant manifolds
  - Periodic solutions
  - Bifurcation, center manifolds

For Part I, the main references are [12, Chapter 5] for regular Sturm-Liouville problem, [13, Chapter 3, 9] for singular Sturm-Liouville problem, [8, Chapter 5, 7] for special functions.

For Part II, the references according to our tenative plan are [10, Chapter 7-10] and [4, Chapter 2].

1. Sturm-Liouville problems

We denote the operator introduced in the previous section by the following operator notation

$$\mathcal{L} = \frac{1}{r}\left(-\frac{d}{dx}p\frac{d}{dx} + q\right),$$

where p, q, r are real-valued functions.

Definition 1.1 (Regular Sturm-Liouville problem)**.** *Suppose $I = (a,b) \subset \mathbb{R}$, $-\infty < a < b < \infty$, where $\mathcal{L}$ is said to be regular if $p, p', q, r \in C(\bar{I};\mathbb{R})$ and $p > 0, r > 0$.*

Then following question is motivated by separation of variable in PDEs, especially in the one-dimensional wave equations. We consider

$$\begin{cases} \mathcal{L}u = zu, \quad z \in \mathbb{C}, \\ + \text{ separated boundary conditions: } BC_a(u) = 0, BC_b(u) = 0, \end{cases} \tag{1.1}$$

where the BC operator of u at a and b are defined as

$$\begin{aligned} BC_a(u) :=& \alpha_0 u(a) - \alpha_1 p(a)\frac{d}{dx}u(a), \quad (\alpha_0, \alpha_1) \neq 0, \\ BC_b(u) :=& \beta_0 u(b) - \beta_1 p(b)\frac{d}{dx}u(b), \quad (\beta_0, \beta_1) \neq 0, \end{aligned}$$

respectively. This boundary condition is called separated boundary conditions. Then our question is trying to determine that for which $z \in \mathbb{C}$, there exists $u \neq 0$ solving (1.1).

The first approach to solve this problem is to note that we can simplify the boundary condition as follows. Without loss of generality, one can assume $|(\alpha_0, \alpha_1)| = 1$ and (α_0, α_1) lies in the upper half circle except $(-1, 0)$. Then by a change of variable, we assume

$$(\alpha_0, \alpha_1) = (\cos\alpha, \sin\alpha), \quad (\beta_0, \beta_1) = (\cos\beta, \sin\beta), \quad \alpha, \beta \in [0, \pi).$$

Let u_a be the solution to

$$\begin{cases} \mathcal{L}u_a = zu_a, \\ u_a(a) = \sin\alpha, \quad p(a)u_a'(a) = \cos\alpha, \end{cases}$$

where u_a exists and is unique due to the fundamental theorem of ODEs. Clearly, u_a satisfies $BC_a(u_a) = 0$. Moreover, any solution u to

$$\begin{cases} \mathcal{L}u = zu, \\ BC_a(u) = 0, \end{cases}$$

satisfies $u = cu_a$ for some constant c, that is, we get a one-parameter family of solutions to this ODE. If we do the same thing by replacing a by b, then we get two one-parameter family of solutions, which sometime may match with each other to be a solution of (1.1). This kind of problem is called a shooting problem. Since (1.1) is a second order linear ODE, all its solutions form a two dimensional vector space. Heuristically, the shooting problem is kind of like matching two stuffs of one-dimensional in a two dimensional space. This is why it is called a shooting problem.

So the problem whether (1.1) has nonzero solutions can be transformed to thinking whether $u_a \propto u_b$. To prove the main theorem of regular Sturm-Liouville problems, that is, there exists eigenvectors, we borrow tools from functional analysis.

1.1. **A primer on bounded symmetric operators.**

Definition 1.2. *We say $\langle u, v\rangle$ is a sesquilinear form on a complex vector space H if it is linear in v and linear conjugation in u, that is,*

$$\langle u, c_1v_1 + c_2v_2\rangle = c_1\langle u, v_1\rangle + c_2\langle u, v_2\rangle, \langle c_1u_1 + c_2u_2, v\rangle = \bar{c_1}\langle u_1, v\rangle + \bar{c_2}\langle u_2, v\rangle.$$

This is in line with the convention in physics.

Definition 1.3. $\langle\, ,\, \rangle$ *is a (complex) inner product if* $\langle u, u\rangle \geq 0$ *and* $= 0$ *iff* $u = 0$, $\langle u, v\rangle = \overline{\langle v, u\rangle}$.

Suppose $(H, \langle\cdot, \cdot\rangle)$ is an inner product space.

Lemma 1.4. *Let $u \in H$, $X \subset H$ such that X is spanned by $(e_1, \dots, e_n)$ and $\langle e_j, e_k\rangle = \delta_{jk}$. Let $u_n = \sum_{j=1}^{n}\langle u, e_j\rangle e_j$ be the orthonormal projection of u to X, then for any $\widetilde{u} \in X$, the error*

$$\|u - \widetilde{u}\| \geq \|u^\perp\|,$$

where the equality holds iff $\widetilde{u} = u_n$, where $u^\perp = u - u_n$.

Proof. The proof is elementary by writing

$$\|u - \widetilde{u}\|^2 = \|u_n + u^\perp - \widetilde{u}\|^2 = \|u_n - \widetilde{u}\|^2 + \|u^\perp\|^2 \geq \|u^\perp\|^2,$$

where the second equality follows from the fact that $\langle u^\perp, e_k\rangle = 0$ for all $1 \leq k \leq n$. □

Example 1.5. *Here are two examples of inner product spaces:*

$$(\mathbb{C}^n, \langle u, v\rangle = \sum \bar{u_j}v_j), \quad (L^2(I; \mathbb{C}), \langle u, v\rangle = \int_I \bar{u}v\, dx).$$

Definition 1.6. *We say the linear operator $A : H \to H$ is bounded if* $\sup_{\|u\|=1} \|Au\| < +\infty$. *$A$ is symmetric if $\langle u, Av\rangle = \langle Au, v\rangle$ for all $u, v \in H$.*

Lemma 1.7. *If A is bounded and symmetric, then any $Au_1 = z_1u_1$, $Au_2 = z_2u_2$ for $u_1 \neq 0, u_2 \neq 0, z_1 \neq z_2$, then $u_1 \perp u_2$ and any eigenvalue is real.*

Remark 1.8 (Spectral theorem on $\mathbb{C}^n$). Suppose A is a Hermitian (conjugate symmetric) matrix, then we know that A is diagonalizable, that is, n eigenvalues are real and there exists eigenvectors that form an orthonormal basis.

The reason why this is interesting is that we can develop functional calculus for A. Suppose $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. For a measurable function $f : \mathbb{C} \to \mathbb{C}$, it is presumably that $f(A)$ is related to $\operatorname{diag}(f(\lambda_1), \dots, f(\lambda_n))$. This gives a formal way to view the operators as constants, which would be extremely useful if one can make it rigorous as shown in the following example. If $\mathcal{L}$ were a positive number, then one can solve the wave equation $\partial_t^2 = -\mathcal{L}u$ by $u = a \sin t\sqrt{\mathcal{L}} + b \cos t\sqrt{\mathcal{L}}$.

Definition 1.9. *$A : H \to H$ is compact if for any bounded sequence $\{u_n\} \subset H$ ($\|u_n\| \le C$), then $\{Au_n\}$ is compact in the sense that Au_n admits a converging subsequence.*

Example 1.10. *$A : H \to H$ is a compact operator if $\dim \mathit{range}\,(A) < +\infty$.*

Remark 1.11. Actually, we have a stronger theorem that tells us that in a separable Hilbert space with countable basis, finite dimensional operators are dense in compact operators with respect to the operator norms.

Theorem 1.12 (Spectral theorem for compact symmetric operators)**.** *Suppose A is a compact symmetric linear operator, then the eigenvalues are discrete except possibly at 0. More precisely,*

$$\|A\| = |\alpha_0| \ge |\alpha_1| \ge \cdots \ge 0,$$

can only accumulate possibly at 0, where α_i are all the eigenvalues. (All eigenvalues are real as discussed before.) And there exists eigenfunctions $\{e_n\}$ that are real-valued and form an orthonormal basis of $\overline{\mathit{range}\,(A)}$

Remark 1.13. In the definition of compactness, we don't assume the completeness of H. However, under completeness, there's even a spectral theorem for compact operators without inner product structure for $A : X \to X$ compact on Banach spaces, where the statement is more like a generalization of Jordan canonical form.

The proof of the spectral theorem uses essentially the same idea as the proof that Hermitian matrices are diagonalizable.

With the same condition for A as in Theorem 1.12, the following lemmas hold.

Lemma 1.14 (Existence of an eigenvalue for A)**.** *Suppose $A : H \to H$ is linear, compact and symmetric. There exists an eigenvalue λ such that $|\lambda| = \|A\|$. And hence by symmetry, $\lambda \in \mathbb{R}$.*

Proof. We want to find the extremizer of $\{\|Au\| : \|u\| = 1\}$. By definition of supremum, there exists u_n such that $\|u_n\| = 1$ and $\|Au_n\| \nearrow \Lambda := \|A\|$. By compactness of A, $Au_n \to v$ by passing to a subsequence. Up to a subsequence, $A^2 u_n \to \tilde{v}$.

Since $\|Au_n\|^2 = \langle Au_n, Au_n\rangle = \langle A^2 u_n, u_n\rangle \to \Lambda^2$, we know

$$\|A^2 u_n\| = \sup_{\|w\|=1} \langle A^2 u_n, w\rangle \ge \langle A^2 u_n, u_n\rangle \to \Lambda^2.$$

On the other hand, $\|A^2 u_n\| \le \Lambda^2$. Hence, $\|A^2 u_n\| \to \Lambda^2$, and in particular, $\|\tilde{v}\| = \Lambda^2$.

Suppose $\tilde{v} = \Lambda^2 u$, $\|u\| = 1$, then we want to show $A^2 u = \Lambda^2 u$. Since

$$\|(A^2 - \Lambda^2)u_n\|^2 = \|A^2 u_n\|^2 - 2Re\langle A^2 \Lambda^2 u_n, u_n\rangle + \Lambda^4 \|u_n\|^2 \to 0$$

as $n \to \infty$, $\Lambda^2 u = \lim_{n\to\infty} A^2 u_n = \lim_{n\to\infty} \Lambda^2 u_n$ and hence $A^2 u = \Lambda^2 u$, which implies $(A - \Lambda)(A + \Lambda)u = 0$, so either Λ or $-\Lambda$ is an eigenvalue. $\square$

Proof of the Spectral theorem, Theorem 1.12. Using Lemma 1.14, we can find a real-valued eigenvalue λ_0 and a corresponding eigenfunction e_0 of A. Without loss of generality, we can assume e_0 is real-valued. Otherwise, we replace by normalizing $e_0 + \bar{e_0}$, which is also an eigenfunction corresponds to λ_0 thanks to the symmetry of $\mathcal{L}$ or more precisely, it follows from the property that p, p', q, r are real-valued. And then consider the restriction $A|_{(e_0)^\perp}$, and by the symmetry of A, we know that $ran(A|_{(e_0)^\perp}) \subset (e_0)^\perp$. Then applying Lemma 1.14 to $A|_{(e_0)^\perp}$ will help us find λ_1, e_1. By repeating this argument, we get possibly infinitely many eigenvalues and eigenfunctions.

If the sequence is infinite, we want to show that $\lambda_n \to 0$. Suppose not, there exists $\delta > 0$ such that $|\lambda_{n_j}| > \delta$ by passing to a subsequence and $Au_{n_j} = \lambda_{n_j} u_{n_j}$. Since $v_j = \frac{1}{\lambda_{n_j}} u_{n_j}$ is a bounded sequence, by compactness, $Av_j = u_{n_j}$ should have a convergent subsequence. However, u_{n_j} is orthonormal to each other, so $\|u_{n_j} - u_{n_l}\|^2 = 2$, therefore it does not have any convergent subsequences, contradiction!

Finally, if $v \in ran(A)$, then there exists w such that $v = Aw$. Let $v_n = \sum_{k=1}^n \langle e_k, v\rangle e_k$ and $w_n = \sum_{k=1}^n \langle e_k, w\rangle e_k$. Since $Aw_n = \sum_{k=1}^n \langle Ae_k, w\rangle e_k = v_n$, we have

$$\|v - v_n\| = \|A(w - w_n)\| \le |\lambda_{n+1}| \|w - w_n\| \le 2|\lambda_{n+1}| \|w\| \to 0.$$

If $v \in \overline{ran(A)}$, then for all $\varepsilon > 0$, there exists $v_\varepsilon \in ran(A)$ such that $\|v - v_\epsilon\| < \varepsilon/2$. Using the result in the preceding paragraph, there exists $v_{\varepsilon,N} = \sum_{k=1}^N c_k e_k$ such that $\|v_\epsilon - v_{\epsilon,N}\| < \varepsilon/2$. From Lemma 1.4 shown in our last lecture, $\|v - v_N\| \le \|v - v_{\epsilon,N}\| < \varepsilon$, so $v_N \to v$, where $v_N = \sum_{k=1}^N \langle e_k, v\rangle e_k$. $\square$

Unfortunately, we cannot apply the spectral theorem directly to $\mathcal{L}$ in our Sturm-Liouville problem. We will show that the resolvent $(\mathcal{L} - z)^{-1}$ for $z \in \mathbb{C}$ turns out to be compact and symmetric and it will suffice to understanding (1.1).

1.2. **Regular Sturm-Liouville problem.**

For the sake of convenience, we restate the problem here. We consider

$$\begin{cases} \mathcal{L}u = zu, \quad z \in \mathbb{C}, \\ + \text{ separated boundary conditions: } BC_a(u) = 0, BC_b(u) = 0, \end{cases} \tag{1.2}$$

where

$$\mathcal{L} = \frac{1}{r}\left(-\frac{d}{dx} p \frac{d}{dx} + q\right),$$

with $p, p', q, r \in C(\bar{I}; \mathbb{R})$ and $p > 0, r > 0$, $I = (a, b) \subset \mathbb{R}$.

Put $H = (L^2(I; r dx); \langle f, g\rangle = \int_I \bar{f} g r\, dx)$. Set $\mathcal{D}(\mathcal{L}) = \{f \in C^2(\bar{I}; \mathbb{C}) : BC_a(f) = 0, BC_b(f) = 0\}$. We say z is an eigenvalue for (1.2) with an eigenfunction u, if there exists $u \neq 0$, $u \in \mathcal{D}(\mathcal{L})$ such that $\mathcal{L}u = zu$.

Theorem 1.15. *For a regular Sturm-Liouville problem, there are countably many eigenvalues, which are real and simple and accumulate only at ∞. And there exists real-valued eigenfunctions which form an orthonormal basis $\{u_j\}$ of H, that is, every $f \in H$ can be written as*

$$f(x) = \sum_{j=0}^\infty \langle u_j, f\rangle u_j(x).$$

In this subsection, our goal is to prove this theorem.

Example 1.16. *Suppose* $\mathcal{L} = -\frac{d^2}{dx^2}$, $BC_a(u) = u(a), BC_b(u) = u(b)$ *with* $I = (0, \pi)$. *Then by solving explicitly, we get*

$$\begin{cases} \lambda_1 = 1, u_1 = \sin x, \quad \dots, \\ \lambda_n = n^2, u_n = \sin nx \end{cases}$$

Now we consider some ODEs and will heavily rely on integration by parts.

Lemma 1.17.

$$\int_\alpha^\beta f \mathcal{L} g r\, dx = \int_\alpha^\beta \mathcal{L} f g r\, dx + p(\beta) f'(\beta) g(\beta) - f(\beta) p(\beta) g'(\beta) - (p(\alpha) f'(\alpha) g(\alpha) - f(\alpha) p(\alpha) g'(\alpha)) .$$

Proof. A direct integration by parts. ☐

Corollary 1.18 (Symmetry of $\mathcal{L}$)**.** *Suppose* $u, v \in \mathcal{D}(\mathcal{L})$, *then* $\langle u, \mathcal{L}v \rangle = \langle \mathcal{L}u, v \rangle$.

Proof. Use the lemma with $f = \bar{u}, g = v, \alpha = a, \beta = b$. ☐

Note that if f, g are solutions to $\mathcal{L} = 0$, then we know from the lemma that the boundary terms shall vanish, which motivates the following definition of the modified Wronskian.

Definition 1.19 (Modified Wronskian)**.** *The modified Wronskian is defined as*

$$W_x(u, v) := u(x) p(x) v'(x) - p(x) u'(x) v(x) = \det \begin{pmatrix} u(x) & v(x) \\ pu'(x) & pv'(x) \end{pmatrix} .$$

Thus the lemma above can be restated as

$$W_\beta(u, v) - W_\alpha(u, v) = \int_\alpha^\beta (\mathcal{L}u)\, vr\, dx - \int_\alpha^\beta u\, (\mathcal{L}v)\, r\, dx. \tag{1.3}$$

Corollary 1.20. *If* $\mathcal{L}u = zu$, $\mathcal{L}v = zv$, *then* $W_x(u, v)$ *is independent of* x. *Moreover,* u *and* v *are linearly dependent if and only if* $W_x(u, v) = 0$ *for some* x.

Recall that respectively, $u_a(x; z)$ and $u_b(x; z)$ satisfies

$$\begin{cases} \mathcal{L} u_a(\ ; z) = z u_a(\ ; z), \\ u_a(a; z) = \sin \alpha, \\ p(a) u_a'(a; z) = \cos \alpha, \end{cases} \qquad \begin{cases} \mathcal{L} u_b(\ ; z) = z u_b(\ ; z), \\ u_b(b; z) = \sin \beta, \\ p(b) u_b'(b; z) = \cos \beta. \end{cases}$$

Corollary 1.21. z *is an eigenvalue of* $\mathcal{L}$ *in* $\mathcal{D}(\mathcal{L})$ *if and only if* $W_x(u_a(\ ; z), u_b(\ ; z)) = 0$ *for some* x.

Definition 1.22. *We define* $W(z) = W_x(u_a(\ ; z), u_b(\ ; z))$.

Let us study

$$\begin{cases} \mathcal{L}u = zu, \\ BC_a(u) = BC_b(u) = 0, \end{cases}$$

if $W(z) \neq 0$, $\mathcal{L}-z$ has trivial kernel. Actually, one can show that $A : \mathcal{D}(A) \to H$ has following properties. For finite dimensional cases, $Ran(A) = (\ker A^*)^\perp$, so when A is symmetric, then it is surjective. In the infinite dimensional cases, we have $\overline{Ran(A)} = (\ker A^*)^\perp$. Moreover, A is coercive, then $\overline{Ran(A)} = Ran(A)$. These properties can be found at [3, Proposition 9.12, 9.14].

We expect, on abstract grounds that

$$\begin{cases} (\mathcal{L}-z)u = f, \\ BC_a(u) = BC_b(u) = 0 \end{cases} \tag{1.4}$$

should be solvable for $f \in H$.

To see this, let us work by hand and construct Green's function:

Definition 1.23. *The function $G(x, y)$ is said to be a Green's function if the solution to is given by*

$$u(x) = \int_a^b G(x, y) f(y) r(y)\, dy.$$

Take $f(y) = \frac{1}{r(y)}\delta_0(y - y_0)$, then the corresponding solution if $u(x) = G(x, y_0)$. Since $r(\cdot)(\mathcal{L}-z)G(\cdot, y_0) = \delta_0(\cdot - y_0)$, where the right hand side vanishes away from y_0. Then from the discussion at the very beginning of the section, the solution for $G(x, y_0)$ is proportional to u_a when $x < y_0$ and proportional to u_b when $x > y_0$. Moreover, from the equation $(\mathcal{L}-z)G(\cdot, y_0) = \delta_0(\cdot - y_0)$, we expect that $G(x, y_0)$ is continuous at $x = y_0$ since if we integrate δ function two times, we will get a continuous function. Thus, we expect

$$G(x, y) = \begin{cases} cu_a(x; z)u_b(y; z), & x < y, \\ cu_a(y; z)u_b(x; z), & x > y, \end{cases}$$

for some c. Now we compute c as follows. By (1.3),

$$\int_a^b u_a(x; z)\left((\mathcal{L}-z)G\right) r\, dx = \int_a^b \left((\mathcal{L}-z)u_a\right) Gr\, dx + W_b(u_a, G) - W_a(u_a, G).$$

Since u_a and G are proportional at $x = a$, we know $W_a(u_a, G) = 0$. Since $(\mathcal{L}-z)u_a = 0$ and $r(\mathcal{L}-z)G = \delta_0(\cdot - y)$, we get

$$u_a(y; z) = W_b(u_a, G) = W_b(u_a(\ ; z), u_b(\ ; z))cu_a(y; z) = cW(z)u_a(y; z)$$

which implies that $c = 1/W(z)$.

Definition 1.24. *Suppose $W(z) \neq 0$, then the operator $R_{\mathcal{L}}(z)$ defined by*

$$R_{\mathcal{L}}(z)f := \int_a^b G(x, y) f(y) r(y)\, dy$$

is a operator $R_{\mathcal{L}}(z) : H \to \mathcal{D}(\mathcal{L})$, where

$$G(x, y) := \begin{cases} \frac{1}{W(z)} u_a(x; z)u_b(y; z), & x < y, \\ \frac{1}{W(z)} u_a(y; z)u_b(x; z), & x > y. \end{cases}$$

One should note that the $R_{\mathcal{L}}(z)$ is well-defined as an operator

$$R_{\mathcal{L}}(z) : H \to H \cap H^2(I) \quad \text{or} \quad R_{\mathcal{L}}(z) : H \cap C(\bar{I}) \to \mathcal{D}(L),$$

where $C(\bar{I}) \cap H$ means the space $C(\bar{I})$ with $\langle f, g\rangle = \int \bar{f} gr\, dx$. This is because that for all $f \in H$, one can compute the distributional derivative of $R_{\mathcal{L}}(z)f$ explicitly up to second order by pairing with a test function and using the definition. We have

$$\partial_x \left(R_{\mathcal{L}}(z)f\right)(x) = \int_a^b \partial_x G(x,y) f(y) r(y)\, dy,$$

where $\partial_x G(x,y)$ is just the pointwise derivative in L^∞ and

$$\begin{aligned}\partial_x^2 \left(R_{\mathcal{L}}(z)f\right)(x) =& \frac{1}{W(z)}\left(\int_x^b \partial_x^2 u_a(x;z) u_b(y;z) f(y) r(y)\, dy + \int_a^x u_a(y;z)\partial_x^2 u_b(x;z) f(y) r(y)\, dy\right)\\ &+ \frac{1}{W(z)}\left(-\partial_x u_a(x;z) u_b(x;z) f(x) r(x) + u_a(x;z)\partial_x u_b(x;z) f(x) r(x)\right)\end{aligned}$$

which implies that $R_{\mathcal{L}}(z)f \in C^2(\bar{I})$ provided $f \in C^0(\bar{I})$ while $R_{\mathcal{L}}(z)f \in H^2(I)$ provided $f \in L^2(I)$ and in both cases, we can find that $(\mathcal{L} - z)R_{\mathcal{L}}(z)f = f$.

And $R_{\mathcal{L}}(z) : C(\bar{I}) \cap H \to \mathcal{D}(\mathcal{L})$ is surjective since for all $u \in \mathcal{D}(\mathcal{L})$, by a direct computation using (1.3), one can see $R_{\mathcal{L}}(z)(\mathcal{L}-z)u = u$. Hence, it is natural to denote $R_{\mathcal{L}}(z)$ by $(\mathcal{L}-z)^{-1}$.

Proposition 1.25. *$(\mathcal{L} - z)^{-1}$ is well-defined on $C(\bar{I}) \cap H$ for z such that $W(z) \neq 0$ and it is compact from $C(I) \cap H$ to $C(\bar{I}) \cap H$ and symmetric when $z = \bar{z}$.*

Proof. The well-definedness of $(\mathcal{L} - z)^{-1}$ is shown in the preceding paragraph.

Now we show the compactness. Fix z such that $W(z) \neq 0$ and note that $G(x,y)$ is continuous on $[a,b] \times [a,b]$ and hence uniformly continuous. For $f \in C(\bar{I}) \cap H$, set $g(x) = (\mathcal{L} - z)^{-1} f$, then by $\partial_x G(x,y) \in L^\infty(I \times I)$ and $g'(x) = \int_a^b \partial_x G(x,y) f(y) r(y)\, dy$, we know that

$$|g(x_1) - g(x_2)| \leq C|x_1 - x_2| \|f\|_{L^2(I;rdx)}.$$

Hence, if $\{f_n\} \subset H$ is a bounded sequence, then $\{g_n\}$ is equicontinuous and hence has a uniformly convergent subsequence by the Arzelà-Ascoli theorem. Moreover,

$$\|f_n - f_m\|^2_{L^2(I;r\,dx)} \leq \sup_{x\in[a,b]} |f_n(x) - f_m(x)|^2 \int_a^b r(x)\, dx \leq C_r \left(\sup_{x\in[a,b]} |f_n(x) - f_m(x)|\right)^2$$

implies the convergence of $\{f_n\}$ in H, which shows that $(\mathcal{L} - z)^{-1}$ is compact.

Finally, we show that $(\mathcal{L} - z)^{-1}$ is symmetric if $z = \bar{z}$. If suffices to show $(\mathcal{L} - \bar{z})^{-1} = ((\mathcal{L} - z)^{-1})^*$. Then it suffices to show $G(x,y;z) = G(y,x;\bar{z})$ since this would imply

$$\begin{aligned}\langle g, (\mathcal{L} - z)^{-1} f\rangle &= \int_a^b \overline{g(x)} \left(\int_a^b G(x,y;z) f(y) r(y)\, dy\right) r(x)\, dx\\ &= \int_a^b \left(\int_a^b G(x,y;z)\overline{g(x)} r(x)\, dx\right) f(y) r(y)\, dy\\ &= \int_a^b \overline{\left(\int_a^b G(y,x;\bar{z}) g(x) r(x)\, dx\right)} f(y) r(y)\, dy = \langle (\mathcal{L} - \bar{z})^{-1} g, f\rangle.\end{aligned}$$

Since $\overline{G(x,y)} = G(y,x)$, it suffices to show $\overline{u_a(x;z)} = u_a(x;\bar{z})$, $\overline{u_b(x;z)} = u_b(x;\bar{z})$ and $W(\bar{z}) = \overline{W(z)}$, which is obvious. Thus, this completes the proof. $\square$

Proof of Theorem 1.15.

- We claim that for $z_0 \in \mathbb{R}$ sufficiently negative, $W(z_0) \neq 0$. It suffices to show there does not exist a solution u to the Sturm-Liouville problem with separated boundary condition. Suppose not, noting that $p, r > 0$ on $\bar{I}$, integration by parts shows that there exists some $M > 0$ such that
$$\langle \mathcal{L}u, u\rangle \geq \int q(x)|u(x)|^2\,dx \geq -M\langle u, u\rangle.$$
Then it is obvious that for $z_0 < -M$, u cannot be a solution to $\mathcal{L}u = z_0 u$, which implies $W(z_0) \neq 0$.
- Then one can apply the spectral theorem to $(\mathcal{L} - z_0)^{-1} : C(\bar{I}) \cap H \to \mathcal{D}(\mathcal{L})$ for $z_0 \in \mathbb{R}$. Therefore, we know that there exists a countable number of eigenvalues $\alpha_n \to 0$ plus corresponding orthonormal eigenfunctions u_n that form a basis of $C(\bar{I}) \cap H = \overline{\mathcal{D}(\mathcal{L})}$.
- Since $(\mathcal{L} - z_0)^{-1}u_n = \alpha_n u_n$, we have $\mathcal{L}u_n = \left(z_0 + \frac{1}{\alpha_n}\right)u_n$, which shows that $E_n = z_0 + \frac{1}{\alpha_n}$ are eigenvalues of $\mathcal{L}$ with corresponding eigenfunctions u_n.
- Now we only need to show all eigenvalues are simple. Suppose not, then there exists eigenvalues λ with two eigenfunctions u, v. In particular, $BC_a(u) = BC_b(v) = 0$ and hence $W_a(u,v) = 0$, which implies that u and v are linearly dependent, contradiction!

$\square$

Remark 1.26. In the statement of Theorem 1.15 and the proof above, we use the inner product space $H \cap C^0(I)$ to be our space. (We don't need completeness throughout this subsection.) And as noted in the calculation of resolvent $R_{\mathcal{L}}(z)$, we konw that we can replace all $H \cap C^0(I)$ by H and all $\mathcal{D}(\mathcal{L})$ by $\{H^2(I) : BC_a = BC_b = 0\}$ at the same time to derive another version of the theorem.

1.3. **Nodal set and zeros of eigenfunctions : Variational approach.**

This approach is more general and can be applied to PDEs.

Theorem 1.27 (Variational characterization of eigenvalues, known as the Rayleigh-Ritz principle)**.** *Let $\mathcal{L}$ be an operator in the setting of a regular Sturm-Liouville problem with BC_a, BC_b. Suppose $\lambda_0 < \lambda_1 < \dots$ are eigenvalues to the following Sturm-Liouville problem*
$$\begin{cases} \mathcal{L}u = \lambda u, \\ BC_a(u) = BC_b(u) = 0, \end{cases}$$
then
$$\lambda_0 = \min_{\mathcal{D}(\mathcal{L})} \frac{\langle u, \mathcal{L}u\rangle}{\langle u, u\rangle},$$
where $\mathcal{D}(\mathcal{L}) = C^2(\bar{I}; \mathbb{R}) \cap \{BC_a(u) = BC_b(u) = 0\}$. The minimum is realized if and only if $u \propto e_0$. Moreover,
$$\lambda_k = \min_{u \in \mathcal{D}(\mathcal{L}), u \in (e_0, \dots, e_{k-1})^{\perp}} \frac{\langle u, \mathcal{L}u\rangle}{\langle u, u\rangle},$$
and the minimum is realized if and only if $u \propto e_k$.

Proof. By Theorem 1.15, we know that each $f \in \mathcal{D}(\mathcal{L})$ can be expressed as

$$f(x) = \sum_{j=0}^{\infty} \langle u_n, f\rangle u_n.$$

Since $Ran((\mathcal{L}-z)^{-1}) = \mathcal{D}(\mathcal{L})$, we take $g \in H$ such that $f = (\mathcal{L}-z)^{-1}g$. Then it follows from $f = (\mathcal{L}-z)^{-1}g$ and the boundedness of $(\mathcal{L}-z)^{-1} : H \to H$ that $\langle u_n, f\rangle = \alpha_n \langle u_n, g\rangle$. Hence, $(\mathcal{L}-z)f = g = \sum_{j=0}^{\infty} \langle u_n, f\rangle \frac{1}{\alpha_n} u_n$, which implies

$$\mathcal{L}f = \sum_{j=0}^{\infty} (z + \frac{1}{\alpha_n}) \langle u_n, f\rangle u_n.$$

Put $\lambda_n = z + \frac{1}{\alpha_n}$, $e_n = u_n/\|u_n\|$, $c_n = \langle u_n, f\rangle$. Then one can find that the Rayleigh quotient

$$\frac{\langle \mathcal{L}f, f\rangle}{\langle f, f\rangle} = \frac{\sum_{j=0}^{\infty} \lambda_n^2 c_n^2}{\sum_{j=0}^{\infty} c_n^2},$$

which can reaches its minimum when taking the minimum over $f \in \mathcal{D}(\mathcal{L})$ at $f \propto e_0$, which completes the proof. □

Definition 1.28. *For $f : I \to \mathbb{R}$, the nodal set of f is $\{x \in I : f(x) = 0\}$. And a nodal domain of f is a component of $I \setminus \{x \in I : f(x) = 0\}$.*

Corollary 1.29 (Courant nodal domain theorem)**.** *Suppose e_k is the eigenfunction to $(k+1)$-th smallest eigenvalue λ_k has at most $(k+1)$ nodal domains (k zeros in (a,b))*

Proof. Suppose $e \in \mathcal{D}(\mathcal{L})$ with n zeros such that $\mathcal{L}e = \lambda e$, then it has $n+1$ nodal domains $I_0, I_1, \ldots, I_n$ whose left endpoints are denoted by $x_0, \ldots, x_n$, respectively. It suffices to show $\lambda \geq \lambda_n$.

Set $u_j = e|_{I_j}$ for $0 \leq j \leq n$. Clearly, u_j is orthogonal to each other since their supports are distinct and $\frac{\langle u_j, \mathcal{L}u_j\rangle}{\langle u_j, u_j\rangle} = \lambda$. Note that $u_j \notin \mathcal{D}(\mathcal{L})$, but we proceed the proof to see what's the main idea first without manipulating the domain. Then pointwisely,

$$\mathcal{L}u_j = \begin{cases} \lambda u_j, & x \in I_j, \\ 0 = \lambda u_j, & x \notin I_j. \end{cases}$$

Set $f = \sum_{j=0}^{n} c_j u_j$. Then by linear algebra, one can choose c_j such that

$$\langle e_j, f\rangle = 0, \quad 0 \leq j \leq n-1,$$

that is, $f \in (e_0, \ldots, e_{n-1})^{\perp}$. And $\langle \mathcal{L}f, f\rangle / \langle f, f\rangle = \lambda$. Hence, $\lambda \geq \lambda_n$, which completes the proof.

Now, we resolve the problem of $u_j \notin \mathcal{D}(\mathcal{L})$. Since $x_1, \ldots, x_n$ are zeros of e, so we know that the pointwise derivative of f almost everywhere is equal to its distributional derivative. Hence, by the fact that f' is well-defined and bounded almost everywhere, we have $f \in H^1(I)$. Moreover, by integration by parts on each subinterval(each nodal domain) respectively,

$$\langle \mathcal{L}f, f\rangle = \int p(x)(f'(x))^2 + q(x)f(x)\,dx$$

makes perfect sense. And one can find $f^\varepsilon \in C_c^\infty(I) \subset \mathcal{D}(\mathcal{L})$ such that $f^\varepsilon \to f$ in H^1, which implies that $\langle \mathcal{L}f^\varepsilon, f^\varepsilon\rangle \to \langle \mathcal{L}f, f\rangle$ and $\langle f^\varepsilon, f^\varepsilon\rangle \to \langle f, f\rangle$. Moreover, by modifying f^ε a little bit to

$$g^\varepsilon = f^\varepsilon - \sum_{j=1}^{n-1}\langle f^\varepsilon, e_j\rangle e_j \in \mathcal{D}(L) \cap (e_0, \ldots, e_{n-1})^\perp,$$

we know that the Rayleigh quotient

$$\frac{\langle \mathcal{L}g^\varepsilon, g^\varepsilon\rangle}{\langle g^\varepsilon, g^\varepsilon\rangle} \to \lambda$$

and hence $\lambda \geq \lambda_n$, that is, there exist at least $n+1$ eigenvalues that are less than or equal to λ, which completes the proof. □

Remark 1.30. This argument is very general and can be applied if the eigenvalues can be realized as Rayleigh quotients. The result concerning the upper bound of the number of nodal domains can be generalized to multi-dimensional settings, where you also need some technical estimates about the on the regularity of the nodal domains in order to use the divergence theorem. This is just a basic result in the study of nodal domains for eigenfunctions of Laplace-Beltrami operator on manifolds, which is a story that is still very active. There's a famous conjecture, Yau's conjecture, on the asymptotic number of nodal domains versus the location of eigenvalues.

1.4. **Nodal set and zeros of eigenfunctions : ODE approach.** For specific problems like Sturm-Liouville problem, we can say more about the nodal domains.

Theorem 1.31 (Sturm oscillation theorem)**.** *Consider the regular Sturm-Liouville problem and order the eigenvalues as $\lambda_0 < \lambda_1 < \ldots$, which corresponds to $e_0, e_1, \ldots$ with $\|e_j\| = 1$, respectively. Then e_k has exactly k zeros in (a, b).*

Throughout this subsection, Our basic tool for the proof is to introduce polar coordinates in phase space, which is known as Prüfer variables:

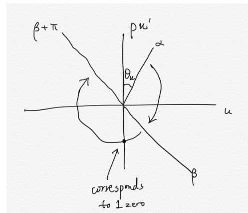
$$\begin{pmatrix} u(x) \\ pu'(x) \end{pmatrix} = \begin{pmatrix} \rho_u(x)\sin(\theta_u(x)) \\ \rho_u(x)\cos(\theta_u(x)) \end{pmatrix},$$

where $u : \bar{I} \to \mathbb{R}$, $p : \bar{I} \to \mathbb{R}$, $p > 0$. Without loss of generality, we assume $\begin{pmatrix} u(x) \\ pu'(x) \end{pmatrix} \neq 0$, that is, $\rho_u(x) \neq 0$. Otherwise, if u is a solution to the regular Sturm-Liouville problem, then $u \equiv 0$. Though $\theta_u(x)$ is defined only up to multiples of 2π, it can be uniquely determined as a continuous function once an initial value at some point c is given. This angle θ_u measures the angle between (u, pu') and the axis pu'.

Note that zeros x_0 of u corresponds to values of x_0 such that $\theta_u(x_0) \equiv 0 \mod \pi$. Thus, counting the zeros of a solution u to the regular Sturm-Liouville problem can be transformed into the counting problem of how many times the θ_u passes the vertical axis. Moreover, $BC_a(u) = \cos\alpha u(a) - \sin\alpha p(a)u'(a) = 0$ if and only if $\theta_u(a) \equiv \alpha \mod \pi$. Similarly, $BC_b(u) = 0$ holds if and only if $\theta_u(b) \equiv \beta \mod \pi$.

Take $u_a(x;\lambda)$ to be solution to the eigenvalue equation $\mathcal{L}u(\cdot;\lambda) = \lambda u(\cdot;\lambda)$ with initial value $u_a(a;\lambda) = \sin\alpha$, $p(a)u_a'(a;\lambda) = \cos\alpha$. Put $\theta_a(x;\lambda) := \theta_{u_a(\cdot;\lambda)}(x)$. Fix $\theta_a(a;\lambda) = \alpha$ with $\alpha \in [0,\pi)$, then $u_a(\cdot;\lambda)$ is an eigenfunction if and only if $\theta_a(b;\lambda) \equiv \beta \bmod \pi$.

Here is a way to visualize the idea of our method, though very heuristic.



However, θ_a is not linear, so we need to make our idea rigorously make sense. Now we derive the ODE for θ_a. Combining the following three formulas

$$\begin{cases} 0 = -(pu')' + qu - \lambda r u, \\ pu' + iu = \rho_u e^{i\theta_u}, \\ (pu')' + iu' = \rho_u' e^{i\theta_u} + i\rho_u\theta_u' e^{i\theta_u}, \end{cases}$$

we have

$$\frac{(-\lambda r + q)u + iu'}{\rho_u} e^{-i\theta_u} = (-\lambda r + q)\sin(\theta_u) + i\frac{1}{p}\cos(\theta_u)e^{-i\theta_u} = \frac{\rho_u'}{\rho_u} + i\theta_u'.$$

By taking the imaginary part, we have

$$\theta_u' = (\lambda r - q)\sin^2\theta_u + \frac{1}{p}\cos^2\theta_u, \tag{1.5}$$

which is the ODE satisfied by θ_u corresponding to a solution u of the eigenvalue problem.

Theorem 1.32 (Comparison for (1.5)). *Consider θ_0, θ_1 solving (1.5) with coefficients p_0, q_0, r_0, λ_0 and p_1, q_1, r_1, λ_1 respectively under the regular Sturm-Liouville assumptions. Suppose*

$$\frac{1}{p_0} \le \frac{1}{p_1} \ (\iff p_1 \le p_0) \ \text{ and } \lambda_0 r_0 - q_0 \le \lambda_1 r_1 - q_1,$$

then

(1) if $\theta_0(x_0) \le \theta_1(x_0)$, then for all $x \in [x_0, b]$, we have $\theta_0(x) \le \theta_1(x)$;
(2) if $\theta_0(x_1) \ge \theta_1(x_1)$, then for all $x \in [a, x_1]$, we have $\theta_0(x) \ge \theta_1(x)$;
(3) if $\theta_0(x_0) = \theta_1(x_0)$ and $\theta_0(x_1) = \theta_1(x_1)$ for $a \le x_0 \le x_1 \le b$, then there is rigidity, that is, $p_0 = p_1$ and $\lambda_0 r_0 - q_0 = \lambda_1 r_1 - q_1$ on $[x_0, x_1]$.

The key tool to prove Theorem 1.32 is to set up some ODE comparisons. This is basically the idea behind weak maximum principle, but here you are just concerned with first order ODEs.

Lemma 1.33 (Comparison lemma). *Let $F(t,y) : [a,b] \times \mathbb{R} \to \mathbb{R}$ satisfies*

- *$g' > F(t, g(t))$ on $[a,b]$, a strict supersolution of the ODE;*
- *$f' = F(t, f(t))$ on $[a,b]$, a solution to the ODE.*

If $g(t_0) \geq f(t_0)$ for some $t_0 \in [a,b]$, then $g(t) \geq f(t)$ for all $t \in [t_0, b]$ and if $g(t_1) \leq f(t_1)$ for some $t_1 \in [a,b]$, then $g(t) \leq f(t)$ for all $t \in [a, t_1]$

Proof. Suppose not, then there exists t^* such that $g(t^*) = f(t^*)$ and $g(t) < f(t)$ for all $t > t^*$. However, $g'(t^*) > F(t^*, g(t^*)) = F(t^*, f(t^*)) = f'(t^*)$, which is a contradiction.

For the second statement, the result follows from reversing the time. □

Proof of the Theorem 1.32. Let

$$F(x,\varphi) = (\lambda_0(x) r_0(x) - q_0(x)) \sin^2\varphi + \frac{1}{p_0(x)} \cos^2\varphi.$$

- Step 1: First, we assume the inequalities in the assumption are strict,

  $$\frac{1}{p_0} < \frac{1}{p_1} \text{ and } \lambda_0 r_0 - q_0 < \lambda_1 r_1 - q_1.$$

  Then

  $$\begin{cases} \theta_1' = (\lambda_1 r_1 - q_1)\sin^2\theta_1 + \frac{1}{p_0}\cos^2\theta_1 > F(x, \theta_1(x)), \\ \theta_0' = F(x, \theta_0(x)), \end{cases}$$

  and the result follows.
- Step 2: Now we eliminate the assumption that inequalities are all strict, and then prove by approximation. We take $(\lambda_1^\varepsilon, p_1^\varepsilon, q_1^\varepsilon, r_1^\varepsilon)$, such that it converges to $(\lambda_1, p_1, q_1, r_1)$ on $[a,b]$ with $\frac{1}{p_1^\varepsilon} \geq \frac{1}{p_1} + \varepsilon$ and $\lambda_1^\varepsilon r_1^\varepsilon - q_1^\varepsilon \geq \lambda_1 r_1 - q_1 + \varepsilon$. Then by applying the result proved in the preceding step, we know $\theta_0(x) \leq \theta_1^\varepsilon(x)$ on $[x_0, b]$ for all $\varepsilon > 0$. By passing to the limit, the result follows.
- Step 3: The second assertion follows from reversing the time.
- Step 4: By the first two assertions, we know $\theta_0(x) = \theta_1(x) := \theta(x)$ for all $x \in [x_0, x_1]$. This implies that

$$(\lambda_0(x) r_0(x) - q_0(x)) \sin^2\theta(x) + \frac{1}{p_0(x)}\cos^2\theta(x) = (\lambda_1(x) r_1(x) - q_1(x))\sin^2\theta(x) + \frac{1}{p_1(x)}\cos^2\theta(x)$$

  holds for all $x \in [x_0, x_1]$. The result follows by plugging in any two distinct x's.

□

Corollary 1.34. *$\theta_a(x;\lambda)$ must be strictly increasing in λ for all $x \in (a,b]$.*

Proof. Suppose $\lambda_0 < \lambda_1$, Since $\theta_a(a, \lambda_0) = \theta_a(a, \lambda_1)$, then by Theorem 1.32(1), $\theta_a(x, \lambda_0) \leq \theta_a(x, \lambda_1)$ and the inequality is strict thanks to Theorem 1.32(3). □

The following lemma tells us θ_u can only cross a multiple of π from below and hence will always increase by π between two consecutive zeros.

Lemma 1.35. *If $\theta_u(x_0) = 0$, then $\theta_u'(x_0) = \frac{1}{p(x_0)} > 0$.*

Proof. This follows directly from (1.5). □

Lemma 1.36. *The function $\theta_a(x;\lambda)$ satisfies*

$$\lim_{\lambda \to -\infty} \theta_a(x;\lambda) = 0,$$

for all $x \in (a,b]$.

Proof. By Lemma 1.34, the limit exists. Moreover, θ_a is always positive since it cannot cross zero from above thanks to Corollary 1.35 and we have $\theta_a(a) \in [0, \pi)$.

Fix $x_0 \in (a, b]$, $\varepsilon > 0$, put $\Theta(x) = \alpha - (\alpha - \varepsilon)\frac{x-a}{x_0-a}$. We want to show

$$\Theta' > (\lambda r - q)\sin^2\Theta + \frac{1}{p}\cos^2\Theta$$

on $[a, x_0]$. Assuming this claim, by comparison lemma, Θ is a supersolution such that $\Theta(a) = \theta_a(a)$, which implies $\theta_a(x; \lambda) \leq \Theta(x)$ on $[a, x_0]$. In particular, $\theta_a(x_0; \lambda) \leq \varepsilon$. Hence, $\limsup_{\lambda\to-\infty} \theta_a(x_0; \lambda) = 0$ and therefore $\lim_{\lambda\to-\infty} \theta_a(x_0; \lambda) = 0$.

Proof of claim: Since $\Theta \geq \varepsilon$, $\sin^2\Theta \geq C\varepsilon^2$. Let λ be sufficiently negative such that $(\lambda r - q)\sin^2\Theta \leq A - \frac{1}{p}$ for some $A \in \mathbb{R}$ to be determined. Since $\Theta' = -(\alpha - \varepsilon)$, the claim holds if $A - \frac{1}{p} + \frac{1}{p}\cos^2\Theta \leq A$ is less than $-(\alpha - \varepsilon)$, and then choosing $A = -\alpha$ would work. □

Now we are armed with all the tools needed to prove the Sturm oscillation theorem.

Proof of Theorem 1.31. By Lemma 1.36, as $\lambda \to -\infty$, $\theta_a(b; \lambda) \to 0$. Thanks to Lemma 1.34, and the existence of eigenvalues, by increasing λ, the minimal λ such that $\theta_a(b, \lambda_0) = \beta_0$ shall be the first eigenvalue $\lambda = \lambda_0$, , where $\beta_0 \equiv \beta \mod \pi$ and $\beta_0 \in (0, \pi]$. Since $\beta_0 \in (0, \pi]$, $\theta_a(\cdot, \lambda_0)$ stays in between $(0, \pi)$ on $(a, b]$, which has no zeros.

Increasing λ again until $\theta_a(b; \lambda) = \beta_0 + \pi$, for some $\lambda = \lambda_1$, which shall be the second eigenvalue.

By performing this argument repeatedly, the theorem follows. □

1.5. **Floquet theory and Sturm Liouville problems with periodic coefficients.** We study the same operator $\mathcal{L}$ with $p, q, r, p' \in C^1(\bar{I}; \mathbb{R})$ with $p, r > 0$ on $\bar{I}$, where $I = (0, l)$. Moreover, we require that p, q, r are l-periodic. There are two naturally associated problems. One is the periodic boundary value problem:

$$\begin{cases} \mathcal{L}u = \lambda u, \\ u \text{ is periodic with period } l, \end{cases} \tag{1.6}$$

and the other is the problem on $(-\infty, \infty)$ with periodic coefficients. We can see later that the first one can be used to study the second.

The typical example of a periodic boundary value problem is $\mathcal{L} = -\frac{d^2}{dx^2}$ with $l = 2\pi$. The solution will be $\lambda = n$ and $e_n = \sin nx, \cos nx$.

1.5.1. *Floquet theory.* This is the structure of solutions to linear ODEs on $(-\infty, \infty)$ with periodic coefficients. In this part, we consider

$$\vec{y}\,' = A(x)\vec{y}, \tag{1.7}$$

where $\vec{y} \in \mathbb{R}^n$, $A : \mathbb{R} \to R^{n\times n}$ has period l, that is, $A(x) = A(x + l)$.

Note that in view of the case $A = const \neq 0$, we need to allow exponential growth or decay of solutions. We will see in a moment that Floquet's theorem tells us once we factor out such exponential behaviors out, the solution is periodic.

Definition 1.37. *We say $\Pi(x, x_0)$ is a principal matrix solution to the problem (1.7) (at x_0) if it satisfies*

$$\frac{d}{dx}\Pi(x, x_0) = A(x)\Pi(x, x_0), \quad \Pi(x_0, x_0) = I, \tag{1.8}$$

which is uniquely defined thanks to the fundamental theorem of ODEs.

For each $\vec{y_0} \in \mathbb{R}^n$, $\vec{y} = \Pi(x, x_0)\vec{y_0}$ is the solution to (1.7) with the initial value $\vec{y}(x_0) = \vec{y_0}$.

Lemma 1.38. *The principal matrix solution satisfies*

$$\Pi(x, x_1)\Pi(x_1, x_0) = \Pi(x, x_0). \tag{1.9}$$

In particular, $\Pi(x_1, x_0)^{-1} = \Pi(x_0, x_1)$.

Proof. Note that both side of (1.9) solves the ODE in (1.8) and coincide at $x = x_1$. □

By periodicity of A, we have $\Pi(x + l, x_0 + l) = \Pi(x, x_0)$, which suggests the key to understand dynamics is the principal solution starting at x_0 and evaluated after l, that is $\Pi(x_0 + l, x_0) := M_{x_0}$.

Definition 1.39. *The matrix $M_{x_0} := \Pi(x_0 + l, x_0)$ is called the monodromy matrix for (1.7).*

Note that if $M_{x_0} = \Pi(x_0 + l, x_0) = I$, then

$$\Pi(x + l, x_0) = \Pi(x + l, x_0 + l)\Pi(x_0 + l, x_0) = \Pi(x, x_0)$$

and hence the solutions to (1.7) will be periodic. But as we mentioned above, this fails to hold in one-dimension with constant matrix A. Fortunately, we have

$$\Pi(x_0+kl, x_0) = \Pi(x_0+kl, x_0+(k-1)l)\Pi(x_0+(k-1)l, x_0) = M_{x_0}\Pi(x_0+(k-1)l, x_0) = \cdots = M_{x_0}^k$$

for any integer k. Thus $\Pi(x, x_0)$ exhibits an exponential behavior if we move on by one period in each step.

So we want to find $P(x, x_0)$ such that $P(x_0, x_0) = I$ and $P(x_0 + l, x_0) = M_{x_0}^{-1}\Pi(x, x_0)$. Heuristically, $P(x, x_0)$ is the matrix we expect after factoring out the exponential behavior of $\Pi(x, x_0)$.

We want to find Q_{x_0} such that $M_{x_0} = \exp(lQ_{x_0})$ and define $P(x, x_0) = \Pi(x, x_0)e^{-(x-x_0)Q_{x_0}}$. If so, then P will be l-periodic, which is because

$$\begin{aligned} P(x + l, x_0) =& \Pi(x + l, x_0)e^{-(x+l-x_0)Q_{x_0}} = \Pi(x + l, x_0 + l)\Pi(x_0 + l, x_0)e^{-(x+l-x_0)Q_{x_0}} \\ =& \Pi(x + l, x_0 + l)M_{x_0}e^{-lQ_{x_0}}e^{-(x-x_0)Q_{x_0}} = \Pi(x + l, x_0 + l)e^{-(x-x_0)Q_{x_0}} = P(x, x_0). \end{aligned}$$

It remains to find Q_{x_0}, which heuristically the log of M_{x_0}. Thanks to (1.9), M_{x_0} has no zero eigenvalues.

Using the Jordan canonical form, $\log M_{x_0}$ is well-defined. A fancy way to do this is holomorphic functional calculus. A more hands-on way to do this is using the series

$$\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^2 - \cdots .$$

Without loss of generality, we assume M_{x_0} only has one Jordan block, $M_{x_0} \sim (\alpha_0 I + N)$. Then $\log(\alpha I + N) = \log \alpha + \log\left(I + \frac{1}{\alpha}N\right)$ can be defined using series expansion, where the sequence is finite since N is nilpotent.

Now we can conclude by the following theorem.

Theorem 1.40 (Floquet's theorem). *Suppose $A(\cdot)$ is l-periodic, then the principal matrix solution has the form $\Pi(x,x_0) = P(x,x_0)e^{l(x-x_0)Q_{x_0}}$, where $P(\cdot,x_0)$ is also l-periodic and $P(x_0,x_0) = I$.*

From the characterization above, one only need to focus on Q_{x_0} to see whether it is bounded. Using this, we can formulate the question of stability of solutions to (1.7), using Q_{x_0}, or equivalently M_{x_0}.

Lemma 1.41. *M_{x_1} and M_{x_2} are similar for all x_1, x_2.*

Proof. We write

$$\begin{aligned} M_{x_0} &= \Pi(x_0+l,x_0) = \Pi(x_0+l,x_1+l)\Pi(x_1+l,x_1)\Pi(x_1,x_0) \\ &= \Pi(x_0,x_1)M_{x_1}\Pi(x_1,x_0) = \Pi(x_1,x_0)^{-1}M_{x_1}\Pi(x_1,x_0). \end{aligned}$$

□

Thus, the eigenvalues of M_{x_0} are independent of the choice of x_0 and the same holds for the eigenvalues of Q_{x_0}.

Definition 1.42. *The Floquet multipliers ρ_j are defined as eigenvalues of M_{x_0}, and the Floquet exponents r_j are defined as eigenvalues of Q_{x_0} with $\rho_j = e^{lr_j}$.*

Definition 1.43. *A linear system is called stable to the future if all solutions remain bounded as $t \to +\infty$.*

Theorem 1.44. *The system (1.7) is stable to the future if and only if $|\rho_j| \le 1$ ($Re(r_j) \le 0$) and the Jordan blocks of M_{x_0} corresponding to $|\rho_j| = 1$ ($Re(r_j) \le 0$) has no nilpotent part. Likewise, to get the characterization for backward stable, you just need to replace $\le$ by $\ge$.*

Proof. By Floquet's theorem, (1.7) is stable to the future if and only if $e^{l(x-x_0)Q_{x_0}}\vec{y_0}$ is stable to the future for all $\vec{y_0}$. Since for any generalized eigenvectors of Q_{x_0}, it is bounded for all $x > x_0$ respectively, then we know the requirements for the eigenvalues of Q_{x_0} are as in the statement of the theorem. □

1.5.2. *Sturm-Liouville problem with periodic coefficients.* Now we specialize to Sturm-Liouville problem with periodic coefficients (1.10). Our goal is to determine the set

$$\{\lambda \in \mathbb{R} : \mathcal{L}u = \lambda u \text{ is stable}\}.$$

And we will see in the following weeks that this set is somehow equal to $Spec(\mathcal{L})$ on $L^2(-\infty,\infty)$.

Our set-up is as follows. The eigenvalue equation $\mathcal{L}u = zu$ is equivalent to

$$\begin{pmatrix} u \\ pu' \end{pmatrix}' = \begin{pmatrix} 0 & \frac{1}{p} \\ zr - q & 0 \end{pmatrix} \begin{pmatrix} u \\ pu' \end{pmatrix} := A(x) \begin{pmatrix} u \\ pu' \end{pmatrix} \tag{1.10}$$

and we take $x_0 = 0$. The corresponding principal matrix solution is given by

$$\Pi(x,0;z) = \begin{pmatrix} c(x;z) & s(x;z) \\ pc'(x;z) & ps'(x;z) \end{pmatrix},$$

where c, s are the solution to

$$\begin{cases} \mathcal{L}c = zc, & c(0) = 1, \quad pc'(0) = 0, \\ \mathcal{L}s = zs, & s(0) = 0, \quad ps'(0) = 1. \end{cases}$$

Here we use the notation c, s to denote since they are initials of $\cos, \sin$.

With a slight abuse of notation, we denote the monodromy matrix by

$$M(z) := \Pi(l,0;z) = \begin{pmatrix} c(l;z) & s(l;z) \\ pc'(l;z) & ps'(l;z) \end{pmatrix}.$$

We are interested in the two eigenvalues of $M(z)$. Here's an observation that $\det M(z) = 1$, which follows from the property of modified Wronskian, Corollary 1.20, that

$$\det M(z) = \det \begin{pmatrix} c(l;z) & s(l;z) \\ pc'(l;z) & ps'(l;z) \end{pmatrix} = W_l(c,s) = W_0(c,s) = 1. \tag{1.11}$$

Definition 1.45. *The Floquet discriminant is defined by $\Delta(z) = \frac{trM(z)}{2}$. Then the Floquet multipliers are given as $\rho_\pm(z) = \Delta(z) \pm \sqrt{\Delta(z)^2 - 1}$.*

Definition 1.46. *The stability set is given by*

$$\Sigma = \{\lambda \in \mathbb{R} : |\Delta(\lambda)| \le 1\}.$$

Lemma 1.47. *The stability set Σ indeed characterizes the stability of* (1.10)*. More precisely,*

$$\Sigma = \overline{\{\lambda \in \mathbb{R} : (1.10) \text{ is forward and backward stable}\}}.$$

Proof. For λ real, $\Delta(\lambda)$ is real, and then

$$\rho_\pm = \begin{cases} \Delta(\lambda) \pm i\sqrt{1 - \Delta(\lambda)^2}, & |\Delta(\lambda)| < 1, \\ \Delta(\lambda), & \Delta(\lambda) = \pm 1, \\ \Delta(\lambda) \pm \sqrt{\Delta(\lambda)^2 - 1}, & |\Delta(\lambda)| > 1. \end{cases} \tag{1.12}$$

Now we apply the characterization for stability in Theorem 1.44 and hence if $|\Delta(\lambda)| < 1$, then $|\rho_\pm| = 1$ implies (1.10) is forward and backward stable while if $|\Delta(\lambda)| > 1$, then $|\rho_+| > 1, |\rho_-| < 1$, implies (1.10) is forward and backward unstable. ☐

Our goal is to study the stability set Σ. The basic approach is to start from $\lambda = -\infty$ and study $\Delta(\lambda)$ as $\lambda \to \infty$. We claim that if $\lambda \in \partial\Sigma$, that is, $|\Delta(\lambda)| = 1$, then λ is an eigenvalue for Sturm-Liouville problem associated with $\mathcal{L}$ for periodic ($\Delta(\lambda) = 1$) functions or anti-periodic ($\Delta(\lambda) = -1$) functions, that is, $f(x + l) = -f(x)$.

Recall that $\Delta(\lambda) = \pm 1$ implies that $M(\lambda)$ has eigenvalues ± 1. If $\Delta(\lambda) = 1$, then there exists a solution such that

$$\begin{pmatrix} u \\ pu' \end{pmatrix}(l) = M(\lambda) \begin{pmatrix} u \\ pu' \end{pmatrix}(0) = \begin{pmatrix} u \\ pu' \end{pmatrix}(0),$$

which is exactly the periodic condition. Likewise, if $\Delta(\lambda) = -1$, then there exists a solution such that

$$\begin{pmatrix} u \\ pu' \end{pmatrix}(l) = M(\lambda)\begin{pmatrix} u \\ pu' \end{pmatrix}(0) = -\begin{pmatrix} u \\ pu' \end{pmatrix}(0).$$

More precisely, we define the periodic (antiperiodic) H^2 functions as

$$H^2_{p,\pm} := \{f \in H^2(0,l) : \exists \widetilde{f} \in H^2_{loc}(\mathbb{R}), \widetilde{f} : \mathbb{R} \to \mathbb{R}, \widetilde{f}(x+l) = \pm\widetilde{f}(x),$$
$$p(x+l)\widetilde{f}'(x+l) = \pm p(x)\widetilde{f}'(x) \text{ s.t. } \widetilde{f}|_{(0,l)} = f\}$$

We construct $L_\pm : \mathcal{D}(L_\pm) \to L^2(0,l)$ with $\mathcal{D}(L_\pm) = H^2_{p,\pm}(0,l)$, such that $L_\pm f = \mathcal{L}f$.

Lemma 1.48. $\Delta(\lambda) = \pm 1$ *if and only if* λ *is an eigenvalue for* $L_\pm$ *and there exists a sequence of real eigenvalues with no finite accumulation point.*

Proof. Obviously, $L_\pm$ is symmetric on $\mathcal{D}(L_\pm)$. Note that by constructing the Green's functions as in Section 1.2, (see [12, Chapter 5, Problem 5.33]) one can show that $L_\pm$ has compact resolvent for z real and sufficiently negative, then $L_\pm$ has countably many eigenvalues for $L_+ : \lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots$ and for $L_- : \mu_0 \le \mu_1 \le \mu_2 \le \ldots$ that accumulates at ∞. □

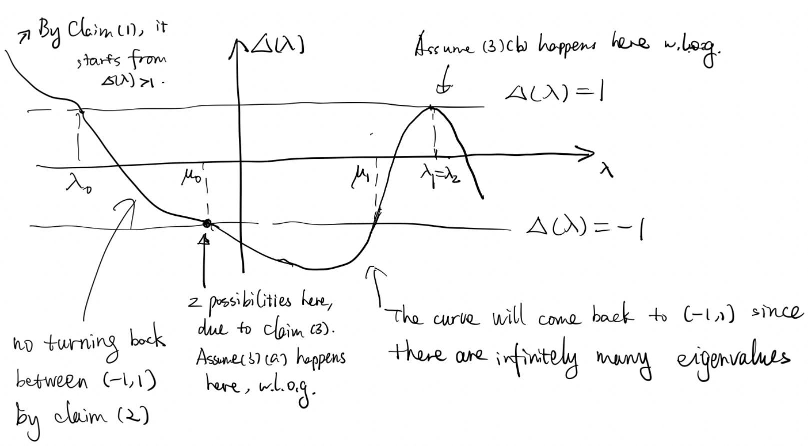
Theorem 1.49. *We have* $\lambda_0 < \mu_0 \le \mu_1 < \lambda_1 \le \lambda_2 < \mu_2 \le \mu_3 < \ldots$ *and* $\Sigma = [\lambda_0, \mu_0] \cup [\mu_1, \lambda_1] \cup [\lambda_2, \mu_2] \cup \ldots$.

To prove this theorem, we make the following claims :

(1) If λ is sufficiently negative, then $\Delta(\lambda) > 1$.
(2) If $\lambda \in \Sigma^{Int}$, that is $|\Delta(\lambda)| < 1$, then $\frac{d}{d\lambda}\Delta(\lambda) \neq 0$.
(3) If $\lambda \in \partial\Sigma$, that is, $|\Delta(\lambda)| = 1$, then either $\frac{d}{d\lambda}\Delta(\lambda) \neq 0$ or $\Delta(\lambda)\frac{d^2}{d\lambda^2}\Delta(\lambda) < 0$, $\frac{d}{d\lambda}\Delta(\lambda) = 0$.

Given these claims, this theorem follows.

Proof of Theorem 1.49. It's a direct but somewhat tedious proof. It's easy to see the idea and how to make it rigorous from the picture below.



□

For the rest of this subsection, we prove these three claims.

Theorem 1.50 (Claim 1). *If λ is sufficiently negative, then $\Delta(\lambda) > 1$.*

Proof.
$$\Delta(\lambda) = \frac{1}{2}\mathrm{tr}M = \frac{1}{2}\left(c(l,\lambda) + p(l)s'(l,\lambda)\right)$$
Take λ sufficiently negative so that $\lambda r - q < 0$ on $[c, l]$, since $(pu')' = -(\lambda r - q)u$.

We claim that if $u(0) \geq 0$, $pu'(0) \geq 0$, then
$$u(x) \geq 0, pu'(x) \geq 0, (pu')'(x) \geq 0, \forall x > 0,$$
which can be proved using continuous induction or proved by contradiction. If at least one in the condition is strict inequality, then all the conclusions are strict. In particular, $c(x;\lambda) > c(0;\lambda) = 1$, $p(x)s'(x;\lambda) > p(0)s(0;\lambda) = 1$, which implies that $\Delta(\lambda) > 1$. $\square$

We write $\vec{u} = \begin{pmatrix} u \\ pu' \end{pmatrix}$, then $M(\lambda) = \begin{pmatrix} \vec{c}(l;\lambda) & \vec{s}(l;\lambda) \end{pmatrix}$. Note $\dot{\Delta}(\lambda) = \frac{1}{2}\left(\dot{c}(l;\lambda) + p\dot{s}(l;\lambda)\right)$ and

$$\begin{cases} (\mathcal{L} - \lambda)\dot{c} = c, \\ \dot{c}(0;\lambda) = 0, \\ p\dot{c}'(0;\lambda) = 0, \end{cases} \qquad \begin{cases} (\mathcal{L} - \lambda)\dot{s} = s, \\ \dot{s}(0;\lambda) = 0, \\ p\dot{s}'(0;\lambda) = 0. \end{cases}$$

A key intermediate goal is to compute $\dot{\Delta}(\lambda)$ and hence we need to compute $\dot{c}(l;\lambda)$ and $\dot{s}(l;\lambda)$. The idea is to use the fact below. For any $\vec{u}, \vec{v}, \vec{w}$ such that $\{\vec{u}, \vec{v}\}$ is linearly independent at $x = l$, then
$$\vec{w} = \frac{W_l(\vec{w}, \vec{v})}{W_l(\vec{u}, \vec{v})}\vec{u} + \frac{W_l(\vec{w}, \vec{u})}{W_l(\vec{v}, \vec{u})}\vec{v}, \tag{1.13}$$
where the coefficients in the equation above are obtained by pairing $\vec{w}$ with $\vec{u}$ and $\vec{v}$ respectively and using the property that $W_l(\vec{u}, \vec{u}) = W_l(\vec{v}, \vec{v}) = 0$. Note that the Wronskian of two solutions u, v to $\mathcal{L} - \lambda = 0$ are easily computed as follows
$$\int_0^l (\mathcal{L} - \lambda)uv\,dx - \int_0^l u(\mathcal{L} - \lambda)v\,dx = W_l(u, v) - W_0(u, v).$$
The obvious choice of $\{u, v\}$ is $\{c, s\}$, which will work but it is tedious to compute. We need to complete the square in some middle step. However, there is a better choice to let $\{u, v\}$ to be the eigenvectors of $M(\lambda)$. Let
$$\begin{pmatrix} 1 \\ m_\pm(\lambda) \end{pmatrix}$$
be eigenvectors of $M(\lambda)$ corresponding to $\rho_\pm$, then
$$M - \rho_\pm I = \begin{pmatrix} c(l;\lambda) - \rho_\pm(\lambda) & s(l;\lambda) \\ p(l)c'(l;\lambda) & p(l)s'(l;\lambda) - \rho_\pm(\lambda) \end{pmatrix}$$
and hence the Weyl-Titchmarsh functions are
$$m_\pm(\lambda) = -\frac{c(l;\lambda) - \rho_\pm(\lambda)}{s(l;\lambda)} = -\frac{p(l)c'(l;\lambda)}{p(l)s'(l;\lambda) - \rho_\pm(\lambda)}. \tag{1.14}$$

Then we say

$$u_\pm(x;\lambda) = c(x,\lambda) + m_\pm(\lambda)s(x;\lambda) \tag{1.15}$$

are the Floquet solutions. Let $\{u,v\} = \{u_+, u_-\}$ and note that

$$\begin{pmatrix} u_\pm(l;\lambda) \\ p(l)u'_\pm(l;\lambda) \end{pmatrix} = M(\lambda)\begin{pmatrix} u_\pm(0;\lambda) \\ p(0)u'_\pm(l;\lambda) \end{pmatrix} = M(\lambda)\begin{pmatrix} 1 \\ m_\pm(\lambda) \end{pmatrix} = \rho_\pm \begin{pmatrix} 1 \\ m_\pm(\lambda) \end{pmatrix} \tag{1.16}$$

and $(\mathcal{L}-\lambda)u_\pm = 0$.

Lemma 1.51. $W(u_+(\cdot;\lambda), u_-(\cdot;\lambda)) = -\frac{2\sqrt{\Delta(\lambda)^2-1}}{s(l;\lambda)}$.

Proof. We write

$$W_0(u_+,u_-) = \det\begin{pmatrix} 1 & 1 \\ m_+ & m_- \end{pmatrix} = m_- - m_+ = \frac{\rho_-(\lambda)-\rho_+(\lambda)}{s(l;\lambda)} = -\frac{2\sqrt{\Delta(\lambda)^2-1}}{s(l;\lambda)}.$$

□

Lemma 1.52. *If $\lambda \in \mathbb{R}$ satisfies $s(l;\lambda) = 0$, then $|\Delta(\lambda)| \geq 1$.*

Proof. If $s(l;\lambda) = 0$, then $M(\lambda)$ is upper triangular and hence has real eigenvalues since $c(l;\lambda)$, $ps'(l;\lambda)$ are real. Hence, by (1.12), $|\Delta(\lambda)| \geq 1$. □

Theorem 1.53 (Claim 2)**.** *If $\lambda \in \Sigma^{Int}$, that is $|\Delta(\lambda)| < 1$, then $\frac{d}{d\lambda}\Delta(\lambda) \neq 0$.*

Proof. Now we compute

$$W_l(\dot{c},u_\pm) - W_0(\dot{c},u_\pm) = \int_0^l (\mathcal{L}-\lambda)\dot{c}u_\pm r\,dx - \int \dot{c}\,(\mathcal{L}-\lambda)\,u_\pm r\,dx.$$

Thanks to $(\mathcal{L}-\lambda)\,u_\pm = 0$, $\dot{c}(0) = 0$ and $(\mathcal{L}-\lambda)\dot{c} = c$, we have

$$W_l(\dot{c},u_\pm) = \int_0^l cu_\pm r\,dx, \qquad W_l(\dot{s},u_\pm) = \int_0^l su_\pm r\,dx,$$

and finally

$$\vec{\dot{c}}(l;\lambda) = \begin{pmatrix} \dot{c}(l;\lambda) \\ p(l)\dot{c}(l;\lambda) \end{pmatrix} = \frac{-s(l;\lambda)}{2\sqrt{\Delta^2-1}}\left(\int_0^l cu_-r\begin{pmatrix} \rho_+ \\ \rho_+m_+ \end{pmatrix} - \int_0^l cu_+r\begin{pmatrix} \rho_- \\ \rho_-m_- \end{pmatrix}\right),$$
$$\vec{\dot{s}}(l;\lambda) = \begin{pmatrix} \dot{s}(l;\lambda) \\ p(l)\dot{s}(l;\lambda) \end{pmatrix} = \frac{-s(l;\lambda)}{2\sqrt{\Delta^2-1}}\left(\int_0^l su_-r\begin{pmatrix} \rho_+ \\ \rho_+m_+ \end{pmatrix} - \int_0^l su_+r\begin{pmatrix} \rho_- \\ \rho_-m_- \end{pmatrix}\right).$$

Hence,

$$
\begin{aligned}
\dot\Delta(\lambda) =& \frac{1}{2}\frac{-s(l;\lambda)}{2\sqrt{\Delta^2-1}}\left(\dot c(l;\lambda)+p(l)\dot s(l;\lambda)\right)\\
=& \frac{1}{2}\frac{-s(l;\lambda)}{2\sqrt{\Delta^2-1}}\left(\int_0^l cu_-r\rho_+ - cu_+r\rho_- + su_-r\rho_+m_+ - su_+r\rho_-m_-\right)\\
=& \frac{1}{2}\frac{-s(l;\lambda)}{2\sqrt{\Delta^2-1}}\left(\int_0^l \rho_+(c+m_+s)u_-r - \rho_-(c+m_-s)u_+r\,dx\right)\\
=& \frac{1}{2}\frac{-s(l;\lambda)}{2\sqrt{\Delta^2-1}}\int_0^l(\rho_+-\rho_-)u_+u_-r\,dx = -\frac{1}{2}s(l;\lambda)\int_0^l u_+u_-r\,dx.
\end{aligned}
\tag{1.17}
$$

Note that if $\lambda\in\Sigma^{int}$, then (1.12) implies $\rho_+(\lambda)=\overline{\rho_-(\lambda)}$ and $u_+(\lambda)=\overline{u_-(\lambda)}$. Thus,

$$\int u_+u_-r\,dx = \int_0^l |u_+|^2 r\,dx > 0.$$

Moreover, $s(l;\lambda)\neq 0$ if $|\Delta(\lambda)|<1$, which completes the proof. $\square$

Theorem 1.54 (Claim 3). *If $\lambda\in\partial\Sigma$, that is, $|\Delta(\lambda)|=1$, then either*
(1) $\frac{d}{d\lambda}\Delta(\lambda)\neq 0$;
(2) $\Delta(\lambda)\frac{d^2}{d\lambda^2}\Delta(\lambda)<0$, $\frac{d}{d\lambda}\Delta(\lambda)=0$.

Proof. Recall that

$$
\begin{aligned}
\dot\Delta(\lambda) =& -\frac{1}{2}s(l;\lambda)\int_0^l (c(x)+m_+s(x))\,(c(x)+m_-s(x))\,r\,dx\\
=& -\frac{1}{2}s(l;\lambda)\int_0^l \left(c(x)^2+(m_++m_-)s(x)+m_+m_-s(x)^2\right)r\,dx
\end{aligned}
$$

By (1.12) and (1.14), we have

$$m_++m_- = \frac{(\rho_+(\lambda)+\rho_-(\lambda))-2c(l;\lambda)}{s(l;\lambda)} = \frac{2\Delta(\lambda)-2c(l;\lambda)}{s(l;\lambda)} = \frac{p(l)s'(l;\lambda)-c(l;\lambda)}{s(l;\lambda)}$$

and

$$
\begin{aligned}
m_+m_- =& \frac{(\rho_+(\lambda)-c(l;\lambda))(\rho_-(\lambda)-c(l;\lambda))}{s(l;\lambda)^2} = \frac{1-(c(l;\lambda)+p(l)s'(l;\lambda))c(l;\lambda)+c(l;\lambda)^2}{s(l;\lambda)^2}\\
=& \frac{1-p(l)s'(l;\lambda)c(l;\lambda)}{s(l;\lambda)^2} = -\frac{p(l)c'(l;\lambda)}{s(l;\lambda)},
\end{aligned}
$$

where we use (1.11) in the last step. Now

$$\dot\Delta(\lambda) = -\frac{1}{2}\int_0^l \left(s(l;\lambda)c(x;\lambda)^2+(p(l)s'(l;\lambda)-c(l;\lambda))\,c(x;\lambda)s(x;\lambda)-p(l)c'(l;\lambda)s(x;\lambda)^2\right)r(x)\,dx. \tag{1.18}$$

Note that in the preceding derivation of (1.18), we didn't use the fact $\lambda \in \partial\Sigma$ and hence we can differentiate this formula with respect to λ. On the other hand, we claim that for $\lambda = E \in \partial\Sigma$, if $\dot{\Delta}(E) = 0$, then $M(E) = \Delta(E)I = \pm I$.

From (1.12), $\rho_+(E) = \rho_-(E) = \pm 1$. Then the claim is true by writing

$$\dot{\Delta}(E) = -\frac{1}{2}s(l;E)\int_0^l u_+u_-r\,dx = -\frac{1}{2}s(l;E)\int_0^l |u_\pm|^2 r(x)\,dx$$

thanks to (1.16) and (1.17), which implies $s(l;E) = 0$. Hence, $M(E) = \Delta(E)I = \pm I$ and in particular,

$$s(l;E) = 0, \quad p(l)c'(l;E) = 0, \quad p(l)s'(l;E) = c(l;E) = \pm 1. \tag{1.19}$$

Now we differentiate (1.18) and evaluating at $\lambda = E \in \partial\Sigma$, using (1.19), we get

$$\ddot{\Delta}(\lambda) = -\frac{1}{2}\int_0^l \left(\dot{s}(l)c(x;\lambda)^2 + (p(l)\dot{s}'(l) - \dot{c}(l))\,c(x;\lambda)s(x;\lambda) - p(l)\dot{c}'(l)s(x)^2\right) r(x)\,dx. \tag{1.20}$$

By (1.3),

$$W_l(\dot{s}, s) = \int_0^l (s + E\dot{s})sr\,dx - \int_0^l \dot{s}Es\,dx = \int_0^l s(x)^2 r(x)\,dx$$

On the other hand, by (1.19),

$$W_l(\dot{s}, s) = p(l)\left(\dot{s}(l;E)s'(l;E) - s(l;E)\dot{s}'(l;E)\right) = \pm\dot{s}(l;E),$$

that is,

$$\dot{s}(l;E) = \pm\int_0^l s(x)^2 r(x)\,dx.$$

Similarly, we have

$$p(l)\dot{s}'(l) = -\dot{c}(l) = \mp\int_0^l c(x)s(x)r(x)\,dx, \quad -p(l)\dot{c}'(l) = \pm\int_0^l c(x)^2 r(x)\,dx.$$

By plugging this into (1.20), we get

$$\Delta\ddot{\Delta} = \left(\int_0^l c(x)s(x)r(x)\,dx\right)^2 - \left(\int_0^l c(x)^2 r(x)\,dx\right)\left(\int_0^l s(x)^2 r(x)\,dx\right),$$

which is strictly negative by the Cauchy-Schwarz inequality.

□

2. Singular Sturm-Liouville problems

Let $\mathcal{L} = \frac{1}{r}\left(-\frac{d}{dx}p\frac{d}{dx} + q\right)$ on I, where either I is finite but p, q, r, p' are singular or I is infinite.

2.1. Unbounded operators.

In this subsection, we assume H is a complex Hilbert space H. An unbounded operator with the domain is $(A, \mathcal{D}(A))$ such that $\mathcal{D}(A) \subset H$, which is assumed to be dense with $A : \mathcal{D}(A) \to H$.

We say B is an extension of A if $A \subset B$, that is, $\mathcal{D}(B) \supset \mathcal{D}(A)$ and $B|_{\mathcal{D}(A)} = A$. Given an unbounded operator A, its graph is defined as

$$\Gamma(A) := \{(u, Au) \in H \times H : u \in \mathcal{D}(A)\}.$$

Then we say A is closed if $\Gamma(A)$ is a closed set and A is closable if there exists a closed extension of A. If A is closable, there exists a smallest closed extension $\bar{A}$, called the closure of A.

Lemma 2.1. *A is closable if and only if $\overline{\Gamma(A)}$ is a graph of an operator. In particular, $\overline{\Gamma(A)} = \Gamma(\bar{A})$.*

Definition 2.2 (Adjoint of A)**.** *Given $A, \mathcal{D}(A)$, define $A^*, \mathcal{D}(A^*)$ as*

$$\mathcal{D}(A^*) := \{u : \exists C > 0 \text{ such that } |\langle u, Av\rangle| \le C\|v\| \text{ for all } v \in \mathcal{D}(A)\},$$

Then for $u \in \mathcal{D}(A^*)$, A^*u is the unique element obtained by Riesz representation theorem. In particular, $\langle u, Av\rangle = \langle A^*u, v\rangle$ for all $u \in \mathcal{D}(A^*), v \in \mathcal{D}(A)$.

We say A is symmetric if for all $u, v \in \mathcal{D}(A)$, $\langle u, Av\rangle = \langle Au, v\rangle$. And we say A is self-adjoint if $A = A^*$, or equivalently, A is symmetric and $\mathcal{D}(A) = \mathcal{D}(A^*)$. An equivalent definition for symmetric operator is that $A \subset A^*$.

Lemma 2.3. *A^* is closed. In fact, $\Gamma(A^*) = (J\Gamma(A))^\perp$.*

Proof. On $H \times H$, the inner product is just $\langle (u, v), (w, z)\rangle = \langle u, w\rangle + \langle v, z\rangle$. Note that $(u, v) \in \Gamma(A^*)$ holds if and only if $\langle u, Aw\rangle = \langle v, w\rangle$ for all $w \in \mathcal{D}(A)$, which is equivalent to $\langle (u, v), (-Aw, w)\rangle = 0$ for all $w \in \mathcal{D}(A)$. Define $J : H \times H \to H \times H$ as $(w, z) \mapsto (-z, w)$, then we know that $\Gamma(A^*) = (J\Gamma(A))^\perp$, which is closed. □

In the proof of the following lemma, the following basic fact for arbirary subspace $W \subset H$ that

$$W^{\perp\perp} = \overline{W} \tag{2.1}$$

will be applied several times.

Lemma 2.4. *A is closable if and only if $\mathcal{D}(A^*)$ is dense. In fact, in this case*

$$\Gamma(\bar{A}) = (J\Gamma(A^*))^\perp \tag{2.2}$$

and $\bar{A} = A^{**}$.

Proof. If $\mathcal{D}(A^*)$ is dense, then A^{**} is well-defined. And $A^{**} \supset A$.

Thanks to Lemma 2.3, (2.1) and $J^2 = 1$, $(J\Gamma(A^*))^\perp = \overline{\Gamma(A)}$. If $\mathcal{D}(A^*)$ is not dense, then there exists $0 \ne u \in \mathcal{D}(A^*)^\perp$. However, this implies $(0, u) \in (J\Gamma(A^*))^\perp = \overline{\Gamma(A)}$, which is impossible for $\overline{\Gamma(A)}$ to be a graph of any linear operator. Hence, $\mathcal{D}(A^*)$ is dense provided A is closable.

Using Lemma 2.3 again, we get

$$\Gamma(A^{**}) = (J\Gamma(A^*))^\perp = \left(J\left(J\Gamma(A)\right)^\perp\right)^\perp = \left(J^2\left(\Gamma(A)^\perp\right)\right) = \Gamma(A)^{\perp\perp} = \overline{\Gamma(A)},$$

and hence $\bar{A} = A^{**}$. □

Example 2.5 (A non-trivial example of non-closed operators)**.** *Let $v \in L^p([0,1]), 1 < p < 2$ and $v \notin L^2([0,1])$. Then for $0 \neq v_0 \in L^2([0,1])$, we define $Au := \langle u, v\rangle v_0$ with domain $\mathcal{D}(A) = L^{p'}([0,1]) \subset L^2([0,1])$.*

For $u \in \mathcal{D}(A^*)$, u need to satisfy

$$|\langle u, Aw\rangle| = |\langle u, v_0\rangle\langle w, v\rangle| \leq C_u\|w\|$$

for all $w \in \mathcal{D}(A)$, which is true only if $u \perp v_0$. That is to say, $\mathcal{D}(A^*) = \{v_0\}^\perp$, which is not dense obviously. Hence, A is not closable thanks to Lemma 2.4.

Theorem 2.6 (Basic criterion for self-adjointness)**.** *If A is a symmetric unbounded operator, then for any fixed $z \in \mathbb{C}\setminus\mathbb{R}$, the following are equivalent:*

- *A is self-adjoint;*
- $\ker(A^* - z) = \ker(A^* - \bar{z}) = \{0\}$*;*
- *$Ran(A - z) = Ran(A - \bar{z}) = H$.*

Proof.
- If A is self-adjoint, then for $z = x + iy$, $y \neq 0$,
$$\|(A^* - z)u\|^2 = \|(A^* - x)u\|^2 + y^2\|u\|^2 \geq y^2\|u\|^2, \tag{2.3}$$
where we use the self-adjointness of $A^* - x$ in the first step. Then the second one holds.
- For (2) $\Rightarrow$ (3), we use the fact $\overline{Ran(A-z)} = \ker(A^* - \bar{z})^\perp$. We claim that if $\|(A-z)u\| \geq \varepsilon\|u\|$ for some $\varepsilon > 0$ and $A - z$ is closed, then $Ran(A-z)$ is closed. This fact follows from the coercivity condition that if $\{(A-z)u_n\} \subset Ran(A-z)$ is Cauchy, then $\{u_n\}$ is Cauchy and hence $Ran(A-z)$ is closed.

  In fact, the assumption in the claim holds since we can show
$$\|(A-z)u\|^2 \geq (Imz)^2\|u\|^2, \tag{2.4}$$
by an analogous argument as (2.3) due to symmetry of A.
- For (3) $\Rightarrow$ (1), we want to show $u \in \mathcal{D}(A)$ for any $u \in \mathcal{D}(A^*)$. For $w \in \mathcal{D}(A) \subset \mathcal{D}(A^*)$, there exists $u \in \mathcal{D}(A)$ such that $(A^* - z)w = (A - z)u = (A^* - z)u$ due to the assumption on the range, which implies $w - u \in \ker(A^* - z) = Ran(A - \bar{z})^\perp = \{0\}$. Therefore, $u = w \in \mathcal{D}(A)$.

□

Definition 2.7. *For a closed operator, the resolvent set of A is defined as*

$$\rho(A) = \{z \in \mathbb{C} : A - z \text{ is bijective } \mathcal{D}(A) \to H\},$$

Then the resolvent of A at $z \in \rho(A)$, $R_A(z) := (A-z)^{-1}$, is a linear operator $H \to H$, which is bounded by the closed graph theorem. If $A - z$ fail to be injective, then we say z is an eigenvalue, then we say z is in the point spectrum

$$\sigma_p(A) := \{z \in \mathbb{C} : z \text{ is an eigenvalue}\}.$$

The spectrum of A is defined as $\sigma(A) := \mathbb{C} \setminus \rho(A)$.

Corollary 2.8. *If A is symmetric, then A is self-adjoint if and only if $\sigma(A) \subset \mathbb{R}$. Moreover, $\|R_A(z)\| \le \frac{1}{|Imz|}$.*

Proof. The first part follows from Theorem 2.6 directly. The second part follows from (2.4). ☐

Example 2.9. *Suppose $H = L^2([0, 2\pi])$ and $A = \frac{1}{i}\frac{d}{dx}$, $\mathcal{D}(A) = C_c^\infty((0, 2\pi))$, then $A^*u = \frac{1}{i}\frac{d}{dx}u$ with $\mathcal{D}(A^*) = H^1([0, 2\pi])$. Moreover, $\sigma(A^*) = \mathbb{C}$ since $u = e^{z \cdot x} \in \mathcal{D}(A^*)$ solves $(A - \frac{1}{i}z)u = 0$.*

2.2. **Spectral theory for unbounded operators.** Now we would follow [13, Chapter 3] to develop the Spectral theory and throughout this subsection, we shall develop the Spectral theory of unbounded (linear) operators on a complex separable Hilbert space H.

For the finite dimensional case, for any symmetric matrix A, there exists a unitary matrix U such that $A = U^{-1}DU$ with D diagonal with real entries. The diagonal matrix in finite dimensional case can be viewed as

$$u : \{1, \dots, n\} \to \mathbb{C},$$

which can be generalized to the infinite dimensional case by a function $u : X \to \mathbb{C}$ given by the multiplication by $a(x)$.

The following example is taken from [13, Chapter 2.2, 2.4].

Proposition 2.10. *For $H = L^2(X, d\mu)$, $a : X \to \mathbb{R}$ is measurable and a is finite μ-a.e. on X. Then set $M_a u = a(x)u(x)$ with $\mathcal{D}(M_a) := \{u \in L^2(X, d\mu) : au \in L^2(X, d\mu)\}$. Then M_a is self-adjoint and $\sigma(M_a)$ equals the essential range of a, denoted by essran(a).*

Proof. By definition, it is obvious that M_a is symmetric. In particular, $\mathcal{D}(M_a) \subset \mathcal{D}(M_a^*)$. Now we want to show the reverse inclusion. If $h \in \mathcal{D}(M_a^*)$, then there is some $g \in L^2(X, d\mu)$ such that

$$\int \overline{h(x)}a(x)f(x)\, d\mu(x) = \int \overline{g(x)}f(x)\, d\mu(x), \forall f \in \mathcal{D}(M_a),$$

and thus

$$\int \overline{(h(x)a(x) - g(x))}f(x)\, d\mu(x) = 0, \forall f \in \mathcal{D}(M_a).$$

If we take $f(x) = \widetilde{f}(x)\chi_{\Omega_n}(x)$ in the equation above, where $\widetilde{f} \in L^2(X, d\mu)$ and $\chi_{\Omega_n}(x)$ is the characteristic function for $\Omega_n = \{x \in X : |a(x)| \le n\}$, then we get

$$\chi_{\Omega_n}(x)\overline{(h(x)a(x) - g(x))} = 0 \in L^2(X, d\mu)$$

since $\widetilde{f} \in L^2(X, d\mu)$ is arbitrary. Moreover, since n is also arbitrary, we know $h(x)a(x) = g(x) \in L^2(X, d\mu)$, which implies $h \in \mathcal{D}(M_a)$. This completes the proof of self-adjointness.

It is obvious that

$$(M_a - z)^{-1} f(x) = \frac{1}{a(x) - z} f(x), \quad \mathcal{D}((M_a - z)^{-1}) = \{f \in L^2(X, d\mu) : \frac{1}{a(x) - z} f(x) \in L^2(X, d\mu)\}$$

whenever $(M_a - z)^{-1}$ is bounded. Note that $\|(M_a - z)^{-1}\| = \left\| \frac{1}{a(x) - z} \right\|_{L^\infty}$ implies that the resolvent set is given by

$$\rho(M_a) = \{z \in \mathbb{C} : \exists \varepsilon > 0 \text{ such that } \mu(\{x : |a(x) - z| < \varepsilon\}) = 0\},$$

and hence

$$\sigma(M_a) = \{z \in \mathbb{C} : \forall \varepsilon > 0, \mu(\{x : |a(x) - z| < \varepsilon\}) > 0\} = essran(a).$$

$\square$

Remark 2.11. Unless a is bounded, M_a is unbounded.

Theorem 2.12 (Spectral theorem - multiplication version)**.** *Let $A : H \to H$ be a self-adjoint operator, there exists some measure space (X, μ) and an unitary operator $U : H \to L^2(X, d\mu)$ such that there exists a function $a : X \to \mathbb{R}$ measurable and finite μ-a.e. as in the preceding proposition such that $A = U^{-1} M_a U$.*

This theorem gives ways to make sense of

$$f(A) = U^{-1} M_{f \circ a} U.$$

Example 2.13. *For $A = \frac{1}{i} \frac{d}{dx}$, $H = L^2(-\infty, \infty)$, then*

$$\mathcal{F} : L^2((-\infty, \infty), dx) \to L^2((-\infty, \infty), d\xi/2\pi)$$

is a unitary operator defined as

$$\mathcal{F}u(x) = \int u(x) e^{-ix\xi}\, dx, \quad u \in \mathscr{S}(\mathbb{R}).$$

Then $A = \mathcal{F}^{-1} M_\xi \mathcal{F}$.

A drawback of this version of spectral theorem is that it is too abstract. The following alternative version is called the functional calculus version, which can be used to make sense of $e^{itA}, \cos t\sqrt{A}, \frac{\sin t\sqrt{A}}{\sqrt{A}}, e^{-tA}$. It is more useful in our setting.

Theorem 2.14 (Spectral theorem - functional calculus version)**.** *Let $A : H \to H$ be a self-adjoint operator. We denote all the Borel measurable, bounded functions by $\mathcal{B}_b$. Then there exists a unique $\mathcal{F}_A$ given by*

$$\mathcal{F}_A : \mathcal{B}_b \to \mathcal{L}(H), \quad f \mapsto f(A) = \mathcal{F}_A(f)$$

such that

- *$\mathcal{F}_A$ is a homomorphism as C^*-algebras, that is, $\mathcal{F}_A(fg) = \mathcal{F}_A(f)\mathcal{F}_A(g)$ and $\mathcal{F}_A(\bar{f}) = \overline{\mathcal{F}_A(f)}$;*

- *$\|\mathcal{F}_A(f)\|_{H\to H} \le \|f\|_{L^\infty}$;*
- *if $f_n \to f$ pointwisely and $\|f_n\|_{L^\infty} \le M$, then for all $u \in H$, $\mathcal{F}_A(f_n)u \to \mathcal{F}_A(f)u$, which is called the strong convergence or convergence in strong operator topology;*
- *if $f_n \in \mathcal{B}_b$ satisfies $f_n(x) \to x$ pointwisely and $|f_n(x)| \le |x|$ for all n, x, then for all $u \in \mathcal{D}(A)$, $\mathcal{F}_A(f_n)u \to Au$.*

Definition 2.15. *Let $\mathbb{B}$ be the Borel σ-algebra on $\mathbb{R}$. A projection-valued measure is a map $P : \mathbb{B} \to \mathcal{L}(H)$ such that*

- *P is an orthogonal projection, that is, $P(\Omega)^2 = P(\Omega), P(\Omega)^* = P(\Omega)$;*
- *$P(\mathbb{R}) = id$;*
- *(strong σ-additivity) if $\Omega = \cup\Omega_k$ and $\Omega_n \cap \Omega_m = \varnothing$ for all $n \ne m$, then for all $u \in H$,*

$$P(\Omega)u = \lim_{N\to\infty} \sum_{n=1}^{N} P(\Omega_n)u.$$

In fact, the operator $\mathcal{F}_A$ in the preceding theorem can be expressed explicitly by assigning a projection-valued measure P_A such that

$$\mathcal{F}_A(f) = \int_{\mathbb{R}} f(\lambda) dP_A(\lambda),$$

where P_A satisfies $P_A(\Omega) = 1_\Omega(A)$. Then we can use this projection-valued measure P_A to extend $\mathcal{F}_A$ to unbounded Borel measurable functions in the following discussion.

For $u \in H$, $P : \mathbb{B} \to \mathcal{L}(H)$ is a projection-valued measure,

$$\mu_u(\Omega) := \langle u, P(\Omega)u\rangle$$

is well-defined as a positive measure thanks to the first property. Given μ_u, we can construct $u, v \in H$

$$\mu_{u,v}(\Omega) = \langle u, P(\Omega)v\rangle.$$

By polarization,

$$\mu_{u,v}(\Omega) = \frac{1}{4}\left(\mu_{u+v}(\Omega) - \mu_{u-v}(\Omega) + \frac{1}{i}\mu_{u+iv}(\Omega) - \frac{1}{i}\mu_{u-iv}(\Omega)\right).$$

Given any $f \in \mathcal{B}_b$, we can make sense of

$$\int_{\mathbb{R}} f\, d\mu_u =: \langle u, \left(\int f\, dP\right) u\rangle,$$

and then using polarization to make sense of

$$\langle u, \left(\int f\, dP\right) v\rangle := \int_R f\, d\mu_{u,v},$$

which implies we can define $\int f\, dP \in \mathcal{L}(H)$ and $|\int f(\lambda)\, d\mu_{u,v}(\lambda)| \le \|f\|_{L^\infty}\|u\|\|v\|$. Furthermore, we obtain the following isometric property.

Lemma 2.16. *Suppose P is a projection-valued measure, then for all $u \in H$, and $f \in \mathcal{B}_b$,*

$$\|\left(\int f\, dP\right) u\|^2 = \int |f|^2\, d\mu_u.$$

And Theorem 2.14 implies $\|\mathcal{F}_A(f)u\|^2 = \int |f|^2\, d\mu_{A,u}$.

Lemma 2.17. *Given an unbounded Borel-measurable function* f *with*

$$\mathcal{D}(\mathcal{F}_A(f)) = \{u \in H : f \in L^2(\mathbb{R}, d\mu)\}$$

and we can make sense of the functional calculus of unbounded functions as

$$\mathcal{F}_A(f)u = \int f(\lambda)\, d\mu_u(\lambda) := \lim_{n\to\infty} f_n(\lambda)\, d\mu_u$$

for $f_n \to f$ *in* $L^2(\mathbb{R}, d\mu)$ *with* $f_n \in \mathcal{B}_b$.

Theorem 2.18 (Spectral theorem - projection-valued version)**.** *Let* $A : H \to H$ *is a self-adjoint operator, then there exists a unique projection-valued measure* $P_A : \mathbb{B} \to \mathcal{L}(H)$ *such that*

$$A = \int_{\mathbb{R}} \lambda\, dP_A(\lambda), \quad \mathcal{D}(A) = \{u \in H : \int \lambda^2\, d\mu_{u,A} < +\infty\}.$$

In particular, $\langle u, Au\rangle = \int \lambda\, d\mu_{u,A}$.

Now we can prove the multiplication version of spectral theorem from the projection-valued measure version.

Given a projection-valued measure P. First, we construct (X, μ) and a unitary operator $U : H \to L^2(X, \mu)$ and a function $a : X \to \mathbb{R}$ such that $A = U^{-1} M_a U$. Given $u \in H$, consider $L^2(\mathbb{R}, d\mu_u)$ and

$$\tilde{H}_u = \{(\int f dP)u : \forall f \in \mathcal{B}_b\}.$$

Lemma 2.16 tells us the map

$$\tilde{H}_u \ni (\int f\, dP)u \mapsto f \in L^2(\mathbb{R}, d\mu_u)$$

extends to a unirary map $H_u \to L^2(\mathbb{R}, d\mu_u)$, where $H_u := \{(\int f\, dP)u : f \in L^2(\mathbb{R}, d\mu_u)\}$.

If there exists $u \in H$ such that $H_u = H$, then the proof is done and we say u is a cyclic vector. In general, this does not need to be true and we need to introduce the spectral bases.

Definition 2.19 (Spectral bases)**.** *Suppose* H *is separable and* $\{u_n\}$ *is a spectral basis if*

- $\|u_n\| = 1$, $u_n \perp u_{n'}$ *if* $n \neq n'$;
- $H = \oplus_n H_{u_n}$.

Theorem 2.20. *For any separable Hilbert space* H *and a projection-valued measure* P*, there exist a spectral basis.*

Proof. We start with a countable basis $\{\tilde{u}_n\}$ of H and do the Gram-Schmidt process, then we can assume $\{\tilde{u}_n\}$ is an orthonormal basis without loss of generality.

The key observation is that if $v \perp H_u$ for some u, then the space generated by v, namely H_v, is orthogonal to H_u. By polarization, Lemma 2.16 implies

$$\langle(\int f\, dP)u, (\int g\, dP)v\rangle = \int \bar{f} g\, d\mu_{u,v}.$$

Then $v \perp H_u$ implies

$$\langle (\int g\,dP)v, (\int f\,dP)u\rangle = \langle v, (\int \bar{g}\,dP)(\int f\,dP)u\rangle = \langle v, (\int \bar{g}f\,dP)u\rangle = 0,$$

and hence $H_v \perp H_u$.

Hence, we can construct a spectral basis by iteration. To be more specific, we choose $u_1 = \tilde{u}_1$ and then move on to the first $\tilde{u}_j$'s which is not in H_{u_1}. Project this element to $H_{u_1}^{\perp}$, normalize it, and we choose the result to be u_2. Proceeding this procedure, we get a set of spectral vectors $\{u_j\}$ such that $span\{\tilde{u}_j\} \subset \oplus_j H_{u_j}$.

Note that H_u is closed since L^2 is and $\psi_n = (\int f_n dP)u$ converges in H if and only if f_n converges in L^2 by Lemma 2.16. Hence, $H = \overline{span\{\tilde{u}_j\}} \subset \oplus_j H_{u_j}$. □

Definition 2.21. *The minimal cardinality of spectral basis is called the spectral multiplicity of P.*

By the theorem, there exists a unitary map $U : H \to \oplus_n L^2(\mathbb{R}, d\mu_{u_n})$ with a spectral basis $\{u_n\}$. If $P = P_A$ is given by the spectral theorem in the projection-valued form, then $(UAU^{-1})_n = M_\lambda$ since $\int \lambda\,dP_A = A$, where M_λ denotes the multiplication operator on $L^2(\mathbb{R}, d\mu_{u_n})$ respectively. Finally, we can combine

$$\oplus_n L^2(\mathbb{R}, d\mu_n) = L^2(\bigsqcup_n (\mathbb{R}, d\mu_n)),$$

which means that the L^2 space is defined on the disjoint union of $\mathbb{R}$'s and assign each $\mathbb{R}$ the measure μ_n.

Now we give a sketch of proof for the projection-valued measure version. The key step is to construct the projection-valued measure.

We observe that to every projection-valued measure P we can assign a self-adjoint operator $\int_{\mathbb{R}} \lambda\,dP$. The question is whether we can invert this map. To do this, we consider the resolvent $R_A(z) := (A - z)^{-1}$ for $z \in \rho(A)$. If A is self-adjoint, then thanks to Theorem 2.6, $R_A(z)$ makes sense for any $z \notin \mathbb{R}$. We write $F_u(z) = \langle u, R_A(z)u\rangle$ when $Imz > 0$. It follows from Corollary 2.8 that $|F_u(z)| = |\langle u, R_A(z)u\rangle| \le \frac{\|u\|^2}{Imz}$. On the other hand, $F_u(z)$ is holomorphic on $\mathbb{H} := \{Im(z) > 0\}$ thanks to the expansion of resolvent operator.

Lemma 2.22. *$F_u(z)$ is holomorphic on $\mathbb{H} := \{Im(z) > 0\}$.*

Proof. First, by calculating in a formal way of writing $R_A(z)$ as $\frac{1}{A-z}$, it is easy to check

$$R_A(z) - R_A(z') = (z - z')R_A(z)R_A(z'), \tag{2.5}$$

and the argument is in fact rigorous thanks to the commutativity of these operators.

Then one can use this iteratively to get

$$R_A(z) = \sum_{j=0}^{n} (z-z_0)^j R_A(z_0)^{j+1} + (z-z_0)^{n+1} R_A(z_0)^{n+1} R_A(z) := \sum_{j=0}^{n} R_n(z) + (z-z_0)^{n+1} R_A(z_0)^{n+1} R_A(z).$$

For $z_0 \in \mathbb{H}$ fixed, $|z - z_0| < \|R_A(z_0)\|^{-1}$, $R_n(z)$ converges to a bounded operator R_∞. One can show that $R_\infty = R_A(z)$ and this implies that $F_u(z)$ is holomorphic by the convergent power series expansion. One can consult [13, Chapter 2.4] for details. □

Moreover, thanks to $R_A(z)^* = R_A(\bar{z})$ and applying (2.5) in the third step, we have

$$\begin{aligned} ImF_u(z) =& Im\langle u, R_A(z)u\rangle = \frac{1}{2i}\langle u, (R_A(z) - R_A(\bar{z}))u\rangle \\ =& (Imz)\langle u, R_A(z)R_A(\bar{z})u\rangle = (Imz)\|R_A(z)u\|^2 \geq 0, \end{aligned}$$

which implies $F_u : \mathbb{H} \to \mathbb{H}$. Then F_u is a Herglotz-Nevanlinna function. Now the claim is that there exists a positive measure μ on $\mathbb{R}$ such that

$$F_u(z) = \langle u, R_A(z)u\rangle = \int_{\mathbb{R}} \frac{1}{\lambda - z}\, d\mu(\lambda).$$

Set $h(z) = ImF_u(z)$. Note $\dfrac{Im(z)}{|\lambda - z|^2} = Im\dfrac{1}{\lambda - z}$, then for $z = x + iy$, we have

$$\int_a^b h(x+iy)\, dx = \int_a^b \int \frac{y}{(x-\lambda)^2 + y^2} d\mu(\lambda)\, dx = \int \left(\int_a^b \frac{y}{(x-\lambda)^2 + y^2}\, dx \right) d\mu(\lambda).$$

Since

$$\lim_{y \to 0_+} \left(\arctan(\frac{b-\lambda}{y}) - \arctan(\frac{a-\lambda}{y}) \right) = \begin{cases} \pi, \lambda \in (a,b), \\ \frac{\pi}{2}, \lambda \in \{a,b\}, \\ 0, \lambda \in [a,b], \end{cases}$$

we know

$$\lim_{y \to 0_+} \int_a^b ImF_u(x+iy)\, dx = \frac{\pi}{2}\left(\mu_u((a,b)) + \mu_u([a,b]) \right),$$

then we define

$$\mu_u(\lambda) = \frac{1}{\pi} \lim_{\delta \to 0_+} \lim_{y \to 0_+} \int_{-\infty}^{\lambda+\delta} ImF_u(x+iy)\, dx$$

which is called the Stieltjes inversion formula. The function $\mu_u(\lambda)$ we get from this formula is right continuous, and hence the integration with respect to $d\mu_u(\lambda)$ is well-defined as a Stieltjes integral. Moreover, polarization gives $\mu_{\varphi,\psi}(\lambda)$ and we can define a projection-valued measure P_A by

$$\langle \varphi, P_A(\Omega)\psi\rangle = \int_{\mathbb{R}} \chi_\Omega(\lambda) d\mu_{\varphi,\psi}(\lambda),$$

which will regenerate A by the uniqueness.

In fact, one can prove an operator-valued version of the Stieltjes inversion formula. We refer to [13, Chapter 4.1] for details of proof. Let $\mu_u(\Omega) = \langle u, P_A(\Omega)u\rangle$, $F_u = \langle u, R_A(z)u\rangle$.

Theorem 2.23 (Stone's formula)**.** *Suppose $z = x + iy$, then*

$$\lim_{y \to 0_+} \frac{1}{\pi} \int_a^b \frac{1}{2i}(R_A(z) - R_A(\bar{z}))\, dx = \frac{1}{2}\left(P_A([a,b]) + P_A((a,b)) \right),$$

where the limit exists in the strong topology.

2.3. **Singular Sturm-Liouville problems.** For $I = (a, b)$, $-\infty \le a < b \le \infty$, we have

$$\mathcal{L} = \frac{1}{r}\left(-\frac{d}{dx}p\frac{d}{dx} + q\right)$$

with $\frac{1}{p}, q, r \in L^1_{loc}(I)$ and $p, r > 0$ almost everywhere on I. Moreover, if a is finite and $\frac{1}{p}, q, r \in L^1((a, a+\delta))$ for some $\delta > 0$, then $\mathcal{L}$ is said to be regular at a. Similarly for b.

Theorem 2.24. *Consider*

$$\begin{cases} \mathcal{L}u = f, \\ u(t_0) = \eta_0, \quad p(t_0)u'(t_0) = \eta_1 \end{cases} \tag{2.6}$$

with $rf \in L^1_{loc}(I)$ for some $t_0 \in (a, b)$, then there exists a unique solution u to (2.6) with regularity $u \in AC_{loc}(I)$ and $pu' \in AC_{loc}(I)$, that is, $u', (pu')' \in L^1_{loc}(I)$. If in addition, $\mathcal{L}$ is regular at a, then $u \in AC((a, a+\delta))$ and $pu' \in AC((a, a+\delta))$, that is, $u', (pu')' \in L^1((a, a+\delta))$. Moreover, t_0 may be a in this case.

Proof. Integrating

$$\frac{d}{dx}\begin{pmatrix} u \\ pu' \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{p} \\ q & 0 \end{pmatrix}\begin{pmatrix} u \\ pu' \end{pmatrix} - \begin{pmatrix} 0 \\ rf \end{pmatrix},$$

we get a Volterra integral equation. Then for any $x \in I$, there exists $x \in (c, d)$ such that we can apply the contraction mapping theorem on the Banach space $C([c, d])$ and hence we get a unique solution on $C([c, d])$. By using the integral equation again, we get $u, pu' \in AC([c, d])$. Finally, by uniqueness, $u, pu' \in AC_{loc}(I)$. □

In view of the following lemma, we take $H = L^2(I, r dx)$.

Lemma 2.25 (Lagrange's formula)**.** *Suppose $[c, d] \subset I$, we have*

$$\int_c^d \bar{u}\mathcal{L}v\, r dx - \int_c^d \overline{\mathcal{L}u} v\, r dx = -W_d[\bar{u}, v] + W_c[\bar{u}, v]$$

provided $u, v, pu', pv' \in AC_{loc}(I)$, where $W_x[u, v] = puv'(x) - pu'v(x)$.

Proof. The proof follows directly from integration by parts, which is in the same spirit as (1.3). □

But $\mathcal{L}$ is not bounded on H, therefore we need to carefully consider the domain of definition of $\mathcal{L}$. In view of Theorem 2.24, we set

$$\mathcal{D}(\mathcal{L}) = \left\{u \in L^2(I; rdx) : \begin{pmatrix} u \\ pu' \end{pmatrix} \in AC_{loc}(I), \mathcal{L}u \in L^2(I; rdx)\right\}.$$

The reason why we ask for the AC_{loc} property is that any solution to (2.6) satisfies this thanks to Theorem 2.24.

Corollary 2.26. *If $u, v \in \mathcal{D}(\mathcal{L})$, then*

$$\langle u, \mathcal{L}v\rangle - \langle \mathcal{L}u, v\rangle = -W_b[\bar{u}, v] + W_a[\bar{u}, v].$$

In particular, $\lim_{x\to a_+} W_x[\bar{u}, v]$ *and* $\lim_{x\to b_-} W_x[\bar{u}, v]$ *exist and we denote these by* $W_a[\bar{u}, v], W_b[\bar{u}, v]$ *respectively for simplicity.*

But $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is not even symmetric since the boundary terms would be nonzero in general. So define

$$\mathcal{D}(\mathcal{L}_c) := \mathcal{D}(\mathcal{L}) \cap L^2_c(I; r dx),$$

where the subscript c means compact support. Then by Lagrange's formula, $(\mathcal{L}_c, \mathcal{D}(\mathcal{L}_c))$ is symmetric.

Now we study the relation between the operators defined just now and show that the domains we gave before make them densely defined on H.

Theorem 2.27. *Let* $\mathcal{L}_0 = \overline{\mathcal{L}_c}$*, then we have*

$$\mathcal{D}(\mathcal{L}_0) := \{u \in \mathcal{D}(\mathcal{L}) : \forall v \in \mathcal{D}(\mathcal{L}), W_b[\bar{u}, v] = W_a[\bar{u}, v] = 0\}, \tag{2.7}$$

and $\mathcal{L} = \mathcal{L}_c^* = \mathcal{L}_0^*$*. In particular,* $\mathcal{D}(\mathcal{L}_c)$ *and* $\mathcal{D}(\mathcal{L})$ *are dense in* H*.*

Here, $\mathcal{L}_0$ is called the minimal operator associated with the differential operator and $\mathcal{L}$ is called the maximal operator associated with the differential operator. We abuse notation for the differential operator and the maximal operator.

Remark 2.28. Any self-adjoint operator extension L of $\mathcal{L}_c$, would satisfy $\mathcal{L}_0 \subset L \subset \mathcal{L}$ since $L = L^*$ implies L^* is closed. Then $\mathcal{L}_0 = \overline{\mathcal{L}_c} \subset L \subset \mathcal{L}_c^* = \mathcal{L}$. This explains why $\mathcal{L}_0$ is called the minimal operator and $\mathcal{L}$ is called the maximal operator.

Proof. **Step 1:**

- First, we assume $\mathcal{D}(\mathcal{L}_c)$ is dense and compute $\mathcal{D}(\mathcal{L}_c^*)$ by examining the definition for the adjoint.
- Recall that $v \in \mathcal{D}(\mathcal{L}_c^*)$ if and only if there exists $g \in H$ such that

$$\langle v, \mathcal{L}u\rangle = \langle g, u\rangle, \quad \forall u \in \mathcal{D}(\mathcal{L}_c). \tag{2.8}$$

  If $v \in \mathcal{D}(\mathcal{L})$, then for all $u \in \mathcal{D}(\mathcal{L}_c)$, one can use Corollary 2.26 to get $\langle v, \mathcal{L}u\rangle = \langle \mathcal{L}v, u\rangle$. Moreover, since u is compactly supported, we can take $g = \chi\mathcal{L}v$ such that $\chi \equiv 1$ on supp$u$ and $\chi \in C_c^\infty(I)$, then $\mathcal{L}v \in L^1_{loc}(I)$ implies $g \in H$, which shows that $v \in \mathcal{D}(\mathcal{L}_c^*)$.
- Now suppose $v \in \mathcal{D}(\mathcal{L}_c^*)$, then there exists $g \in H$ such that (2.8) holds. Since

$$\left|\int_c^d rg\, dx\right| \le \left(\int_c^d r\, dx\right)^{\frac{1}{2}} \left(\int_c^d g^2 r\, dx\right)^{\frac{1}{2}}, \forall [c, d] \subset I,$$

  that is, $rg \in L^1_{loc}(I)$, and by Theorem 2.24, we know that there exists some $\tilde{v}$ satisfying all the regularity required in that theorem such that $\mathcal{L}\tilde{v} = g$. Let $v_n = v - \tilde{v}$. Now it suffices to prove v_n is a solution to $\mathcal{L} = 0$.

  Define a linear functional

$$l : L^2_c(I; r dx) \to \mathbb{C}, \quad f \mapsto \langle v - \tilde{v}, f\rangle$$

and introduce a fundamental system of solutions u_1, u_2 for $\mathcal{L}$ on I, that is, $\mathcal{L}u_j = 0$, $\begin{pmatrix} u_1(c) \\ pu_1'(c) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} u_2(c) \\ pu_2'(c) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for some $c \in I$. Moreover, we define

$$l_j : L^2_c(I; rdx) \to \mathbb{C}, \quad f \mapsto \langle u_j, f\rangle,$$

then we want to show $l = c_1 l_1 + c_2 l_2$, which is equivalent to $\ker l_1 \cap \ker l_2 \subset \ker l$.

- Now we show $\ker l_1 \cap \ker l_2 \subset \ker l$. Suppose $f \in \ker l_1 \cap \ker l_2 \subset L^2_c(I; rdx)$, that is, $\int_a^b \bar{u}_1 f\, rdx = \int_a^b \bar{u}_2 f\, rdx = 0$. In fact, we do not need bars in the formula since u_1, u_2 are real solutions. Thanks to Duhamel's principle, We take a solution u to $\mathcal{L}u = f$ given by

$$\begin{aligned} u(x) &= \pm \int_a^x (u_1(x)u_2(y) - u_2(x)u_1(y)) f(y)r(y)\, dy \\ &= \pm\, u_1(x) \int_a^x u_2(y)f(y)r(y)\, dy \mp u_2(x) \int_a^x u_1(y)f(y)r(y)\, dy \\ &= \mp\, u_1(x) \int_x^b u_2(y)f(y)r(y)\, dy \pm u_2(x) \int_x^b u_1(y)f(y)r(y)\, dy. \end{aligned}$$

  In view of the last two equivalent formulas respectively at a and b, we know $u \in L^2_c(I; rdx) \cap \mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{L}_c)$. Hence, $l(f) = \langle v - \tilde{v}, \mathcal{L}u\rangle = 0$ by (2.8) and the construction $\mathcal{L}\tilde{v} = g$.
- Since $l = c_1 l_1 + c_2 l_2$ on $L^2_c(I; rdx)$, we know $v - \tilde{v} = c_1 u_1 + c_2 u_2$, which implies that $v_n = v - \tilde{v}$ is a solution to $\mathcal{L} = 0$.
- Now it suffices to show $\ker l_1 \cap \ker l_2 \subset \ker l$ implies $l = c_1 l_1 + c_2 l_2$. Assume $l_1, \dots, l_n$ are linear functionals $V \to \mathbb{C}$ such that $\cap \ker l_j \subset \ker l$, then we show that $l \in span\{l_j\}$. Without loss of generality, we can assume $l_1, \dots, l_n$ are linearly independent since we can just discard some l_j's such that those left are linearly independent. The map $L : V \to \mathbb{C}^n$ given by $f \mapsto (l_1(f), \dots, l_n(f))$ is surjective since $x \in Ran(L)^\perp$ implies $\sum_{j=1}^n x_j l_j(f) = 0$ for all f.

  Hence, there are vectors $f_k \in V$ such that $l_j(f_k) = 0$ for $j \neq k$ and $l_j(f_j) = 1$. Then $f - \sum_{j=1}^n l_j(f) f_j \in \cap_{j=1}^n \ker l_j$ and hence $l(f) = l(f_1)l_1(f) + l(f_2)l_2(f) + \dots + l(f_n)l_n(f)$, which implies $l = l(f_1)l_1 + \dots + l(f_n)l_n$.

Step 2: So far, under the assumption that $\mathcal{D}(\mathcal{L}_c)$ is dense, we showed $\mathcal{D}(\mathcal{L}_c^*) = \mathcal{D}(\mathcal{L})$. Now suppose $\mathcal{D}(\mathcal{L}_c)$ is not dense, then $\mathcal{D}(\mathcal{L}_c)^\perp$ is nonempty. Hence, there exists $g \neq g_* \in H$ such that

$$\langle v, \mathcal{L}u\rangle = \langle g, u\rangle = \langle g_*, u\rangle$$

for all $u \in \overline{\mathcal{D}(\mathcal{L}_c)}$. Likewise, we can obtain $\tilde{v}, \tilde{v}_*$ from Theorem 2.24 as in step 1 and using exactly the same argument implies

$$v = \tilde{v} + c_1 u_1 + c_2 u_2 = \tilde{v}_* + c_{*1} u_1 + c_{*2} u_2,$$

for some constant c_1, c_2, c_{*1}, c_{*2}. Then $\mathcal{L}(\tilde{v} - \tilde{v}_*) = 0$ and hence $g = \mathcal{L}\tilde{v} = \mathcal{L}\tilde{v}_* = \tilde{g}$, which contradicts the assumption.

Step 3: Denote the right hand side of (2.7) by $\mathcal{D}$. Finally, we will show $\mathcal{D}(\overline{\mathcal{L}_c}) = \mathcal{D}$.

- Thanks to the Lagrange formula in Corollary 2.26, it is easy to see $\mathcal{D} \subset \mathcal{D}(\mathcal{L}_c^{**}) = \mathcal{D}(\overline{\mathcal{L}_c})$.

- For every $v \in \mathcal{D}(\overline{\mathcal{L}_c}) \subset \mathcal{D}(\mathcal{L}^*)$, $\langle v, \mathcal{L}u\rangle = \langle \mathcal{L}v, u\rangle$ for all $u \in \mathcal{D}(\mathcal{L})$, that is, $W_b[\bar{v}, u] = W_a[\bar{v}, u]$ for all $v \in \mathcal{D}(\overline{\mathcal{L}_c}), u \in \mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{L}_c^*)$. Note that for all $u \in \mathcal{D}(\mathcal{L})$, there exists $\tilde{u} \in \mathcal{D}(\mathcal{L})$ such that $u = \tilde{u}$ near a and $\tilde{u} = 0$ near b by introducing a cut-off function, and hence $W_a[\bar{v}, u] = W_a[\bar{v}, \tilde{u}] = W_b[\bar{v}, \tilde{u}] = 0$. Similarly for b. Hence, $v \in \mathcal{D}$.

□

2.4. **Limit circles and limit points.** Self-adjointness has something to do with making the boundary terms vanish, which is related to terms W_a and W_b. Note that $u(a)$ and $pu'(a)$ are not well-defined for $u \in \mathcal{D}(L)$, therefore, we need to introduce something else to describe the boundary behavior.

Given any $\vec{u}(x) = \begin{pmatrix} u(x) \\ pu'(x) \end{pmatrix}$, $\vec{v}$, $\vec{w}$ and $\vec{\sigma}$. What we expect is

$$\vec{u}(x) = \frac{W_x[u, w]}{W_x[v, w]}\vec{v}(x) - \frac{W_x[u, v]}{W_x[v, w]}\vec{w}(x), \tag{2.9}$$

like what we did as in (1.13) for the regular Sturm-Liouville problem. However, for $u, v \in \mathcal{D}(\mathcal{L})$, $\lim_{x\to a_+} u(x)$ and $\lim_{x\to b_-} u(x)$ are not well-defined, but the Wronskian can be made sense of at the boundary thanks to Corollary 2.26. What we have is the relation between Wronskians

$$W_x[u, \sigma]W_x[v, w] = W_x[u, w]W_x[v, \sigma] - W_x[u, v]W_x[w, \sigma], \quad \forall x \in \bar{I}, \tag{2.10}$$

which is called the Plücker's formula and this also holds for boundary points. The proof is quite simple. Since (2.9) holds for interior points, we obtain (2.10) for interior points, and it is still valid after we take the limit.

Definition 2.29 (Limit circle, limit point)**.** *We define the limit circle and limit point of $\mathcal{L}$ respectively as follows.*

- *We say $\mathcal{L}$ is a limit circle (LC) at a (resp. b) if there exists a real-valued $v \in \mathcal{D}(\mathcal{L})$ such that there exists at least one $w \in \mathcal{D}(\mathcal{L})$ such that $W_a[v, w] \neq 0$ (resp. $W_b[v, w] \neq 0$).*
- *$\mathcal{L}$ is a limit point (LP) at a if it is not a limit circle at a (resp. b).*

Example 2.30. *If $\mathcal{L}$ is regular at a, then $\mathcal{L}$ is a limit circle of a since we can talk about the solution at a thanks to Theorem 2.24. More specifically, we take v to be a solution to $\mathcal{L}v = 0$ with $v(a) = 1, pv'(a) = 0$ and w to be a solution to $\mathcal{L}w = 0$ with $w(a) = 0, pw'(a) = 1$, then $W_a[v, w] \neq 0$.*

Example 2.31. *Suppose $\mathcal{L} = -\frac{d^2}{dx^2}$, $I = (-\infty, \infty)$. In view of the Fourier side, we know $\mathcal{D}(\mathcal{L}) = H^2(-\infty, \infty)$. One can easily check $\mathcal{L}$ is a limit point at $+\infty$ by the regularity given by the domain $\mathcal{D}(\mathcal{L})$. In this case, $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is already self-adjoint and we do not need to prescribe any boundary conditions. Heuristically speaking, this phenomenon just follows from the regularity given by the domain.*

In fact, the self-adjointness result of $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is a direct conclusion of Theorem 2.32. And the phenomenon that we need to have some prescribed boundary condition if $\mathcal{L}$ is LC at a gives an intuitive explanation for Weyl alternative, Theorem 2.35.

Theorem 2.32. *If $\mathcal{L}$ is LC at a, then set v_a to be the v in the definition. If $\mathcal{L}$ is LC at b, then set v_b to be the v in the definition.*

Let $Lu = \mathcal{L}u$ with

$$\mathcal{D}(L) = \{u \in \mathcal{D}(\mathcal{L}) : \text{ if } \mathcal{L} \text{ is LC at } a, W_a[v_a, u] = 0; \text{ if } \mathcal{L} \text{ is LC at } b, W_b[v_b, u] = 0\}$$

is self-adjoint.

Proof. We proceed in several steps.

- First, we show $(L, \mathcal{D}(L))$ is symmetric. For any $u_1, u_2 \in \mathcal{D}(L)$, if $\mathcal{L}$ is LP at a, then $W_a[u_1, u_2] = 0$. On the other hand, if $\mathcal{L}$ is LC at a, then for the real-valued $v_a \in \mathcal{D}(\mathcal{L})$ there exists a function $w \in \mathcal{D}(\mathcal{L})$ $W_a[v_a, w] \neq 0$ and by the definition of $\mathcal{D}(L)$, $W_a[v_a, u_1] = W_a[v_a, u_2] = 0$. Since v_a is real-valued, we have $W_a[v_a, \bar{u}_1] = W_a[v_a, \bar{u}_2] = 0$. Then thanks to (2.10), we have
$$W_a[\bar{u}_1, u_2]W_a[v_a, w] = W_a[\bar{u}_1, w]W_a[v_a, u_2] - W_a[\bar{u}_1, v_a]W_a[w, u_2] = 0,$$
which implies $W_a[\bar{u}_1, u_2] = 0$. Similarly, we get $W_b[\bar{u}_1, u_2] = 0$. Therefore, $(L, \mathcal{D}(L))$ is symmetric thanks to Corollary 2.26.
- Hence, $L \subset L^* \subset \mathcal{L}_c^*$ Now it suffices to show $\mathcal{D}(L^*) \subset \mathcal{D}(L)$.
  For any $g \in \mathcal{D}(L^*)$, there exists $v \in H$ such that
$$\langle g, Lf \rangle = \langle v, f \rangle, \quad \forall f \in \mathcal{D}(L).$$
On the other hand, for $g \in \mathcal{D}(L^*) \subset \mathcal{D}(\mathcal{L}_c^*) = \mathcal{D}(\mathcal{L})$, we have
$$\langle g, \mathcal{L}f \rangle = \langle \mathcal{L}g, f \rangle, \quad \forall f \in \mathcal{D}(\mathcal{L}_c) \subset \mathcal{D}(L),$$
we know $v = \mathcal{L}g$. Hence, for any $u \in \mathcal{D}(L^*)$, $\langle u, Lv \rangle = \langle \mathcal{L}u, v \rangle$ for all $v \in \mathcal{D}(L)$.
- And it's equivalent to $W_a[\bar{u}, v] = W_b[\bar{u}, v]$ for all $v \in \mathcal{D}(L)$. Take any $u \in \mathcal{D}(L^*)$, if $\mathcal{L}$ is LP at a, then there is nothing to prove. If $\mathcal{L}$ is LC at a, then there exists $w \in \mathcal{D}(\mathcal{L})$ such that $W_a[v_a, w] \neq 0$. And then we just truncate w to produce $\tilde{w} \in \mathcal{D}(L)$ such that $w = \tilde{w}$ near a and $\tilde{w} = 0$ near b. And we still use w to denote the function $\tilde{w} \in \mathcal{D}(L)$, we know $W_a[\bar{u}, w] = 0$.
- Then the basic idea is: $W_a[\bar{u}, w] = 0$ and $W_a[v_a, w] = 0$ will imply $v_a \propto w \propto \bar{u}$ and hence $W_a[\bar{u}, v_a] = 0$. However, this argument does not work for singular Sturm-Liouville problem directly. Fortunately, we still can use the Plücker's formula (2.10) to derive $W_a[\bar{u}, v_a] = 0$, which follows directly from
$$W_x[\bar{u}, v_a]W_x[v_a, w] = W_x[\bar{u}, w]W_x[v_a, v_a] - W_x[\bar{u}, v_a]W_x[w, v_a] = 0.$$
- And repeat the process in the last two bullets for b, we get $W_b[\bar{u}, v_b] = 0$, then $u \in \mathcal{D}(L)$, which completes the proof.

□

In the following discussion, we show a connection between LC, LP and square integrability of solutions called the Weyl alternative theorem, stated as in Theorem 2.35.

The proof is based on the solvability of $\mathcal{L}u = z_0 u$ for $z_0 \in \rho(L)$ in $\mathcal{D}(L)$.

Proposition 2.33. *Suppose $z \in \rho(L)$, then there exists a nontrivial $u_a = u_a(x; z)$ such that u_a is in $L^2((a, a+\delta), rdx)$ for some $\delta > 0$, $u_a \in \mathcal{D}(L)$ and $Lu_a = zu_a$. Moreover, if $\mathcal{L}$ is LC at a, then u_a satisfies the boundary condition in the sense that $W_a[u_a, v_a] = 0$ where v_a is the v in Definition 2.29.*

Similar result is true for b, there exists a nontrivial $u_b = u_b(x; z)$ such that u_b is in $L^2((b-\delta, b), rdx)$ for some $\delta > 0$, $u_b \in \mathcal{D}(L)$ and $Lu_b = zu_b$. Moreover, if $\mathcal{L}$ is LC at b, then u_b satisfies the boundary condition.

Proof. We only prove for the existence of such u_a. Similar argument works for u_b.

- For $g \in C_c^\infty(I)$, set $\tilde{u} = R_L(z)g$. Then by the definition of resolvent set $\rho(L)$ as in Definition 2.7, $\mathcal{L}\tilde{u} = z\tilde{u} + g$ and $\tilde{u} \in \mathcal{D}(L) \subset H$. Hence, $W_a[\bar{\tilde{u}}, v_a] = 0$, where v_a is the v in the definition of LC, provided $\mathcal{L}$ is LC. Then if one set
$$u_a(x) = \begin{cases} \tilde{u}(x), & x \text{ near } a \text{ such that } g = 0 \text{ on this region,} \\ \text{extended by using the differential equation,} & \end{cases}$$
we know $W_a[\bar{\tilde{u}}, v_a] = 0$ still holds.
- Now it suffices to check that $\tilde{u} \neq 0$ near a for a suitable choice of $g \in C_c^\infty(I)$ in order to make u_a not identically zero on I. Moreover, we need to show the local $L^2(rdx)$ property near a for u_a.
- To do this, we use Duhamel's principle. Fixed any $c \in (a, b)$, $(\mathcal{L} - z)u_j = 0$ with $\vec{u_1}(c) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{u_2}(c) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thanks to Corollary 2.26, $W_x[u_1, u_2] = W_c[u_1, u_2] = 1$ for all $x \in I$. Set
$$\begin{aligned} \tilde{u} =& \alpha u_1 + \beta u_2 + \int_c^x (u_1(x)u_2(y) - u_1(y)u_2(x))\, g(y)\, rdy \\ =& u_1(x)\left(\alpha + \int_c^x u_2(y)g(y)\, rdy\right) + u_2(x)\left(\beta - \int_c^x u_1(y)g(y)\, rdy\right), \end{aligned}$$
which is a solution to $(\mathcal{L} - z)\tilde{u} = g$. Without loss of generality, we assume supp$g \subset (c, d) \subsetneq I$, then
$$\begin{aligned} \tilde{u} =& u_1(x)\left(\alpha + \int_a^x u_2(y)g(y)\, rdy\right) + u_2(x)\left(\beta - \int_a^x u_1(y)g(y)\, rdy\right) \\ =& u_1(x)\left(\tilde{\alpha} + \int_a^x u_2(y)g(y)\, rdy\right) + u_2(x)\left(\tilde{\beta} + \int_x^b u_1(y)g(y)\, rdy\right). \end{aligned} \tag{2.11}$$
- Put $\alpha' = \tilde{\alpha} + \int_a^b u_2(y)g(y)\, rdy$ and $\beta' = \tilde{\beta} + \int_a^b u_1(y)g(y)\, rdy$. If $\tilde{u}(x) \equiv 0$ near a (for $a < x < c$), then $\tilde{\alpha}u_1(x) + \beta' u_2(x) = 0$ for all $a < x < c$. Then $0 = W_x[u_1, \tilde{\alpha}u_1 + \beta' u_2] = \beta W_x[u_1, u_2] = \beta'$ and hence $\tilde{\alpha} = 0$.

  On the other hand, using the same argument, if $\tilde{u} \equiv 0$ near b (for $d < x < b$), then $\tilde{\beta} = 0$ and $\alpha' = 0$.
- If $\tilde{u} \equiv 0$ near a and b, then $\tilde{\alpha} = \tilde{\beta} = 0$, $\int_a^b u_1 g\, rdy = \int_a^b u_2 g\, rdy = 0$. However, this can be easily avoided by choosing g appropriately. For such a g, $\tilde{u} \neq 0$ near a or $\tilde{u} \neq 0$ near

b. Without loss of generality, we assume $\tilde{u} \neq 0$ near a, that is, we choose g appropriately such that $\beta' \neq 0$.
- Now we define u_a using the argument in the first bullet, which completes the proof for the existence of u_a.
- Similar argument in the last step by choosing a different g will let us find u_b as desired.

□

Now we can write down the Green's function for $z \in \rho(L)$.

Corollary 2.34. *For $z \in \rho(L)$, the Green's function defined as*

$$G(x, y; z) = -\frac{1}{W[u_a, u_b]} \begin{cases} u_a(x)u_b(y), & x < y, \\ u_a(y)u_b(x), & y < x, \end{cases}$$

satisfies $(L - z)^{-1}g(x) = \int_a^b G(x, y; z)g(y)r(y)\, dy$.

Proof. We can prove this corollary following the similar manner as in the proof of the preceding theorem, see [13, Lemma 9.7].

A more direct way is to use distributional theory to check by a direct computation. We omit the details here. □

Now we can prove the Weyl's alternative.

Theorem 2.35 (Weyl alternative). *$\mathcal{L}$ is LC at a (resp. b) if and only if there exists $z_0 \in \mathbb{C}$ such that all solutions u to $\mathcal{L}u = z_0 u$ are in $L^2((a, a + \delta); rdx)$ (resp. $L^2((b - \delta, b); rdx)$). (Here, "any solution" can be any solution to the differential equation and do not require this solution to be in $\mathcal{D}(\mathcal{L})$.)*

Proof.
- The "if" part is quite easy. We take two arbitrary solutions v, w to $\mathcal{L}u = z_0 u$ such that $W[v, w] \neq 0$ and in particular, $W_a[v, w] \neq 0$. One should note that we have implicitly used our assumption that $v, w \in L^2((a, a + \delta); rdx)$ since this assumption, combined with Theorem 2.24, allows us to make a cut-off near a for v, w such that they are in $\mathcal{D}(\mathcal{L})$ and hence W_x is independent of x near a thanks to Corollary 2.26. Such v, w exist since we can take arbitrary linearly independent solutions to ensure $W[v, w]$ does not vanish.

  Then at least one of $W_a[\operatorname{Re} v, w] \neq 0$, $W_a[\operatorname{Im} v, w] \neq 0$ holds, which implies $\mathcal{L}$ is LC at a.
- Without loss of generality, assume $\mathcal{L}$ is regular at b, otherwise, we can choose some $c \in (a, b)$ such that $\mathcal{L}$ is regular at c and replace b by c. In particular, $\mathcal{L}$ is LC at b by Example 2.30. Since $\mathcal{L}$ is LC at a, there exist at least two different real-valued $v_a, \tilde{v}_a \in \mathcal{D}(\mathcal{L})$ such that $W_a[v_a, \tilde{v}_a] \neq 0$. Consider the extension of L_c to two self-adjoint operators L and $\tilde{L}$ by using v_a and $\tilde{v}_a$ by Theorem 2.32 (using the same w_a for definition) respectively. We assume $W_a[v_a, \tilde{v}_a] \neq 0$, then $\mathcal{D}(L)$ and $\mathcal{D}(\tilde{L})$ are not the same thanks to the Plucker's formula(or see [13, Lemma 9.5]).

  By Proposition 2.33, we construct $u_a(; z)$ and $\tilde{u}_a(; z)$ from L and $\tilde{L}$, respectively. Since $W_a(v_a, u_a) = 0, W_a(\tilde{v}_a, \tilde{u}_a) = 0$, we know $W_a(u_a, \tilde{u}_a) \neq 0$. Otherwise, $\mathcal{D}(L) = \mathcal{D}(\tilde{L})$

thanks to the Plucker's formula(or see [13, Lemma 9.5]). That is to say, these two solutions are linearly independent. Since any other solution can be written as a linear combination of those two near a, every solution is square integrable near a.

□

Example 2.36. *As an application of the Weyl alternative, we consider $\mathcal{L} = -\frac{d^2}{dx^2}$ with $I = (0, \infty)$. Then we know all solutions are of the form $u = c_1 + c_2 x$, which implies $\mathcal{L}$ is LC of a and $\mathcal{L}$ is LP of b.*

2.5. **Spectral Transformation.** In this subsection, we want to introduce a fundamental tool called the Spectral transformation for investigating the spectra of Sturm-Liouville operators and, at the same time, give some nice illustrations of the spectral theorem.

Example 2.37. *Suppose $\mathcal{L} = -\frac{d^2}{dx^2}$, $I = (-\infty, \infty)$, $\mathcal{D}(\mathcal{L}) = \{u \in L^2 : u, u' \in AC_{loc}, u'' \in L^2\} = H^2(-\infty, \infty)$ is self-adjoint. This is the case that $\mathcal{L}$ is LP at $\pm\infty$.*

By taking the unitary Fourier transform $\mathcal{F} : L^2 \to L^2((-\infty, \infty); \frac{1}{\sqrt{2\pi}}\, d\xi)$, we have $\mathcal{F} L \mathcal{F}^{-1} = M_{\xi^2}$.

On the other hand, the multiplication version of the spectral theorem, Theorem 2.12, states that a function λ and a measure μ such that $L^2(\mathbb{R}; d\mu)$ and $\mathcal{L}$ is conjugate to M_λ. What we expect is that we can transform the multiplication by M_{ξ^2} we get to the multiplication by M_λ. A naive try is to take $\lambda = \xi^2$.

Set $U : L^2((-\infty, \infty), dx) \to \oplus_{j=1}^{2} L^2((-\infty, \infty), d\mu_j)$ with

$$u \mapsto \begin{pmatrix} \int e^{-i\sqrt{\lambda}x} u(x)\, dx \\ \int e^{+i\sqrt{\lambda}x} u(x)\, dx. \end{pmatrix}$$

We choose a change of variable such that

$$d\mu_1 = d\mu_2 = \frac{1}{2\pi} \chi_{[0,\infty)}(\lambda)\, d\sqrt{\lambda} = \frac{1}{4\pi\sqrt{\lambda}} \chi_{[0,\infty)}(\lambda)\, d\lambda.$$

Our goal is to generalize such a U. What we care are the spectral basis $\{e^{-i\sqrt{\lambda}x}, e^{+i\sqrt{\lambda}x}\}$ and the spectral measure $d\mu_1, d\mu_2$. Observe that $e^{\pm\sqrt{\lambda}x}$ solves the eigenvalue equation with eigenvalue λ, it will turn out that this is something that can be generalized to find a general U.

2.5.1. *Spectral transformation.*

Proposition 2.38 (Spectral transformation)**.** *Let $\mathcal{L}$ be as above and L is a self-adjoint realization as discussed before. By the spectral theorem, there exists a unitary map*

$$U : L^2(I; r dx) \to \oplus_{j=1}^{k} L^2(\mathbb{R}, d\mu_j)$$

such that

$$ULU^{-1} = \begin{pmatrix} M_\lambda & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & M_\lambda \end{pmatrix}$$

Then

(1) *there exists* $\vec{u}(x,\lambda) = \begin{pmatrix} u_1(x,\lambda) \\ \vdots \\ u_k(x,\lambda) \end{pmatrix}$ *such that*

$$Uf(\lambda) = \int_I \vec{u}(x,\lambda) f(x)\, r dx.$$

such that for μ_j*-almost every* λ, $\mathcal{L}u_j(x,\lambda) = \lambda u_j(x,\lambda)$*. Moreover, the preceding integral is taken in the sense that*

$$\lim_{c\to a, d\to b} \int_c^d \vec{u}(x,\lambda) f(x)\, r dx$$

in $\oplus_{j=1}^k L^2(\mathbb{R}, d\mu_j)$*. (This is similar to the Fourier transform on* L^2 *can only be expressed by integral formula using such kind of limit.)*

If $\mathcal{L}$ *is LC at one of the endpoint, then* $u_j(x,\lambda)$ *would satisfy the boundary condition* $(W_a[v_a, u_j] = 0)$*.*

(2) U^{-1} *has the form*

$$U^{-1}F(x) = \sum_{j=1}^k \int_{\mathbb{R}} \overline{u_j(x,\lambda)} F_j(\lambda)\, d\mu_j(\lambda)$$

with $\vec{F}(\lambda) = (F_1(\lambda), \ldots, F_k(\lambda))^T$ *and the integral is taken in the sense of*

$$\int_{\mathbb{R}} d\mu_j(\lambda) = \lim_{c\to -\infty, d\to +\infty} \int_c^d d\mu(\lambda)$$

in $L^2(I; r dx)$*.*

(3) *Assume furthermore that* μ_j*'s are ordered, that is,* $\mu_j \ll \mu_{j-1}$ *for all* $j > 1$*. Then for* μ_l*-almost every* λ, $\{u_j(x,\lambda)\}_{j=1}^l$ *are linearly independent.*

Corollary 2.39. *Suppose* μ_j*'s are ordered, then* k *is at most* 2*. Moreover, if* $\mathcal{L}$ *is LC at one endpoint, then* $k = 1$*.*

Proof. By Proposition 2.38 (3), $k \le 2$ since there are at most two linearly independent solutions to $(\mathcal{L} - \lambda)u = 0$.

Moreover, there is only one linearly independent solution if there is a prescribed boundary condition. □

Corollary 2.40. *Given a self-adjoint* L*, there exists* U *such that* μ_j*'s are ordered.*

Proof. Recall that U is constructed using a spectral basis $\{u_j\}$ constructed by Theorem 2.20. with $u_j \perp H_{u_{j'}}$ for $j' < j$, and $H = \oplus_{j=1}^\infty H_{u_j}$.

We say v is a maximal spectral vector for H if $\mu_u \ll \mu_v$ for all $u \in H$. It is easy to check that there always exists a maximal spectral vector for a self-adjoint operator L by setting $v = \sum_j \varepsilon_j u_j$ with $\varepsilon_j \neq 0$ such that $\sum_j |\varepsilon_j|^2 = 1$. (See [13, Lemma 3.15].)

Using this observation, we can find a spectral basis $\{\tilde{u}_j\}$ such that $\tilde{u}_k$ is maximal spectral vector for L restricted to $(\oplus_{j=1}^{k-1} H_{\tilde{u}_j})^\perp$ during the Gram-Schmidt process as in Theorem 2.20. Then μ_j's are ordered by definition, which completes the proof. □

Proof of Proposition 2.38. We claim that for any compact interval $[c,d] \subset I$, there exist measurable $u_j(x,\lambda)$'s on $I \times \mathbb{R}$ such that $U(\chi_{[c,d]} f) = \int_c^d \vec{u}(x,\lambda) f(x)\, r dx$ Moreover, if $\mathcal{L}$ is LC at a, then we can take c to be a. Assuming this claim, the rest of the proposition is proved as follows.

- Since U is chosen from the spectral theorem, $ULf = \lambda U f$ for all $f \in \mathcal{D}(\mathcal{L}_c)$. Thanks to the claim above, this is equivalent to
$$\int \vec{u}(x,\lambda) Lf(x)\, r dx = \int \lambda \vec{u}(x,\lambda) f(x)\, r dx,$$
which implies
$$\int_c^d u_j(x,\lambda)(\mathcal{L}-\lambda) f(x) r(x)\, dx = 0, \quad \forall f \in \mathcal{D}((\mathcal{L}|_{[c,d]})_c), \text{ for all } \mu_j - a.e.\, \lambda,$$
then $u_j(x,\lambda) \in \mathcal{D}((\mathcal{L}|_{[c,d]})_c^*)$ and hence $\mathcal{L}u_j|_{[c,d]} = (\mathcal{L}|_{[c,d]})_c^* u_j|_{[c,d]} = \lambda u_j$. In particular, u_j is a solution to $\mathcal{L} - \lambda = 0$.
  If $\mathcal{L}$ is LC at a, then c can be taken to be a and f can be taken to be all functions in $\mathcal{D}(\mathcal{L})$ which satisfy boundary condition at a and vanish near b, then u_j satisfies BC near a.
- Part (2) is quite easy, which follows readily from $U^{-1} = U^*$ by writing out
$$\langle F, Ug\rangle = \langle U^{-1}F, g\rangle$$
explicitly as
$$\sum_j \int_{\mathbb{R}} \overline{F_j(\lambda)} \int_a^b u_j(x,\lambda) g(x)\, r dx\, d\mu(\lambda) = \int_a^b \overline{(U^{-1}F)(x)} g(x)\, r dx$$
for F, g with compact support. Then part (2) is true since compactly supported functions are dense in L^2.
- Suppose $\{\mu_j\}$ are ordered and fix some $l \le k$. Suppose $\lambda \in \operatorname{supp}\mu_l$ and $\sum_{j=1}^l c_j(\lambda) u_j(x,\lambda) = 0$, which implies $\sum_{j=1}^l c_j(\lambda) F_j(\lambda) = 0$ since $F_j(\lambda) = \int_I u_j(x,\lambda) f(x) r(x)\, dx$.
  For any fixed j_0, since $\lambda \in \operatorname{supp}\mu_l$, $\mu_l \ll \mu_{l-1} \ldots \ll \mu_1$ and U is surjective, we can arrange $F_j(\lambda) \equiv \delta_{j,j_0}$ on $\operatorname{supp}\mu_l$. (Recall that $\mu_j(\mathbb{R}) = \|u_j\| = 1$ is finite.) Then $c_{j_0} = 0$. Thus $\{u_j\}$ is linearly independent.

Now we prove the claim. We want to prove that for all $[c,d] \subset I$, there exists $\vec{u}$ such that
$$U(1_{[c,d]} f) = \int_c^d \vec{u}(x,\lambda) f(x)\, r dx.$$
Since $UR_L(z)U^{-1} = \dfrac{1}{\lambda - z}$, we have $U = (\lambda - z) U R_L(z)$. Then
$$U1_{[c,d]} = (\lambda - z) U R_L(z) 1_{[c,d]},$$
where $R_L(z)1_{[c,d]}$ can be expressed by the Green's function
$$R_L(z) 1_{[c,d]} f = \int G(x,y,z) 1_{[c,d]}(y) f(y)\, r dy.$$

Then one can easily check $R_L(z)1_{[c,d]}$ is a Hilbert-Schmidt operator since

$$\int_c^d \int_a^b G^2(x,y,z) r(x)\, dx r(y)\, dy < +\infty,$$

which follows from a direct computation using the expression for $G(x,y,z)$ in Corollary 2.34 and the properties of u_a, u_b derived from Proposition 2.33.

Moreover, since U is unitary, $(UR_L(z)1_{[c,d]})^* UR_L(z)1_{[c,d]} = (R_L(z)1_{[c,d]})^* R_L(z)1_{[c,d]}$, we know $U1_{[c,d]} = (\lambda - z)UR_L(z)1_{[c,d]}$ is a Hilbert-Schmidt operator. Thus, there exists a corresponding kernel $\vec{u}$ defined on $[c,d]$ such that

$$(U_j 1_{[c,d]} f)(\lambda) = \int_c^d u_j(x,\lambda) f(x)\, r dx.$$

Moreover, one can easily check that for $[c', d'] \supset [c,d]$, the new u_j's we construct using this procedure coincide with the previous ones on $[c,d]$. Therefore, we get a measurable $\vec{u}$ on I satisfying the property of the claim. □

We provide the definition of a Hilbert-Schmidt operator and some crucial properties here for reader's references.

Theorem 2.41. *Suppose $A : H \to H$ is a bounded linear operator on a Hilbert space H, then the following are equivalent:*

(1) $\{e_j\}$ is an orthonormal basis for H, $\sum_j \|Ae_j\|^2 < +\infty$;

(2) A is compact and $\{s_j(A)\}$ are eigenvalues of A^*A such that $\sum_j s_j(A)^2 < \infty$;

(3) if $H = L^2(X, d\mu(x))$, then

$$Af(x) = \int_X K(x,y) f(y)\, d\mu(y),$$

where $\iint |K(x,y)|^2\, d\mu(x)\, d\mu(y) < +\infty$.

Then we say such an A is a Hilbert-Schmidt operator, and the three quantities above are equal to each other and we denote it by $\|A\|_{HS}^2$.

Proof. The proof of the implication (2) ⇒ (3) would use the singular value decomposition. The other implications are easy to check. □

Henceforth we stick to the case μ_j's are ordered and hence $k \geq 2$.

2.5.2. *Computing spectral measures with $\mathcal{L}$ LC at one endpoint.* We will start with a simpler case. Suppose $\mathcal{L}$ is LC at a and $\mathcal{D}(L)$ is defined with a boundary condition at a, then we know $k = 1$ thanks to Corollary 2.39. By Proposition 2.33, u_a satisfying the boundary condition at a with $u_a \in L^2((a, a+\delta); r dx)$.

Using the same notation as before, v_a is the one in the definition of LC, then $W_a[u_a, v_a] = 0$ with $v_a \in \mathcal{D}(\mathcal{L})$. Moreover, take $\tilde{v}_a$ to be another solution such that $\{v_a, \tilde{v}_a\}$ is linearly independent and $W_a[u_a, \tilde{v}_a] = 1$. (Such $\tilde{v}_a$ exists since we can solve the differential equation at $c \in I$ such that $W_c[u_a, \tilde{v}_a] = 1$ and W_x is independent of x.)

Since $k = 1$, for any $u(x,\lambda)$ solving $\mathcal{L}u - \lambda u = 0$ with desired boundary condition, $u(x,\lambda) \propto u_a$, that is, $u(x,\lambda) = \gamma_a(\lambda) u_a(x,\lambda)$ for some $\gamma_a(\lambda)$. This implies that for $u = u_1$ in

Proposition 2.38, there is some measure $\tilde{\mu}$ such that $d\tilde{\mu}(\lambda) = |\gamma_a(\lambda)|^2 d\mu_1(\lambda)$. Furthermore, this corresponds to a unitary operator $\tilde{U} : L^2(I; rdx) \to L^2(\mathbb{R}, d\tilde{\mu})$,

$$\tilde{U}f = \int_a^b u_a(x, \lambda) f(x)\, rdx,$$

which is a spectral transformation. This rescaling gets rid of the unknown factor γ_a. From now on, we remove the tilde notation. One should keep in mind that μ here may not be a finite measure anymore.

In order to compute $d\mu$, we take $z \in \mathbb{C} \setminus \mathbb{R}$ such that there exists a $u_b(x, z) \in L^2((b - \delta, b); rdx)$ by Proposition 2.33 such that u_b satisfies the boundary condition at b if $\mathcal{L}$ is LC at b. Note that $\{u_a, u_b\}$ is linearly independent. Suppose not, then $u = u_b = cu_a$ will imply that u satisfies $\mathcal{L}u = zu$ and u satisfy the boundary condition at a and b if there is any. Hence, $u \in \mathcal{D}(L)$ and it is an eigenfunction, which contradicts $z \in \mathbb{C} \setminus \mathbb{R} \subset \rho(L)$.

Hence u_b is a linear combination of $\tilde{v}_a$ and u_a with nonzero coefficient corresponding to $\tilde{v}_a$, that is,

$$u_b(x, z) = \gamma_b(z) \left(\tilde{v}_a(x, z) + m_b(z) u_a(x, z)\right)$$

for some $\gamma_b(z)$.

Let us u_b be normalized such that $\gamma_b(z) \equiv 1$, that is,

$$u_b = \tilde{v}_a(x, z) + m_b(z) u_a(x, z).$$

This is called a shooting problem or a connection formula since it connects functions with good bahavior at a and functions with good bahavior at b. The function $m(z)$ is called the Weyl-Titchmarsh function.

Recall that

$$Uf = \int_a^b u_a(x, \lambda) f(x) r(x)\, dx, \quad \int_a^b \overline{Uf} Uf\, d\mu(\lambda) = \int_a^b \overline{f(x)} f(x) r(x)\, dx.$$

In order to compute $d\mu$, we need to select some f such that the expression for Uf is fairly simple. To do that, we first compute in a formal way to give some heuristic idea. Since $UL = M_\lambda U$,

$$Uh(L) = M_{h(x)} U,$$

and hence $Uh(L)f = h(\lambda) Uf$. Then formally, we write

$$U(\frac{\delta(x-y)}{r(x)}) = \int u_a(x, \lambda) \delta(x - y)\, dx = u_a(y, \lambda).$$

Let $h(L) = \frac{1}{L-z}$ for $z \in \rho(L)$ and $f_y(x) = \frac{\delta(x-y)}{r(x)}$, then

$$Uh(L)f_y = h(\lambda) Uf_y = \frac{1}{\lambda - z} u_a(y, \lambda).$$

Since $(L - z)G = \dfrac{\delta_0(x-y)}{r(x)}$, $h(L)f_y = G(\cdot, y, z)$ and hence

$$UG(\cdot, y, z) = \frac{1}{\lambda - z} u_a(y, \lambda).$$

Now it would be a nice formula since G is in L^2 though the computation above is just heuristics.

Note that we can normalized u_b such that $W[u_a, u_b] = 1$. Then the Green function of L can be expressed by

$$G(x,y,z) = \begin{cases} u_a(x,z)u_b(y,z), & y \geq x, \\ u_a(y,z)u_b(x,z), & y < x. \end{cases}$$

Lemma 2.42. *For $z \in \rho(L)$, the following identities hold:*

$$(UG(\cdot,y,z))(\lambda) = \frac{u_a(y,\lambda)}{\lambda - z}, \quad U(p(y)\partial_y G(\cdot,y,z))(\lambda) = \frac{p(y)\partial_y u_a(y,\lambda)}{\lambda - z}.$$

Proof. Since $U : L^2(I; rdx) \to L^2(\mathbb{R}, d\mu)$ is unitary and

$$R_L(z)f = U^{-1}\frac{1}{\lambda - z}Uf,$$

we have

$$\int_a^b G(x,y,z)f(x)r(x)\,dx = \int_a^b \frac{u_a(y,\lambda)}{\lambda - z}Uf(\lambda)\,d\mu(\lambda), \quad \forall f \in \mathcal{D}(\mathcal{L}_c)$$

in the sense of L^2. Note that $\mathcal{D}(\mathcal{L}_c)$ is dense in L^2 and we use the definition of adjoint, the fact $G(\cdot,y,z) \in L^2(I; rdx)$ for all $y \in I$ and the fact $U^{-1} = U^*$ to see

$$(UG(\cdot,y,z))(\lambda) = \frac{u_a(y,\lambda)}{\lambda - z}.$$

By differentiating this formula,

$$U(\partial_y G(\cdot,y,z))(\lambda) = \frac{\partial_y u_a(y,\lambda)}{\lambda - z}.$$

We multiply both sides by $p(y)$ just for our convenience. $\square$

By Plancherel's formula,

$$\int \frac{|u_a(y,\lambda)|^2}{|\lambda - z|^2}\,d\mu(\lambda) = \int |UG|^2\,d\mu = \int_a^b |G(x,y,z)|^2 r(x)\,dx. \tag{2.12}$$

On the other hand, we would expect the following result.

Lemma 2.43. *For $z \in \rho(L)$,*

$$(Im\,z)\int_a^b |G(x,y,z)|^2\,rdx = Im\,G(y,y,z).$$

Remark 2.44. Before we give a rigorous proof, we see why this is true in a heuristic way. Note that $G(\cdot,y,z)$ satisfies the boundary conditions at both a and b if there is any and $\mathcal{L}G(\cdot,y,z) = \dfrac{\delta_0(x-y)}{r(x)}$ at least formally, which is the unique one with such properties.

Moreover, $G(x,y,\bar{z}) = \overline{G(x,y,z)}$ and we denote $G = G(x,y,z)$, $\bar{G} = \overline{G(x,y,z)}$ for simplicity. We write

$$\int_a^b \overline{G(x,y,z)}G(x,y,z)\,rdx = \frac{1}{\bar{z}-z}\int_a^b \left(\bar{z}G(x,y,\bar{z})G(x,y,z) - zG(x,y,\bar{z})G(x,y,z)\right)\,rdx$$
$$=\frac{1}{\bar{z}-z}\int_a^b \left(\left(\mathcal{L}G(x,y,\bar{z}) - \frac{\delta_0(x-y)}{r(x)}\right)G(x,y,z) - G(x,y,\bar{z})\left(\mathcal{L}G(x,y,z) - \frac{\delta_0(x-y)}{r(x)}\right)\right)\,rdx$$
$$=\frac{1}{\bar{z}-z}\int_a^b \left(\mathcal{L}\bar{G}G - \bar{G}\mathcal{L}G - \frac{\delta_0(x-y)}{r(x)}G + \frac{\delta_0(x-y)}{r(x)}\bar{G}\right)\,rdx$$
$$=-\frac{1}{2i(\mathrm{Im}z)}(-2i\mathrm{Im}(G(y,y,z))) = \frac{1}{\mathrm{Im}z}\mathrm{Im}(G(y,y,z)),$$

where we use

$$\int \left((\mathcal{L}\bar{G})G - \bar{G}(\mathcal{L}G)\right)\,rdx = 0$$

in the third step, which follows from integration by parts and the fact that $G, \bar{G}$ both satisfy boundary conditions.

Note that $\dfrac{Imz}{|\lambda - z|^2}$ is the Poisson kernel, or more specifically, for $z = \lambda_0 + i\varepsilon$, $\dfrac{\varepsilon}{(\lambda-\lambda_0)^2+\varepsilon^2}$ is an approximation of $\delta_0(\lambda-\lambda_0)$. Hence, with the help of Lemma 2.43, we would expect to recover $d\mu$ from the equation (2.12).

Now we give a rigorous proof for Lemma 2.43.

Proof of Lemma 2.43. Recall that

$$G(x,y,z) = \begin{cases} u_a(x,z)u_b(y,z), & y \geq x, \\ u_a(y,z)u_b(x,z), & y < x, \end{cases}$$

where we normalized u_b such that $W[u_a, u_b] = 1$. We write

$$\int_a^b \overline{G(x,y,z)}G(x,y,z)\,rdx = \frac{1}{\bar{z}-z}\int_a^b \left(\bar{z}G(x,y,\bar{z})G(x,y,z) - zG(x,y,\bar{z})G(x,y,z)\right)\,rdx.$$

Then we know

$$\bar{z}G(x,y,\bar{z}) = \begin{cases} \mathcal{L}u_a(x,\bar{z})u_b(y,\bar{z}), & y > x, \\ u_a(y,\bar{z})\mathcal{L}u_b(x,\bar{z}), & y < x, \end{cases}$$

and similar equation holds for $zG(x,y,z)$. Hence,

$$\int_a^b |G(x,y,z)|^2\,rdx$$
$$=\frac{1}{\bar{z}-z}\int_a^y \left(\mathcal{L}u_a(x,\bar{z})u_b(y,\bar{z})u_a(x,z)u_b(y,z) - u_a(x,\bar{z})u_b(y,\bar{z})\mathcal{L}u_a(x,z)u_b(y,z)\right)\,rdx+$$
$$\frac{1}{\bar{z}-z}\int_y^b \left(u_a(y,\bar{z})\mathcal{L}u_b(x,\bar{z})u_a(y,z)u_b(x,z) - u_a(y,\bar{z})u_b(x,\bar{z})u_a(y,z)\mathcal{L}u_b(x,z)\right)\,rdx.$$

For the first integral, we compute by using Lagrange's identity,

$$\begin{aligned}
&u_b(y,\bar z)u_b(y,z)\int_a^y \left(\mathcal{L}u_a(x,\bar z)u_a(x,z) - u_a(x,\bar z)\mathcal{L}u_a(x,z)\right) r dx \\
=&u_b(y,\bar z)u_b(y,z)\left(-W_y\left[u_a(x,\bar z),u_a(x,z)\right]\right) \\
=&u_b(y,\bar z)u_b(y,z)\left(p(y)\partial_y u_a(y,\bar z)u_a(y,z) - u_a(y,\bar z)p(y)\partial_y u_a(y,z)\right) \\
=&u_a(y,z)u_b(y,z)\left(p(y)\partial_y u_a(y,\bar z)u_b(y,\bar z)\right) - u_a(y,\bar z)u_b(y,\bar z)\left(p(y)\partial_y u_a(y,z)u_b(y,z)\right)
\end{aligned}$$

where in the second step, the boundary term $W_a\left[u_a(x,\bar z),u_a(x,z)\right]=0$ since

$$W_a\left[u_a(x,\bar z),v_a(x)\right] = W_a\left[u_a(x,z),v_a(x)\right] = 0$$

with the v_a in Theorem 2.32. Likewise, the second integral is given by

$$\begin{aligned}
&u_a(y,\bar z)u_a(y,z)\int_y^b \left(\mathcal{L}u_b(x,\bar z)u_b(x,z) - u_b(x,\bar z)\mathcal{L}u_b(x,z)\right) r dx \\
=&u_a(y,z)u_b(y,z)\left(p(y)u_a(y,\bar z)\partial_y u_b(y,\bar z)\right) - u_a(y,\bar z)u_b(y,\bar z)\left(p(y)u_a(y,z)\partial_y u_b(y,z)\right).
\end{aligned}$$

By combining these two and using $W\left[u_a,u_b\right]=1$, we get

$$\int_a^b |G(x,y,z)|^2 r dx = \frac{1}{\bar z - z}\left(-u_a(y,z)u_b(y,z) + u_a(y,\bar z)u_b(y,\bar z)\right) = \frac{1}{\bar z - z}\left(\overline{G(y,y,z)} - G(y,y,z)\right),$$

which completes the proof. □

Using the same type of argument as in Lemma 2.43, we can show

$$\begin{aligned}
&\int_a^b |G(x,y,z)|^2 r dx = \frac{1}{\operatorname{Im} z}\operatorname{Im} G(y,y,z), \\
&\int_a^b \operatorname{Re}\left(\overline{G(x,y,z)}p(y)G(x,y,z)\right) r dx = \frac{1}{2}\frac{1}{\operatorname{Im} z}\operatorname{Im}\left(p(x)\partial_x G(x,y,z) + p(y)\partial_y G(x,y,z)\right)|_{x=y}, \\
&\int_a^b \left|p(y)\partial_y G(x,y,z)\right|^2 r dx = \frac{1}{\operatorname{Im} z}\operatorname{Im}\left(p(x)\partial_x p(y)\partial_y G(x,y,z)\right).
\end{aligned}$$

In the second identity, though $\lim_{x\to y_+} p(x)\partial_x G(x,y,z) \neq \lim_{x\to y_-} p(x)\partial_x G(x,y,z)$ (both exists since $u_a, u_b \in \mathcal{D}(\mathcal{L})$ but are not equal to each other), the right hand side is well-defined since the sum of these two discontinous function $p(x)\partial_x G(x,y,z) + p(y)\partial_y G(x,y,z)$ turns out to be continuous.

Now we can compute spectral measure when $\mathcal{L}$ is LC at a. We restate what we discussed at the very beginning of this subsection to refresh our memory. We stick to the notation v_a we used in Theorem 2.32 and we got $u_a(\cdot,z)$ and $\tilde v_a(\cdot,z)$ in $\mathcal{D}(L)$ satisfying the boundary condition such that $\tilde v_a \in \mathcal{D}(L)$, $\mathcal{L}u_a = zu_a$, $\mathcal{L}\tilde v_a = z\tilde v_a$ with $W_a\left[u_a,v_a\right] = W_a\left[\tilde v_a,v_a\right]=0$ and $W_a\left[u_a,\tilde v_a\right]=1$ for any z. Moreover, $\lambda\in\mathbb{R}$, $u_a(x,\lambda)$ and $\tilde v_a(x,\lambda)$ will be real-valued. Fix $u=u_a$ in

$$Uf(x) = \int_a^b u(x,\lambda)f(x)r(x)\,dx, \quad U: L^2(I, rdx)\to L^2(\mathbb{R}, d\mu),$$

then our spectral measure μ is fixed. Furthermore, $u_b(x,z) = \tilde v_a(x,z) + m_b(z)u_a(x,z)$ for $z\in\rho(A)$.

Since

$$G(y,y,z) = u_a(y,z)u_b(y,z) = u_a(y,z)v_a(y,z) + m_b(z)u_a(y,z)^2, \tag{2.13}$$

take $z = t + i\varepsilon$, then

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\int_{t_0}^{t_1+\delta}\int \frac{\varepsilon}{(\lambda-t)^2+\varepsilon^2}u_a(y,\lambda)^2\,d\mu(\lambda)\,dt = \lim_{\varepsilon\searrow 0}\int_{t_0}^{t_1+\delta}\operatorname{Im}G(y,y,t+i\varepsilon)\,dt\\
=&\lim_{\varepsilon\searrow 0}\int_{t_0}^{t_1+\delta}\operatorname{Im}\left(u_a(y,t+i\varepsilon)v_a(y,t+i\varepsilon)+m_b(t+i\varepsilon)u_a(y,t+i\varepsilon)^2\right)\,dt\\
=&\lim_{\varepsilon\searrow 0}\int_{t_0}^{t_1+\delta}(\operatorname{Im}m_b(t+i\varepsilon))\,|u_a(y,t)|^2\,dt.
\end{aligned}$$

On the other hand, by Fubini's theorem,

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\int_{t_0}^{t_1+\delta}\int \frac{\varepsilon}{(\lambda-t)^2+\varepsilon^2}|u_a(y,\lambda)|^2\,d\mu(\lambda)\,dt\\
=&\lim_{\varepsilon\searrow 0}\int \arctan\left(\frac{t_1+\delta-\lambda}{\varepsilon}\right)-\arctan\left(\frac{t_0-\lambda}{\varepsilon}\right)|u_a(y,\lambda)|^2\,d\mu(\lambda)\\
=&\frac{1}{2\pi}\int\left(\chi_{[t_0,t_1+\delta]}(\lambda)+\chi_{(t_0,t_1+\delta)}(\lambda)\right)|u_a(y,\lambda)|^2\,d\mu(\lambda)
\end{aligned}$$

and hence

$$\frac{1}{2\pi}\int\left(\chi_{[t_0,t_1+\delta]}(\lambda)+\chi_{(t_0,t_1+\delta)}(\lambda)\right)|u_a(y,\lambda)|^2\,d\mu(\lambda) = \lim_{\varepsilon\searrow 0}\int_{t_0}^{t_1+\delta}(\operatorname{Im}m_b(t+i\varepsilon))\,|u_a(y,t)|^2\,dt. \tag{2.14}$$

which will help us recover $d\mu(\lambda)$.

If $\mathcal{L}$ is regular at a, it is easy to recover μ as following. Then we can choose $u_a(a,\lambda)$ normalized to be $|u_a(a,\lambda)| = 1$ for all $\lambda\in\mathbb{R}$ provided $u_a(a,\lambda)\neq 0$. If $u_a(a,\lambda) = 0$, then we apply a similar argument to $p(a)u_a'(a,\lambda) = 0$ by using the second identity in Lemma 2.42. Moreover, by taking $y = a$ in (2.13), we get $\lim_{\varepsilon\to 0}\operatorname{Im}G(a,a,t+i\varepsilon) = \operatorname{Im}m_b(z)$ and hence $\operatorname{Im}m_b(t+i\varepsilon) > 0$ if $\varepsilon > 0$ thanks to Lemma 2.42.

Then (2.14) implies

$$\mu(t_0) = \frac{1}{\pi}\lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0}\int_{-\infty}^{t_0+\delta}\operatorname{Im}m_b(t+i\varepsilon)\,dt, \tag{2.15}$$

which will be a right continuous monotone non-decreasing function and the Stieltjes measure is given by the extension of $\mu((c,d]) = \mu(d)-\mu(c)$.

It is a bit tricky when $\mathcal{L}$ is LC at a. Let $d\nu_y(\lambda) = |u_a(y,\lambda)|^2\,d\mu(\lambda)$ be a new measure, then $\nu_y \ll \mu$. If we use the same type of argument to recover ν, we get

$$\nu_y(t_0) = \frac{1}{\pi}\lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0}\int_{-\infty}^{t_0+\delta}\operatorname{Im}m_b(t+i\varepsilon)|u_a(y,t)|^2\,dt,$$

which still depends on u_a. In fact, we can recover $d\mu$ directly by combining the two identities in Lemma 2.42 as follows. We compute

$$p(y)u_b'(y,\lambda)(UG(\cdot,y,z))(\lambda) - u_b(y,\lambda)U(p(y)\partial_y G(\cdot,y,z))(\lambda) = \frac{W_y[u_a,u_b]}{\lambda - z}.$$

Furthermore, since $W_y[u_a,u_b]$ is continuous in y and in fact $W[u_a,u_b]=1$, we can assign $y=a$ on both sides, which implies

$$(Uu_b(\cdot,z))(\lambda) = \frac{1}{\lambda - z}.$$

Then (2.15) follows from exactly the same argument as in the case for regular points.

Example 2.45. *Let $\mathcal{L} = -\frac{d^2}{dx^2}$ with $I=(0,+\infty)$. $\mathcal{L}$ is LC at 0 and LP at ∞ since we can find two linearly independent solutions both L^2 near 0. Let v_a be the v in Definition 2.29, $v_a(0)=\sin\alpha$, $v_a'(0)=\cos\alpha$, which is well-defined since $\mathcal{L}$ is actually regular at 0. The boundary condition $W_a[u,v_a]=0$ for u turns out to become*

$$BC_a(u) = \cos\alpha u(0) - \sin\alpha u'(0) = 0,$$

which coincides with the case for regular Sturm-Liouville problems. Likewise, there is a solution $\tilde{v}_a$ such that $\tilde{v}_a(0) = -\cos\alpha$, $p(a)\tilde{v}_a'(0)=\sin\alpha$.

Now our goal is to find $u_a(\cdot,z)$, $u_b(\cdot,z)$ and $m(z)$. Then for $z\in\mathbb{H}$, fundamental solutions are $e^{\sqrt{-z}x}$ and $e^{-\sqrt{-z}x}$. Here, we choose the branch of $\sqrt{-z}$ such that Re $\sqrt{-z}>0$ for $z\in\mathbb{H}$ in the upper half plane.

In order to make u_b decay at ∞, we need to choose $u_b(x,z)$ to be

$$u_b(x,z) = c\exp(-\sqrt{-z}x). \tag{2.16}$$

In order to find $u_a(x,z) = ce^{\sqrt{-z}x} + de^{-\sqrt{-z}x}$ with boundary condition $BC_a(u_a)=0$, we solve

$$\begin{pmatrix}\sin\alpha\\ \cos\alpha\end{pmatrix} = \begin{pmatrix}1 & 1\\ \mu+i\nu & -\mu-i\nu\end{pmatrix}\begin{pmatrix}c\\ d\end{pmatrix}, \quad \sqrt{-z}=\mu+i\nu$$

explicitly, we get

$$\begin{pmatrix}c\\ d\end{pmatrix} = \frac{1}{-2u-2iv}\begin{pmatrix}-\mu-i\nu & -1\\ -\mu-i\nu & 1\end{pmatrix}\begin{pmatrix}\sin\alpha\\ \cos\alpha\end{pmatrix}$$

$$u_a(x,z) = \left(\frac{1}{2}\sin\alpha + \frac{1}{2\sqrt{-z}}\cos\alpha\right)e^{\sqrt{-z}x} + \left(\frac{1}{2}\sin\alpha - \frac{1}{2\sqrt{-z}}\cos\alpha\right)e^{-\sqrt{-z}x}.$$

Let $\lambda>0$, then for $z=\lambda$, $\sqrt{-z}=\sqrt{-\lambda}=-i\sqrt{\lambda}$ due to the branch we take. Hence,

$$u_a(x,\lambda) = \sin\alpha\cos\sqrt{\lambda}x + \frac{1}{\sqrt{\lambda}}\cos\alpha\sin\sqrt{\lambda}x. \tag{2.17}$$

Moreover, one can get $\tilde{v}_a$ by a similar computation, and in fact, we just need to replace $\sin\alpha$ by $-\cos\alpha$ and $\cos\alpha$ by $\sin\alpha$ in (2.17), that is,

$$\tilde{v}_a(x,\lambda) = -\cos\alpha\cos\sqrt{\lambda}x + \frac{1}{\sqrt{\lambda}}\sin\alpha\sin\sqrt{\lambda}x.$$

Therefore,

$$\begin{pmatrix} u_a \\ \tilde{v}_a \end{pmatrix} = \begin{pmatrix} \sin\alpha & \frac{\cos\alpha}{\sqrt{\lambda}} \\ -\cos\alpha & \frac{\sin\alpha}{\sqrt{\lambda}} \end{pmatrix} \begin{pmatrix} \cos\sqrt{\lambda}x \\ \sin\sqrt{\lambda}x \end{pmatrix}$$

Hence,

$$u_b(x,z) = \begin{pmatrix} m_b(z) & 1 \end{pmatrix} \begin{pmatrix} u_a \\ \tilde{v}_a \end{pmatrix} = \begin{pmatrix} m_b(z) & 1 \end{pmatrix} \begin{pmatrix} \sin\alpha & \frac{\cos\alpha}{\sqrt{\lambda}} \\ -\cos\alpha & \frac{\sin\alpha}{\sqrt{\lambda}} \end{pmatrix} \begin{pmatrix} \cos\sqrt{\lambda}x \\ \sin\sqrt{\lambda}x \end{pmatrix} \tag{2.18}$$

On the other hand, from (2.16)*, we know*

$$u_b(x,z) = c\begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} \cos\sqrt{\lambda}x \\ \sin\sqrt{\lambda}x \end{pmatrix}. \tag{2.19}$$

Equating (2.18) *and* (2.19) *gives*

$$m_b(z) = -\frac{i\sqrt{\lambda}\cos\alpha + \sin\alpha}{i\sqrt{\lambda}\sin\alpha - \cos\alpha} = \frac{\sin\alpha - \sqrt{-z}\cos\alpha}{\cos\alpha + \sqrt{-z}\sin\alpha}.$$

A simple calculation reveals

$$\lim_{\varepsilon\searrow 0} Im\, m_b(t+i\varepsilon) = -Im\,\frac{i\sqrt{\lambda}\cos\alpha + \sin\alpha}{i\sqrt{\lambda}\sin\alpha - \cos\alpha} = \frac{\sqrt{\lambda}}{\cos^2\alpha + \lambda\sin^2\alpha}$$

It turns out that $d\mu$ does not have atomic part and

$$d\mu(\lambda) = \frac{1}{\pi}\frac{\sqrt{\lambda}}{\cos^2\alpha + \lambda\sin^2\alpha}\,d\lambda.$$

The unitary map U constructed using the spectral measure is named as the distorted Fourier transform, which is useful in recent research, like in [2], [6]. They use this machinery to study of long time bahavior of the wave equation, and to construct blow-ups at finite time.

2.5.3. *Computing spectral measures in general case.* We only give a brief introduction of the idea and one can refer to [13, Section 9.6] for a discussion in detail. In general, our spectral transformation map

$$U : L^2(I, r dx) \to L^2(\mathbb{R}, d\mu_1) \oplus L^2(\mathbb{R}, d\mu_2)$$

is given by

$$Uf = \int_a^b \begin{pmatrix} u_1(x,\lambda) \\ u_2(x,\lambda) \end{pmatrix} f(x) r(x)\,dx.$$

If L has simple spectrum, we just choose $\mu_2 = 0$. The procedure is as follows :

- For $c_0 \in I$, we define $c(x,z), s(x,z)$ to solve $\mathcal{L}u = zu$ with $s(c_0,z) = 0$, $p(c_0)s'(c_0,z) = 1$ and $c(c_0,z) = 1$, $p(c_0)c'(c_0,z) = 0$.
- Note that $u_j, j = 1,2$ solve $\mathcal{L}u = zu$, then there exists $C(\lambda)$ such that

$$\begin{pmatrix} u_1(x,\lambda) \\ u_2(x,\lambda) \end{pmatrix} = C(\lambda)\vec{S}(x,\lambda),$$

where $\vec{S}(x,\lambda) = \begin{pmatrix} c(x,\lambda) \\ s(x,\lambda) \end{pmatrix}$.

- Set $\tilde{\mu} = \mu_1 + \mu_2$, then $\mu_j \ll \tilde{\mu}$, and hence $d\mu_1 = r_1 d\tilde{\mu}$, $d\mu_2 = r_2 d\tilde{\mu}$. If we express $U^*U = I$ in terms of integral kernels thanks to Proposition 2.38, then we get

$$\begin{aligned}
\delta_0(x-y) &= \int \overline{u_1(x,\lambda)} u_1(y,\lambda) r(y)\, d\mu_1(\lambda) + \int \overline{u_2(x,\lambda)} u_2(y,\lambda) r(y)\, d\mu_2(\lambda) \\
&= \int \begin{pmatrix} \overline{u_1(x,\lambda)} & \overline{u_2(x,\lambda)} \end{pmatrix} \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \begin{pmatrix} u_1(y,\lambda) \\ u_2(y,\lambda) \end{pmatrix} r(y) d\tilde{\mu}(\lambda) \\
&= \int (C(\lambda)\vec{S}(x,\lambda))^* \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} C\vec{S}(y,\lambda) r(y) d\tilde{\mu}(\lambda) = \int \vec{S}(x,\lambda)^* R(\lambda) \vec{S}(y,\lambda) r(y) d\tilde{\mu}(\lambda),
\end{aligned}$$

where $R(\lambda) := C(\lambda)^* \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} C(\lambda)$. δ_0 is the integral kernel of I

- Now we define $\tilde{U} : L^2(I, r dx) \to L^2(\mathbb{R}, R(\lambda)\, d\tilde{\mu})$ by

$$\tilde{U} f(\lambda) := \int_I S(x,\lambda) f(x) r(x)\, dx,$$

where the inner product of $L^2(\mathbb{R}, R(\lambda)\, d\tilde{\mu})$ is given by $\langle \vec{F}, \vec{G} \rangle = \int_{\mathbb{R}} \vec{F}^* R \vec{G}\, d\tilde{\mu}(\lambda)$. Note that R is symmetric and positive definite. By renormalizing $\tilde{\mu}$, we may assume $\mathrm{tr} R = 1$.

- Now we characterize $R d\tilde{\mu}$. Put $G = G(\cdot, y; z)$, $p\partial_y G = p(y)\partial_y G(\cdot, y; z)$. We will have

$$\begin{cases} \langle G, G \rangle = \int \frac{1}{|z-\lambda|^2} R_{11}(\lambda) d\tilde{\mu}(\lambda), \\ \langle G, p\partial_y G \rangle = \int \frac{1}{|z-\lambda|^2} R_{12}(\lambda) d\tilde{\mu}(\lambda), \\ \langle p\partial_y G, p\partial_y G \rangle = \int \frac{1}{|z-\lambda|^2} R_{22}(\lambda) d\tilde{\mu}(\lambda), \end{cases}$$

- If $m_a(z), m_b(z)$ are defined so that $z \in \mathbb{C} \setminus \mathbb{R}$,

$$u_a(z) = c(x,z) - m_a(z) s(x,z), \quad u_b(z) = c(x,z) + m_b(z) s(x,z),$$

then

$$\begin{pmatrix} \langle G, G \rangle & \langle G, p\partial_y G \rangle \\ \langle G, p\partial_y G \rangle & \langle p\partial_y G, p\partial_y G \rangle \end{pmatrix} = \mathrm{Im} M(z),$$

which $M(z)$ is the Weyl-Titchmarsh M-matrix, given by

$$M(z) = \frac{1}{m_a(z) + m_b(z)} \begin{pmatrix} -1 & \frac{m_a(z) - m_b(z)}{2} \\ \frac{m_a(z) - m_b(z)}{2} & m_a(z) m_b(z) \end{pmatrix} = \begin{pmatrix} G(c_0, c_0, z) & \frac{p\partial_x + p\partial_y}{2} G(c_0, c_0, z) \\ \frac{p\partial_x + p\partial_y}{2} G(c_0, c_0, z) & p\partial_x p\partial_y G(c_0, c_0, z) \end{pmatrix}.$$

- We can recover the measure using $M(z)$ as what we did before.

3. ODEs in complex domains and special functions

First, we do a quick review for ODEs in complex domains. Let

$$\mathcal{L}u = \frac{d^2}{dz^2}u + P(z)\frac{d^2}{dz^2}u + Q(z)u = u'' + P(z)u' + Q(z)u = 0 \tag{3.1}$$

with $P, Q : U \to \mathbb{C}$ and $U \subset \mathbb{C}$ is simply connected.

Theorem 3.1 (Picard-Lindelof theorem)**.** *Suppose P and Q are analytic on U, then*

$$\begin{cases} u'' + P(z)u' + Q(z)u = 0, \\ u(z_0) = u_0, \\ u'(z_0) = u_1 \end{cases}$$

has a unique solution $u : U \to \mathbb{C}$ which is analytic on I.

Proof. We still use Picard iteration on curves from z_0. Thanks to simply connectedness, we can show that the value we choose is independent of the choice of paths. □

Remark 3.2. We can also prove analytic dependence on parameters. Suppose A is an open subset of $\mathbb{C}$ and if $P, Q : A \times U \to \mathbb{C}$ are continuous. For $a \in A$, we write $P = P(a, z)$, $Q = Q(a, z)$ and $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u_0(a) \\ u_1(a) \end{pmatrix}$. If $P(a_0, \cdot), P(\cdot, z_0)$, $Q(a_0, \cdot), Q(\cdot, z_0)$ are analytic for all $a_0 \in A, z_0 \in U$, then the solution $u = u(a, z)$ is analytic in A for all fixed z.

3.1. **Classification of singularities.**

Definition 3.3. *We say $z_0 \in U$ is a regular point of $\mathcal{L}$ if P and Q are analytic at z_0. If not, z_0 is a singular point of $\mathcal{L}$. If z_0 is the removable singular point of $(z - z_0)P(z)$ and $(z - z_0)^2 Q(z)$, then z_0 is said to be a regular singularity. If not, z_0 is an irregular singularity.*

Now we can find a local solution near a regular singular point by using the Frobenius method. Without loss of generality, we assume $z_0 = 0$ is a regular point and the corresponding Laurent series converge for $|z| < \rho$. We do the expansion

$$P(z) = \frac{P_0}{z} + P_1 + P_2 z + \cdots, \quad Q(z) = \frac{Q_0}{z^2} + \frac{Q_1}{z} + Q_2 + \cdots.$$

Then we denote the highest order terms by

$$\mathcal{L}_0 u = u'' + \frac{P_0}{z}u' + \frac{Q_0}{z^2}u.$$

Any equation of the form

$$u'' + \frac{P_0}{z}u' + \frac{Q_0}{z^2}u = 0$$

is called the Euler's equation, where 0 is a regular point provided that P_0, Q_0 are holomorphic near 0. In view of its homogeneity, we try the ansatzs $u = z^\alpha$ and this motivates the following indicial equation at 0,

$$\alpha(\alpha - 1) + P_0\alpha + Q_0 = 0. \tag{3.2}$$

We denote solutions to this indicial equation by α_1, α_2 and suppose $\operatorname{Re}\alpha_1 \geq \operatorname{Re}\alpha_2$. The basic idea is to construct solutiosn of this kind

$$u = z^\alpha \sum_{n=0}^{\infty} a_n z^n,$$

solving $\mathcal{L}u = 0$. To illustrate in details, here are three cases to consider.

(1) If $\alpha_1 - \alpha_2 \notin \mathbb{Z}$, then we can plug in the ansatzs

$$u_j = z^{\alpha_j} \sum_{n=0}^{\infty} a_{j,n} z^n$$

and equate the coefficients.

(2) If $\alpha_1 = \alpha_2 = \alpha$, then we plug in

$$u_1 = z^\alpha \sum_{n=0}^{\infty} a_{1,n} z^n.$$

In order to derive the form for u_2, by plugging $u_2 = hu_1$ into $\mathcal{L}u_2 = 0$, we get

$$0 = h'' u_1 + 2h' u_1' + P h' u_1,$$

which is a first order ODE. By setting $H = h'$, we have

$$\begin{aligned} H' + \left(2\frac{u_1'}{u_1} + \frac{P_0}{z} + P_1 + P_2 z + \cdots\right) H =& H' + (2\frac{\alpha}{z} + \frac{P_0}{z} + \text{ higher order terms})H \\ =& H' + \frac{1}{z}\left(1 + \text{ higher order terms}\right) H, \end{aligned}$$

where in the last step, we use $\alpha = \frac{1-P_0}{2}$, thanks to our assumption. Solving the approximate equation $H' = -\frac{1}{z}H$ gives $H = \frac{1}{z}$ and hence, $h = \log z$. That is to say, $h(z) = \log z +$ regular terms. Thus,

$$u_2 = z^\alpha \sum_{n=0}^{\infty} a_{2,n} z^n + u_1 \log z.$$

(3) If $\alpha_1 - \alpha_2 \in \mathbb{Z}_{>0}$, then

$$u_1 = z^{\alpha_1} \sum_{n=0}^{\infty} a_{1,n} z^n, \quad u_2 = z^{\alpha_2} \sum_{n=0}^{\infty} a_{2,n} z^n + c u_1 \log z,$$

where c can also be determined via equating coefficients.

Theorem 3.4. *These solutions u_1, u_2 constructed above converge within $\{|z| < \rho\}$, where ρ is the radius of convergence for the Laurent series of P, Q.*

Now we study the singularities at ∞. Let $\tilde{z} = \frac{1}{z}$, then $d\tilde{z} = -\frac{1}{z^2} dz$ and

$$\partial_z = -\frac{1}{z^2}\partial_{\tilde{z}} = -\tilde{z}^2 \partial_{\tilde{z}}, \quad \partial_z^2 = \tilde{z}^4 \partial_{\tilde{z}}^2 + 2\tilde{z}^3 \partial_{\tilde{z}},$$

which transform the original equation to

$$\tilde{u}'' + \left(\frac{2}{\tilde{z}} - \frac{1}{\tilde{z}^2}P(\frac{1}{\tilde{z}})\right)\tilde{u}' + \frac{1}{\tilde{z}^4}Q(\frac{1}{\tilde{z}})\tilde{u} = 0. \tag{3.3}$$

Then it is natural to classify the singularities as follows.

Definition 3.5. *We say ∞ is a regular point for $\mathcal{L}$ if $2z - z^2P(z)$ and $z^4Q(z)$ are bounded for $|z| \gg 1$. Otherwise, we say ∞ is a singular point for $\mathcal{L}$.*

Moreover, if $|z^2P(z)| = O(|z|)$ and $|z^4Q(z)| = O(|z|^2)$, then we say ∞ is a regular singularity for $\mathcal{L}$. Equivalently, ∞ is a regular singularity if and only if

$$|P(z)| \lesssim \frac{1}{|z|}, \quad |Q(z)| \lesssim \frac{1}{|z|^2}$$

when P, Q are analytic.

Then for (3.1), if ∞ is a regular singularity, then we can write

$$P = \frac{1}{z}\sum_{n\geq 0}\frac{P_n}{z^n}, \quad Q = \frac{1}{z^2}\sum_{n\geq 0}\frac{Q_n}{z^n},$$

for $|z| \geq R_0$. Thanks to (3.3), the corresponding indicial equation is given by

$$\alpha(\alpha-1) + (2 - P_0)\alpha + Q_0 = 0.$$

3.2. **Hypergeometric equations.**

3.2.1. *Associated Legendre equations and Hypergeometric functions.* As an motivating example, we derive the Legendre and associated Legendre equations. Both equations are originated from the spectral theory of $\mathbb{S}^2$. The laplacian operator in the spherical polar coordinates $g_{\mathbb{S}^2} = d\theta^2 + \sin^2\theta d\varphi^2$ is given by

$$\begin{aligned}
-\Delta_{\mathbb{S}^2} =& -\frac{1}{\det g}\frac{\partial}{\partial_i}(g^{ij}\sqrt{\det g}\frac{\partial}{\partial_j}) = -\frac{1}{|\sin\theta|}\left(\partial_\theta(|\sin\theta|\partial_\theta) + \frac{1}{|\sin\theta|}\partial_\varphi^2\right) \\
=& -\partial_\theta^2 - \frac{\cos\theta}{\sin\theta}\partial_\theta - \frac{1}{\sin^2\theta}\partial_\varphi^2.
\end{aligned}$$

Out starting point is the ansatz

$$u = \sum_{m\in\mathbb{Z}} u_m(\theta)e^{im\varphi}.$$

We compute

$$-\Delta_{\mathbb{S}^2}\left(u_m(\theta)e^{im\varphi}\right) = e^{im\varphi}\left(-\partial_\theta^2 - \frac{\cos\theta}{\sin\theta}\partial_\theta + \frac{m^2}{\sin^2\theta}\right)u_m$$

Then the eigenvalue problem $(L_m - \lambda)u_m = 0$ is equivalent to

$$-\partial_\theta^2 - \frac{\cos\theta}{\sin\theta}\partial_\theta + \frac{m^2}{\sin^2\theta} - \lambda = 0.$$

To study this, we transform it into an ODE via a change of variable. Let $z = \cos\theta$, and in the following computation, we stick to $z \in \mathbb{R}$ for simplicity. Then $\partial_\theta = -\sin\theta\partial_z = -\sqrt{1-z^2}\partial_z$ and $\partial_\theta^2 = (1-z^2)\partial_z^2 - z\partial_z$. If we put this in, we get an ODE

$$-(1-z^2)\partial_z^2 u_m + z\partial_z u_m + z\partial_z u_m + \frac{m^2}{1-z^2}u_m - \lambda u_m = 0.$$

By replacing $\lambda = l(l+1)$, we get the associated Legendre equation

$$(1-z^2)u'' - 2zu' + \left(l(l+1) - \frac{m^2}{1-z^2}\right)u = 0. \tag{3.4}$$

If $m = 0$, it is said to be the Legendre equation

$$(1-z^2)u'' - 2zu' + l(l+1)u = 0. \tag{3.5}$$

Note that all the three singularities ± 1 and ∞ are all regular singularities. It is important to notice that three points uniquely determine a Mobius transformation in the complex plane. With this vital property, one can reduce this to a canonical form to make the solutions look nicer so that we can develop the connection form. This is why the hypergeometric functions are of great importance.

Theorem 3.6. *If (3.1) has at most three singularities in $\mathbb{C}\cup\{\infty\}$, all of which are regular singularities, then it can be transformed, via a change of variable and a conjugation, to any other equation of the same form.*

Let

$$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{3.6}$$

where $(a)_n = a(a+1)\cdots(a+(n-1))$ is the Pochhammer's notation with the convention $(a)_0 = 1$. Functions of the form (3.6) are called hypergeometric functions. Such hypergeometric functions are supposed to be the Frobenius solution to a hypergeometric differential equation at $z = 0$ with index $\alpha = 0$. In view of this, we can give a derivation of the hypergeometric equation.

To keep track of the construction of a desired series solution, we list the properties of a series solution as follows.

$$z\frac{d}{dz}\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} n a_n z^n, \quad c\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} c a_n z^n, \quad z\sum_{n=0}^{\infty} a_n z^n = \sum_{n=1}^{\infty} a_{n-1} z^n.$$

A naive guess for the form of the hypergeometric equation will be

$$z\frac{d}{dz}u = zu,$$

whose power series solution near $z = 0$ is given by $na_n = a_{n-1}$, that is, $a_n = \frac{1}{n!}a_0$, which corresponds to an exponential. To produce the Pochhammer's notation in our coefficients, we find

$$z\frac{d}{dz}(z\frac{d}{dz} + c - 1)u = zu,$$

shall imply $n(n+c-1)a_n = a_{n-1}$, which generates $a_n = \frac{1}{(c)_n n!}a_0$. Finally, in order to get the terms in the numerator correctly, we notice that

$$z\frac{d}{dz}(z\frac{d}{dz}+c-1)u = z(z\frac{d}{dz}+a)(z\frac{d}{dz}+b)u, \tag{3.7}$$

is equivalent to

$$n(n+c-1)a_n = (n+a-1)(n+b-1)a_{n-1},$$

which corresponds to $a_n = \dfrac{(a)_n(b)_n}{(c)_n n!}a_0$.

Later, we will study the confluent hypergeometric function

$$M(a,c,z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^n, \tag{3.8}$$

which solves the confluent hypergeometric equation

$$z\frac{d}{dz}\left(z\frac{d}{dz}+c-1\right)u = z\left(\frac{d}{dz}+a\right)u. \tag{3.9}$$

It appears to be simpler than the hypergeometric equation at the first glance. However, its singularities may not be regular.

By an explicity computation, one can check (3.7) is equivalent to the following differential equation

$$z(1-z)u'' + (c-(a+b+1)z)u' - abu = 0, \tag{3.10}$$

which is called the hypergeometric equation. It has exactly three singularities $0, 1, \infty$ with corresponding indices as shown in the table.

singularity	0	1	∞
corresponding indices	$0, 1-c$	$0, c-a-b$	a, b

Table 1. Correspondence between singularities and indices

The indicial equations at $0, 1, \infty$ are

$$\begin{cases} \alpha(\alpha-1) + c\alpha = \alpha(\alpha-1+c) = 0, \\ -\alpha(\alpha-1) + (c-a-b-1)\alpha = 0, \\ \alpha(\alpha-1) + (a+b+1)\alpha + ab = (\alpha+a)(\alpha+b) = 0, \end{cases}$$

respectively. Though the indices found from the third equation for $z = \infty$ are $-a, -b$, we need to take the positive sign due to the change of variable $z \mapsto \frac{1}{z}$ when we define the singularities at ∞.

Now let's see the proof of Theorem 3.6.

Proof of Theorem 3.6. We start from the study of an ODE with three regular singularities $z_0, z_1, z_2 \in \mathbb{C}$. As a remark, if (3.1) has less than three singularities, we need to view some arbitrary ordinary points as regular singularities with indices 0 and 1, that is, $P_0 = Q_0 = 0$ in (3.2).

Henceforth, we use the Riemann notation ot describe the correspondence of the singularities and their indices.

$$\begin{Bmatrix} z_0 & z_1 & z_2 & \\ \alpha_1 & \beta_1 & \gamma_1 & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{Bmatrix}$$

Let

$$P(z) = \frac{\tilde{P}(z)}{(z-z_0)(z-z_1)(z-z_2)}, \quad Q(z) = \frac{\tilde{Q}(z)}{(z-z_0)^2(z-z_1)^2(z-z_2)^2}$$

with $\tilde{P}, \tilde{Q}$ holomorphic on $\mathbb{C}$. Since (3.1) is an ordinary point at ∞,

$$|P - \frac{2}{z}| \lesssim \frac{1}{|z|^2}, \quad |Q(z)| \lesssim \frac{1}{|z|^4}$$

thanks to Definition 3.5. Then by Liouville theorem,

$$\tilde{Q} = q_0 + q_1 z + q_2 z^2, \quad \tilde{P} = p_0 + p_1 z + p_2 z^2.$$

where $p_2 = 2$. In view of this, there are only five numbers of possibilities, since they satisfy some compatibility for the indices as follows. Suppose $P(z) = \frac{p_0+p_1z+p_2z^2}{(z-z_0)(z-z_1)(z-z_2)}$, and we know $P(z) \sim \frac{2}{z}$ as $z \to \infty$. We write $P(z)$ in terms of partial fractions, that is,

$$P(z) = \frac{p_0 + p_1 z + p_2 z^2}{(z-z_0)(z-z_1)(z-z_2)} = \frac{A_0}{z-z_0} + \frac{A_1}{z-z_1} + \frac{A_2}{z-z_2},$$

which implies $A_0 + A_1 + A_2 = 2$. The indicial equation at z_0 is

$$\rho(\rho-1) + (z-z_0)P(z)|_{z=z_0}\rho + \cdots = 0,$$

which implies

$$\alpha_1 + \alpha_2 = (1 - (z-z_0)P(z)|_{z=z_0}) = 1 - A_0.$$

Similarly, we have

$$\beta_1 + \beta_2 = (1 - (z-z_1)P(z)|_{z=z_1}) = 1 - A_1, \quad \gamma_1 + \gamma_2 = (1 - (z-z_2)P(z)|_{z=z_2}) = 1 - A_2,$$

and hence the sum of the indices is

$$\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 = 3 - (A_1 + A_2 + A_3) = 1.$$

We make a transformation

$$(z_0, z_1, z_2) \mapsto (0, 1, \infty)$$

given by

$$\tilde{z} = \frac{z-z_0}{z-z_2} \cdot \frac{z_1-z_2}{z_1-z_0}.$$

Let $\tilde{u} = u(z(\tilde{z}))$, then $\tilde{u}'' + \tilde{P}(\tilde{z})\tilde{u}' + \tilde{Q}(\tilde{z})\tilde{u} = 0$ is a differential equation with exactly the same indices as the original one.

Let $v = \tilde{z}^{\mu}(\tilde{z}-1)^{\nu}\tilde{u}(\tilde{z})$, then one can observe that this increases the indices at 0 by μ and the indices at 1 by ν by keeping track of the change of the coefficient $P(\tilde{z}), Q(\tilde{z})$ due to this subsequent change, (One can plug $\tilde{u} = \tilde{z}^{-\mu}(\tilde{z}-1)^{-\nu}v$ into the equation to see what happens.) Hence, the indices are

$$\alpha_k + \mu, \beta_k + \nu, \gamma_k - \mu - \nu$$

at $0, 1, \infty$ respectively, $k = 1, 2$.

□

Now we study the hypergeometric function (3.6). For simplicity, we suppose $c \notin \mathbb{Z}$ to avoid the appearance of log in the solutions. First, we study linearly independent solutions at 0. Let $u = z^{1-c}U(z)$, then U will satisfy an ODE with regular singularities and indices

$$U = \left\{ \begin{matrix} 0 & 1 & \infty & \\ c-1 & 0 & a+1-c & z \\ 0 & c-a-b & b+1-c & \end{matrix} \right\}$$

where we use the Riemann notation, which follows easily from the definition of u and Table 1. Hence,

$$G(a, b, c, z) := z^{1-c} F(a+1-c, b+1-c, 2-c, z)$$

is the other solution to (3.10) at $z = 0$ corresponding to the other exponent $\alpha = 1 - c$. In fact, one can check this explicitly by writing $u = \sum_n a_n z^{n+1-c}$ and plugging it into (3.7) to derive the recursive relation $(n - c + 1)na_n = (n + a - c)(n + b - c)a_{n-1}$.

Lemma 3.7. *For $c \notin \mathbb{Z}$,*

$$W_z[F, G] = (1-c)z^{-c}(1-z)^{c-a-b-1}.$$

Proof. Since

$$F'' = -\frac{c-(a+b+1)z}{z(1-z)}F' + \frac{ab}{z(1-z)}F, \quad G'' = -\frac{c-(a+b+1)z}{z(1-z)}G' + \frac{ab}{z(1-z)}G$$

we compute

$$\begin{aligned} W_z'[F, G] &= -\frac{c-(a+b+1)z}{z(1-z)}W_z[F, G] \\ &= -\left(\frac{c}{z(1-z)} - \frac{a+b+1}{1-z}\right)W_z[F, G] = -\left(\frac{c}{z} + \frac{c-a-b-1}{1-z}\right)W_z[F, G]. \end{aligned}$$

Hence,

$$W_z[F, G] = Cz^{-c}(1-z)^{c-a-b-1}.$$

Evaluating F, F', G, G' at $z = 0$ gives $C = 1 - c$. □

Now we study the solution near 1. Take $u = U(1 - z)$, then

$$U = \left\{ \begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a\ , & z \\ c-a-b & 1-c & b & \end{matrix} \right\}.$$

So it follows from Table 1 that the Frobenius solution for indices 0 and $c - a - b$ at 1 are

$$F(a, b, 1 + a + b - c, 1 - z), \quad G(a, b, 1 + a + b - c, 1 - z),$$

respectively, provided that $a + b - c \notin \mathbb{Z}$.

Finally, we study the linearly independent solutions at ∞. Let $u = z^{-\alpha}U(\frac{1}{z})$ and suppose $b - a \notin \mathbb{Z}$.

$$z^{-a}F(a, 1+a-c, 1+a-b, \frac{1}{z}), \quad z^{-b}F(b, 1+b-c, 1+b-a, \frac{1}{z})$$

3.2.2. *Connection formula with a review for Gamma functions.* In order to derive the connection formula, we recall some basic properties of Gamma functions. Let

$$\Gamma(z) = \int_0^\infty e^{-t} t^z \frac{dt}{t}, \quad \operatorname{Re} z > 0.$$

It follows from integration by parts that $z\Gamma(z) = \Gamma(z+1)$. Moreover, from this recursion relation and $\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$, we know $\Gamma(n) = (n-1)!$ and

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

As it is written, $\Gamma(z)$ is defined on $\{\operatorname{Re} z > 0\}$ and we can extend it to the whole complex plane by analytic continuation except integers less than or equal to zero.

Theorem 3.8. *Gamma functions can be expressed by an infinite product form*

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\cdots(z+n)}.$$

Proof. Since

$$e^{-t} = \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n \chi_{\{0<t<n\}}(t),$$

where the limit is monotone with respect to n and hence

$$\begin{aligned}
\Gamma(z) &= \int_0^\infty e^{-t} t^{z-1}\, dt = \lim_{n\to\infty} \int_0^n (1 - \frac{t}{n})^n t^{z-1}\, dt \\
&= \lim_{n\to\infty} \frac{1}{z} \int_0^n -\partial_t \left(1 - \frac{t}{n}\right)^n t^z\, dt = \lim_{n\to\infty} \frac{n}{nz} \int_0^n \left(1 - \frac{t}{n}\right)^{n-1} t^z\, dt \\
&= \lim_{n\to\infty} \frac{n(n-1)}{n^2 z(z+1)} \int_0^n \left(1 - \frac{t}{n}\right)^{n-2} t^{z+1}\, dt = \lim_{n\to\infty} \frac{n!}{n^n z(z+1)\cdots(z+n-1)} \int_0^n t^{z+n-1}\, dt \\
&= \lim_{n\to\infty} \frac{n!}{n^n z(z+1)\cdots(z+n-1)(z+n)} n^{z+n},
\end{aligned}$$

where $\operatorname{Re} z > 0$. ∎

Theorem 3.9. *Gamma functions satisfy the reflection identity*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

for $z \notin \mathbb{Z}$.

Proof. Let us formally evaluate

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \lim_{n\to\infty} \frac{z(z+1)\cdots(z+n)}{n!n^z}\frac{(1-z)\cdots(n+1-z)}{n!n^{1-z}}$$

$$= \lim_{n\to\infty} \left(1+\frac{z}{n}\right)\left(1+\frac{z}{n-1}\right)\cdots(1+z)\,z\left(1-\frac{z}{n+1}\right)\left(1-\frac{z}{n}\right)\cdots(1-z)\,\frac{n+1}{n}$$

$$= \lim_{n\to\infty} \frac{n+1}{n} z\left(1-\frac{z}{n+1}\right)\prod_{j=1}^{n}\left(1-\frac{z^2}{j^2}\right).$$

Hence, it suffices to show

$$\frac{\sin(\pi z)}{\pi z} = \prod_{j=1}^{n}\left(1-\frac{z^2}{j^2}\right).$$

Since $\dfrac{\sin(\pi z)}{\pi z}$ is an entire function of order 1, we know

$$\frac{\sin(\pi z)}{\pi z} = e^{p(z)}\prod_j E_1(\frac{z}{z_j}) = e^{az+b}\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right)$$

where $p(z) = az+b$ is a polynomial of degree at most 1, $z_j = j$ for all $j \in \mathbb{Z}\setminus\{0\}$, and

$$E_1\left(\frac{z}{j}\right) = \left(1-\frac{z}{j}\right)e^{z/j}$$

is the Weierstrass canonical factor (3.11). The last step follows from

$$E_1\left(\frac{z}{k}\right)E_1\left(\frac{z}{-k}\right) = 1-\frac{z^2}{k^2}.$$

Then evaluating at $z=0$ gives $b=0$. Moreover, thanks to the evenness, we know $a=0$, which completes the proof. $\square$

We restate the Hadamard factorization theorem (see [11, Theorem 22]) as a reference.

Theorem 3.10. *Let f be a nontrivial entire function of order ρ, that is,*

$$\rho = \inf\{\rho' : |f(z)| \le e^{|z|^{\rho'}} \text{ as } |z|\to\infty\}.$$

Let k be the integer such that $k \le \rho < k+1$. Then

$$f(z) = e^{p(z)}z^m\prod_n E_k\left(\frac{z}{z_n}\right),$$

where $p(z)$ is a polynomial of order at most k, m is the order of vanishing of f at the origin, $\{z_n\}$ enumerates the non-zero zeros of f (by multiplicity) and E_k is the Weierstrass canonical factor

$$E_k(z) = (1-z)\exp\left(\sum_{l=1}^{k}\frac{1}{l}z^l\right). \tag{3.11}$$

Theorem 3.11. *Gamma funtions satisfy the multiplication formula*

$$\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt = B(p,q),$$

which is the beta function.

Proof. We do a direct computation using Fubini's theorem

$$\Gamma(p)\Gamma(q) = \int_0^\infty e^{-t}t^{p-1}\,dt \int_0^\infty e^{-s}s^{q-1}\,ds = \int_0^\infty \int_0^\infty e^{-t-s}t^{p-1}s^{q-1}\,dt\,ds.$$

Now we make a change of variable $t = \frac{u+v}{2}$, $s = \frac{u-v}{2}$ and we get

$$\begin{aligned}\Gamma(p)\Gamma(q) &= \frac{1}{2}\int_0^\infty \int_{-u}^{u} e^{-u}\left(\frac{u+v}{2}\right)^{p-1}\left(\frac{u-v}{2}\right)^{q-1}\,dv\,du \\ &= \int_0^\infty e^{-u}u^{p-1+q}\int_{-u}^{u}\left(\frac{1+\frac{v}{u}}{2}\right)^{p-1}\left(\frac{1-\frac{v}{u}}{2}\right)^{q-1}\frac{dv}{2u}\,du \\ &= B(p,q)\int_0^\infty e^{-u}u^{p-1+q}\,du = \Gamma(p+q)B(p,q).\end{aligned}$$

□

Using these properties, we write

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^\infty \frac{(a)_n z^n}{n!}\frac{\Gamma(b+n)}{\Gamma(c+n)} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{n=0}^\infty \frac{(a)_n z^n}{n!}\int_0^1 t^{b+n-1}(1-t)^{c-b-1}\,dt,$$

and then by using the binomial formula $(1-z)^{-a} = \sum \frac{(a)_n z^n}{n!}$, we get

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^b(1-t)^{c-b-1}(1-zt)^{-a}\,dt.$$

By symmetry of a, b, we can also swap a and b to get another formula

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^a(1-t)^{c-a-1}(1-zt)^{-b}\,dt. \tag{3.12}$$

Theorem 3.12. *For $c \notin \mathbb{Z}$, $Re(c-a-b) > 0$, we have*

$$\lim_{z\to 1^-} F(a,b,c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}.$$

Proof. We compute

$$\begin{aligned}\lim_{z\to 1^-} F(a,b,c;z) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^b(1-t)^{c-a-b-1}\,dt \\ &= \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}.\end{aligned}$$

□

Now we derive the connection formula for the hypergeometric equation. We want to show that we can express any one of $F(a,b,c;z)$, $G(a,b,c;z)$, $F(a,b,1+a+b-c;1-z)$ and $G(a,b,1+a+b-c;1-z)$ by the other three.

Proposition 3.13. *Suppose* $c, a+b-c \notin \mathbb{Z}$ *and* $\mathrm{Re}(c-b-a) > 0, \mathrm{Re} b > 0$*, we have*

$$F(a,b,c;z) = AF(a,b,1+a+b-c;1-z) + BG(a,b,1+a+b-c;1-z), \tag{3.13}$$

where

$$A = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad B = -\frac{\pi}{\sin \pi(c-a-b)} \frac{\Gamma(c)\Gamma(c-a-b+1)}{\Gamma(a)\Gamma(b)}.$$

Proof. The key step is to use Theorem 3.12, By the theory of second order ODE, we know there exists some A, B such that (3.13) holds on $z \in (0,1)$. Take the limit $z \to 1^-$, then we know

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = A.$$

To compute B, we want to take $z \to 0^+$,

$$\begin{aligned} 1 =& A \lim_{z \to 1^-} F(a,b,1+a+b-c;z) + B \lim_{z \to 1^-} G(a,b,1+a+b-c;z) \\ =& A \frac{\Gamma(1+a+b-c)\Gamma(1-c)}{\Gamma(1+a-c)\Gamma(1+b-c)} + B \frac{\Gamma(c-a-b+1)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)}. \end{aligned}$$

□

3.3. **Confluent hypergeometric functions.** Our motivation is to study the eigenfunctions of Laplacian on $\mathbb{R}^2$. We express $-\Delta_{\mathbb{R}^2}$ for radial functions, which is

$$-\Delta_{\mathbb{R}^2} u(r) = k^2 u \iff -u'' - \frac{1}{r}u' - k^2 u = 0. \tag{3.14}$$

The theory developed so far cannot be applied directly since this equation has an irregular singularity at ∞.

We look for a generalized hypergeometric function which obeys an equation with irregular singularities as (3.14). One can find by construction that

$$M(a,c;z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} \tag{3.15}$$

solves (3.9). Here, $M(a,c;z)$ is called a Kummer's function or a confluent hypergeometric function with a regular singlarity at 0 and an irregular singularity at ∞. Formally, (3.15) is a limit case of some hypergeometric functions since $M(a,c;z)$ can be seen as $\lim_{b\to\infty} F(a,b,c;\frac{z}{b})$ by taking the limit of each term though it is not rigorous at all.

3.3.1. *ODEs with irregular singularities.* First, we study in full generality for $u''+Pu'+Qu = 0$. We introduce v and write $u = e^{\lambda z} v$ and compute

$$\begin{aligned} u'' + Pu' + Qu =& \lambda^2 e^{\lambda z} v + 2\lambda e^{\lambda z} v' + e^{\lambda z} v'' + P\lambda e^{\lambda z} v + Pe^{\lambda z} v' + Qe^{\lambda z} v \\ =& (P - P_0)\lambda e^{\lambda z} v + e^{\lambda z}(2\lambda + P)v' + e^{\lambda z} v'' + (Q - Q_0)e^{\lambda z} v, \end{aligned}$$

where we choose λ satisfying

$$\lambda^2 + P_0 \lambda + Q_0 = 0, \tag{3.16}$$

which is called the characteristic equation. Hence,

$$0 = v'' + (2\lambda + P)v' + ((P - P_0)\lambda + (Q - Q_0))\, v.$$

If we formally drop the first term and try to balance the last two after making another approximation

$$(2\lambda + P_0)v' + \left(\frac{P_1}{z}\lambda + \frac{Q_1}{z}\right) v = 0,$$

one will easy to see an ansatz $v = z^{\mu}$, which leads to

$$\mu(2\lambda + P_0) + (P_1\lambda + Q_1) = 0. \tag{3.17}$$

For λ, μ defined as in (3.16), (3.17), one can find a formal expansion

$$w \sim \sum \frac{c_n}{z^n}$$

such that $u = e^{\lambda z} z^{\mu} w$ solves $u'' + Pu' + Q = 0$ by substituting this expression in, where c_n is given by the recurrence relation

$$\begin{aligned}(P_0 + 2\lambda)nc_n =&(n-\mu)(n-1-\mu)c_{n-1} + (\lambda P_2 + Q_2 - (n-1-\mu)P_1)\, c_{n-1} \\ &+ (\lambda P_3 + Q_3 - (n-2-\mu)P_2)\, c_{n-2} + \ldots + (\lambda P_{n+1} + Q_{n+1} + \mu P_n)\, c_0.\end{aligned} \tag{3.18}$$

Actually, we need to assume $P_0^2 \neq 4Q_0$ in order to guarantee that two formal expansions are linearly independent due to the following reasons. Thanks to (3.16), $P_0^2 = 4Q_0$ shall imply $2\lambda + P_0 = 0$, which means that we cannot get a unique μ from (3.17).

Remark 3.14. In general, for $P_0^2 \neq 4Q_0$,

$$e^{\lambda z} z^{\mu} \sum \frac{c_n}{z^n} \tag{3.19}$$

will not be converegent. One will find that $c_{n+1} = O(n)c_n + \cdots$. The best thing one can hope for is that (3.19) is an asymptotic expansion for an actual soltution.

Definition 3.15 (Asymptotic expansion)**.** *Let $\mathcal{A}$ be a range for arguments of complex numbers. For a sector given by $\Gamma = \{\arg z \in \mathcal{A}, |z| > A\}$ and a function $u : \Gamma \to \mathbb{C}$, we say*

$$u \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n}$$

on Γ as $z \to \infty$ if and only if for all $N \geq 0$,

$$\left| z^{N+1} \left(u - \sum_{n=0}^{N} \frac{a_n}{z^n} \right) \right| \lesssim_N 1$$

on Γ.

Though the constants for different N may grow with N extremely fast, if we fix N and the divergent series will provide a good approximation if z is large enough. For more details about asymptotic expansions, see [8, Chapter 1.7].

For $P_0^2 = 4Q_0$, we need to use Fabry's idea. To find a formal ansatz for the solutions, we consider t with $z = t^2$. Then $dz = 2t\,dt$ and $\partial_z = \frac{1}{2t}\partial_t$ transform $0 = u'' + Pu' + Qu$ equivalently to

$$\partial_t^2 u + \left(-\frac{1}{t} + 2P(t^2)t\right)\partial_t u + 4t^2 Q(t^2)u = 0.$$

In this equation, the problematic terms are $2P_0 t$ and $4t^2 Q_0$. In order to eliminate the term $2P_0 t$, we set $u = e^{-\frac{1}{2}P_0 t^2}U(t)$. This is motivated as follows. Suppose

$$u = WU, \quad u' = WU' + W'U, \quad u'' = WU'' + 2W'U' + W''U, \quad \tilde{P} = -\frac{1}{t} + 2P(t^2)t, \quad \tilde{Q} = 4t^2 Q(t^2),$$

then we get

$$u'' + \tilde{P}u' + \tilde{Q}u = W\left(U'' + \left(2\frac{W'}{W} + \tilde{P}\right)U' + \left(\frac{W''}{W} + \tilde{P}\frac{W'}{W} + \tilde{Q}\right)U\right).$$

Hence, we choose W such that

$$\frac{W'}{W} = -\frac{1}{2}2P_0 t \iff (\log W) = -\frac{1}{2}P_0 t^2,$$

which will make the leading order terms of both $2\frac{W'}{W} + \tilde{P}$ and $\frac{W''}{W} + \tilde{P}\frac{W'}{W} + \tilde{Q}$ vanish. Then we can construct formal series solution for W as we did before.

Theorem 3.16. *Let $P(z), Q(z)$ are analytic functions of z having the convergent series expansion $P = \sum_{n\geq 0}\frac{P_n}{z^n}$ and $Q = \sum_{n\geq 0}\frac{Q_n}{z^n}$ for $|z| > R_0$. Suppose $P_0^2 \neq 4Q_0$ and*

$$\begin{matrix} \lambda_1 & \mu_1 & c_{1,n} \\ \lambda_2 & \mu_2 & c_{2,n} \end{matrix}$$

are constructed using (3.16), (3.17) and (3.18) as we discussed before.

For any $\delta > 0$, set

$$\Gamma_1 := \left\{z \in \mathcal{R} : \arg((\lambda_2 - \lambda_1)z) < \frac{3}{2}\pi - \delta, |z| > R_0\right\},$$

$$\Gamma_2 := \left\{z \in \mathcal{R} : \arg((\lambda_1 - \lambda_2)z) < \frac{3}{2}\pi - \delta, |z| > R_0\right\},$$

where $\mathcal{R}$ is the Riemann surface of $\log z$, the helix shaped surface. Then there exists unique holomorphic solutions u_j $(j = 1, 2)$ to

$$u'' + P(z)u' + Q(z)u = 0 \tag{3.20}$$

with the constructed formal series being their asymptotic expansions on Γ_j, respectively.

Proof. **Step 1:** First, we focus on $j = 1$ and we just work on a closure of a branch cut-off of the complex plane

$$\tilde{\Gamma}_1 = \{z : |\arg((\lambda_2 - \lambda_1)z)| \leq \pi, |z| \geq R_0\}.$$

Moreover, we drop all the subscripts $j = 1$ unless stated otherwise. By truncating the asymptotic expansion of u, we construct an approximating solution

$$u_{\leq N}(z) = e^{\lambda z} z^{\mu} \sum_{n=0}^{N} \frac{c_n}{z^n}.$$

Suppose $u = u_{\leq N} + \varepsilon_N$ solves (3.20), then ε_N satisfies

$$\varepsilon_N'' + P(z)\varepsilon_N' + Q(z)\varepsilon_N = -R_N, \quad R_N = u_{\leq N}'' + P(z)u_{\leq N}' + Q(z)u_{\leq N},$$

where R_N is the error term. Since $e^{\lambda z} z^{\mu} \sum_{n=0}^{\infty} \frac{c_n}{z^n}$ formally solves (3.20), we know that in the resulting formal sequence of

$$R_N = u_{\leq N}'' + P(z)u_{\leq N}' + Q(z)u_{\leq N},$$

all the coefficients of terms in the form of $e^{\lambda z} z^{\mu - n}$ vanish for $n = 0, 1, \dots, N$. In fact, the coefficients of $e^{\lambda z} z^{\mu - N - 1}$ in the formal sequence of R_N also vanishes. This is because one can observe that the coefficient of $e^{\lambda z} z^{\mu - N - 1}$ in the higher order summations $e^{\lambda z} z^{\mu} \sum_{n=N+1}^{\infty} \frac{c_n}{z^n}$ can be computed explicitly as

$$\lambda^2 c_{N+1} + P_0 \lambda c_{N+1} + Q_0 c_{N+1} = 0,$$

which vanishes thanks to (3.16). Hence,

$$R_N = e^{\lambda z} z^{\mu} O(\frac{1}{z^{N+2}}). \tag{3.21}$$

Heuristically speaking, though the remaining terms of $u_{\leq N}$ is $O(\frac{1}{z^{N+1}})$ and the remaning terms of R_N seems to be $O(\frac{1}{z^{N+3}})$ by plugging it into a second order ODE, but by our selection process of λ, μ, you only need to integrate once instead of twice, so we can obtain $O(\frac{1}{z^{N+1}})$ from integrating $O(\frac{1}{z^{N+2}})$. This is like what we did for the Green's functions - we only integrate once to get a solution to the second order ODE.

Step 2: The equation used to solve ε_N is

$$\varepsilon_N'' + P_0\varepsilon_N' + Q_0\varepsilon_N = -R_N - (P(z) - P_0)\varepsilon_N' - (Q(z) - Q_0)\varepsilon_N, \tag{3.22}$$

and we want to rewrite this by using Duhamel's principle so that we can apply the method of Picard iteration.

If ε_N has sufficient decay,

$$\varepsilon_N(z) = \int_z^{\infty e^{i\omega}} G(z, z') \left(R_N(z') + (P(z') - P_0)\, \varepsilon_N' + (Q(z') - Q_0)\varepsilon_N\right) dz', \tag{3.23}$$

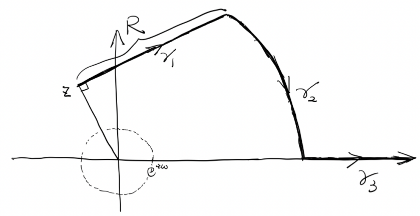
where $G(z, z') = \frac{e^{\lambda(z-z')} - e^{\lambda_2(z-z')}}{\lambda - \lambda_2}$ is the Green's function of the second order constant coefficients ODE derived by variation of parameters and ω satisfies $\arg((\lambda_2 - \lambda_1)e^{i\omega}) = 0$.

The choice of ω is motivated as below. We write $e^{\lambda_1 z} = e^{\lambda_2 z} e^{(\lambda_1 - \lambda_2)z}$, then the decay rate of $e^{(\lambda_1 - \lambda_2)z}$ is fastest if $\arg((\lambda_2 - \lambda_1)z) = 0$.

Moreover, we need to choose a specific path γ of integration for (3.23) as follows. We pick the direction $e^{i\omega}$ as our x-axis. Then let $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$, where

$$\begin{cases} \gamma_1 := \{z - itz : t \in [0, R]\}, \quad \gamma_3 := \{re^{i\omega} : r > r(R), r \in \mathbb{R}\}, \\ \gamma_2 := \text{ an arc of a circle centered at the origin joining } z - iRz \text{ and a point } r(R)e^{i\omega} \text{ with } r(R) > 0, \end{cases}$$

which is shown in the following figure.



On this path γ, $\mathrm{Re}((\lambda_2-\lambda_1)z')$ is nondecreasing as shown in the figure and hence

$$|e^{(\lambda_2-\lambda_1)(z-z')}| = e^{\mathrm{Re}((\lambda_2-\lambda_1)z)}e^{-\mathrm{Re}((\lambda_2-\lambda_1)z')}|_{z'=z} = 1,$$

which implies

$$|G(z,z')| \le \frac{|e^{\lambda_1(z-z')}|\left(1-e^{(\lambda_2-\lambda_1)(z-z')}\right)}{|\lambda_1-\lambda_2|} \le \frac{2|e^{\lambda_1(z-z')}|}{|\lambda_1-\lambda_2|}, \quad |\partial_z G(z,z')| \le \frac{|\lambda_1|+|\lambda_2|}{|\lambda_1-\lambda_2|}|e^{\lambda_1(z-z')}|.$$

Moreover,

$$|z^\mu| = e^{\mathrm{Re}(\mu\log z)} = e^{\mathrm{Re}\mu\log|z|+\mathrm{Im}\mu\arg z} \le C|z|^m,$$

where $m=\mathrm{Re}\mu$.

From (3.21), we know

$$|R_N(z')| \le C_N|e^{\lambda_1 z'}||z'|^{m-N-2}.$$

Step 3: Now we want to run Picard iteration. We define

$$T\varepsilon(z) = \int_\gamma G(z,z')\left(R_N(z')+(P(z')-P_0)\varepsilon'(z')+(Q(z')-Q_0)\varepsilon(z')\right)dz'.$$

First, we need to figure out what space we will work on. For $\varepsilon(z')\equiv 0$, we know that $T0(z)$ is holomorphic in $\tilde{\Gamma}_1$ and

$$\begin{aligned}|T0(z)| &\lesssim_N \int_\gamma \frac{2|e^{\lambda_1(z-z')}|}{|\lambda_1-\lambda_2|}|e^{\lambda_1 z'}||z'|^{m-N-2}\,dz' \lesssim_N |e^{\lambda_1 z}|\int_\gamma |z'|^{m-N-2}|dz'| \\ &\lesssim_{N,m} |e^{\lambda_1 z}|\lim_{R\to\infty}\int_{\gamma_1}|z'|^{m-N-2}|dz'| \lesssim_{N,m} |e^{\lambda_1 z}||z|^{m-N-1}\int_0^\infty |1-it|^{m-N-2}\,dt.\end{aligned}$$

Similarly, by the estimates for $\partial_z G$, we get

$$|(T0)'(z)| \lesssim_{N,m} |e^{\lambda_1 z}||z|^{m-N-1}\int_0^\infty |1-it|^{m-N-2}\,dt.$$

This motivates us to define

$$B_A = \{u : u \text{ is holomorphic on } \tilde{\Gamma}_1, |u(z)| \le A|e^{\lambda_1 z}||z|^{m-N-1}, |u'(z)| \le A|e^{\lambda_1 z}||z|^{m-N-1}\},$$

which is ball in the naturally induced Banach space with weighted C^0 topology. For any $\varepsilon_1, \varepsilon_2 \in B_A$, we estimate the difference by writing

$$|T(\varepsilon_1 - \varepsilon_2)(z)| \leq \int_\gamma |G(z,z')| \, |(P(z') - P_0)(\varepsilon_1 - \varepsilon_2)'(z') + (Q(z') - Q_0)(\varepsilon_1 - \varepsilon_2)(z')| \, |dz'|$$
$$\leq C|e^{\lambda_1 z}| \int_\gamma \frac{1}{|z'|} |\Delta\varepsilon| |z'|^{m-N-1} \, |dz'| \leq C|e^{\lambda_1 z}| |\Delta\varepsilon| |z|^{m-N-1} \int_0^\infty \frac{dt}{|1-it|^{N+2-m}},$$

where C is independent of N, obtained from the estimate

$$|P(z') - P_0| \leq C'|z'|^{-1}, |Q(z') - Q_0| \leq C'|z'|^{-1},$$
$$\Delta\varepsilon := \sup_{z \in \tilde{\Gamma}} \left\{ |\varepsilon_1 - \varepsilon_2|(z) e^{-\lambda_1 z} |z|^{-m+N+1} + |\varepsilon_1' - \varepsilon_2'|(z) e^{-\lambda_1 z} |z|^{-m+N+1} \right\}$$

and in the last step, we estimate by letting $R \to \infty$ and compute the integral along γ_1. Since

$$\int_0^\infty \frac{dt}{|1-it|^{N+2-m}} = \int_0^\infty (1+t^2)^{-\frac{N+2-m}{2}} \, dt \to 0$$

as $N \to \infty$, we know that for some sufficiently large N,

$$\sup_{z \in \tilde{\Gamma}} |e^{-\lambda_1 z}| |z|^{-m+N+1} |T(\varepsilon_1 - \varepsilon_2)(z)| \leq \delta |\Delta\varepsilon|,$$

for some $\delta \ll 1$.

Hence, by contraction mapping theorem, we know that there exists a unique $\varepsilon_N \in B_A$ satisfying (3.23) and hence (3.22).

Step 4: We check that $u_N = u_{\leq N} + \varepsilon_N$ is independent of N. For the sake of distinction, we stick to the notation with subscript and note that the preceding argument works for both $j = 1$ and $j = 2$. From the discussion above, we know that there exist solutions to (3.20) with the behavior

$$u_{j,N}(z) = e^{\lambda_j z} z^{\mu_j} \left(\sum_{n=0}^N \frac{c_{j,n}}{z^n} + O(\frac{1}{z^{N+1}}) \right)$$

in $\tilde{\Gamma}_j$ as $z \to \infty$ thanks to the result in preceding step. For the sake of distinction, we stick to the notation with subscript. Since (3.20) is of second order, we know $u_{2,N}$ is a linear combination of $u_{1,N}$ and $u_{1,N'}$ provided that $u_{1,N}$ and $u_{1,N'}$ are linearly independent. However, this is impossible by considering the asymptotics. Hence, $u_{1,N} = cu_{1,N'}$ for some constant c and for $N, N' \gg 1$. Moreover, by letting $z \to e^{i\omega}\infty$ along $\arg((\lambda_2 - \lambda_1)z) = 0$, we know $c = 1$.

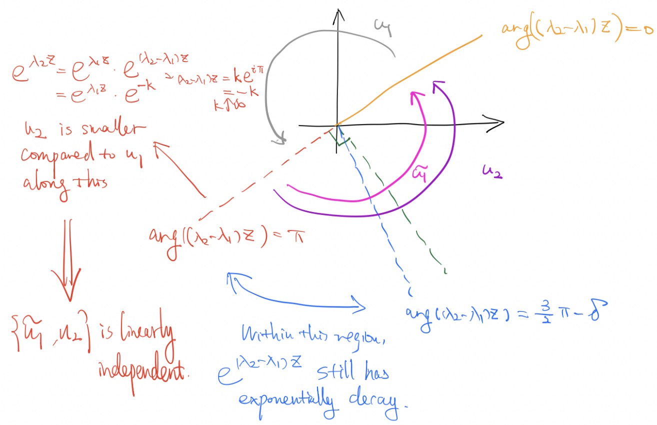
So we set $u_j := u_{j,N}$ for $N \gg 1$.

Step 5: Finally, we need to extend u_j from $\tilde{\Gamma}_j$ to Γ_j. Let $\tilde{u}_1(z) = u_1(ze^{-2\pi i})$, which is defined on the sector

$$\tilde{\Gamma}_3 := \{-\pi \leq \arg\left((\lambda_2 - \lambda_1)ze^{-2\pi i}\right) \leq \pi\} = \{\pi \leq \arg\left((\lambda_2 - \lambda_1)z\right) \leq 3\pi\}.$$

By considering the asymptotics along the ray $\arg((\lambda_2 - \lambda_1)z) = \pi$, we know that $\{\tilde{u}_1, u_2\}$ is linearly independent and hence there exist constants A and B such that

$$u_1(z) = A\tilde{u}_1(z) + Bu_2(z). \tag{3.24}$$



As we can see from the derivation in the picture, since u_2 is comparably smaller than u_1 and $\tilde{u}_1$ along $\arg((\lambda_2-\lambda_1)z)=\pi$, the asymptotics of u_1 and $\tilde{u}_1$ shall be the same, that is,

$$A\tilde{u}_1 \sim Ae^{\lambda_1 z}(e^{-2\pi i}z)^{\mu_1}\dots,$$

which implies $A = e^{2\pi\mu_1 i}$. Note that $\tilde{u}_1$ and u_2 are both well-defined within the region $\pi \leq \arg((\lambda_2-\lambda_1)z) < 2\pi$, which allows us to extend the expansion of u_1 to $\arg((\lambda_2-\lambda_1)z) \leq \frac{3}{2}\pi-\delta$ in terms of (3.24) without affecting its asymptotic series since $\tilde{u}_1$ still dominates within this region compared to u_2.

By extending in the same way to $\arg((\lambda_2-\lambda_1)z) \geq -\frac{3}{2}\pi+\delta$, u_1 can be continued to Γ_1 and the proof is complete. □

3.3.2. *Confluent hypergeometric equations.* Recall that (3.15) solves the confluent hypergeometric equation (3.9)

$$zu'' + (c-z)u' - au = 0,$$

which plays an important role in mathematical physics. In terms of transformations (fractional linear transformations for z, $z = \tilde{z}^\alpha$, or conjugation of u, i.e. $u(z) = w(z)U(z)$), confluent hypergeometric equations can be transformed into radial Laplacian ($d \geq 2$) and Airy's equations.

By writing (3.9) in the form we discussed in the preceding subsubsection, we have

$$u'' + (\frac{c}{z}-1)u' - \frac{a}{z}u = 0. \tag{3.25}$$

By changing the variables $u(z) = z^{1-c}U(z)$, a direct computation shows that

$$U'' + (-1+(2-c)z^{-1})U' + (-(1+a-c)z^{-1})U = 0,$$

which is a confluent hypergeometric equation with indices $1+a-c, 2-c$. Then

$$U(z) = M(1+a-c, 2-c, z)$$

is a power series solution and we set

$$N(a,c,z) := z^{1-c}M(1+a-c, 2-c, z),$$

which is another solution to (3.25) corresponding to different indices at $z=0$.

Lemma 3.17. *If $c \notin \mathbb{Z}$, then*

$$W_z[M(a,c;z), N(a,c;z)] = (1-c)e^z z^{-c}.$$

In particular, M and N are linearly independent.

Proof. This can be proved by a similar argument as in Lemma 3.7. Since M, N both satisfy (3.25), we get

$$W_z'[M,N] = -\left(\frac{c}{z} - 1\right) W_z[M,N],$$

which implies $W_z[M,N]$ is a constant multiple of $e^z z^{-c}$. By examining the behavior at $z=0$, the constant is $1-c$. □

Proposition 3.18. *For $c \notin \mathbb{Z}$, $\mathrm{Re} a, \mathrm{Re}(c-a) > 0$, $M(a,c,z)$ can be expressed by*

$$M(a,c,z) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{tz}\, dt.$$

Proof. We write

$$\begin{aligned} M(a,c,z) =& \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)} \frac{z^n}{n!} \\ =& \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 \sum_{n=0}^{\infty} \frac{(tz)^n}{n!} t^{a-1}(1-t)^{c-a-1}\, dt \\ =& \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{tz}\, dt. \end{aligned}$$

□

Remark 3.19. Formally,

$$F(a,b,c;\frac{z}{b}) \to M(a,c,z), \qquad \text{as } b \to \infty$$

in the sense that each summand in the series converge, that is,

$$\frac{(a)_n (b)_n}{(c)_n n!} \frac{z^n}{b^n} \to \frac{(a)_n}{(c)_n} \frac{z^n}{n!}.$$

In fact, this formal limit allows us to derive Proposition 3.18 from (3.12) since

$$\lim_{b\to\infty} \left(1 - \frac{zt}{b}\right)^{-b} = e^{tz}.$$

From the form (3.25) of the confluent hypergeometric equation, we know

$$P_0 = -1, \quad P_1 = c, \quad Q_0 = 0, \quad Q_1 = -a$$

and all other terms are zero. Thanks to the recurrence relation (3.18), we know that for $\lambda_1 = 0$ and $\mu_1 = -a$,

$$-nc_{1,n} = (a-c+n)(a+n-1)c_{1,n-1} \Rightarrow c_{1,n} = \frac{(a-c+1)_n (a)_n}{n!}(-1)^n c_{1,0}$$

and for $\lambda_2 = 1$ and $\mu_2 = a - c$,

$$nc_{2,n} = (-a+n)(-a+c+n-1)c_{2,n-1} \Rightarrow c_{2,n} = \frac{(-a+1)_n(-a+c)_n}{n!}c_{2,0}.$$

By Theorem 3.16, for all $\delta > 0$, there exist solutions U and V such that

$$\begin{aligned} U(a,c,z) &\sim z^{-a}\sum_{n=0}^{\infty}(-1)^n\frac{(a)_n(a-c+1)_n}{n!}\frac{1}{z^n} \text{ as } z\to\infty \text{ on } \Gamma_1, \\ V(a,c,z) &\sim e^z z^{a-c}\sum_{n=0}^{\infty}\frac{(1-a)_n(c-a)_n}{n!}\frac{1}{z^n} \text{ as } z\to\infty \text{ on } \Gamma_2, \end{aligned} \tag{3.26}$$

where

$$\Gamma_1 := \{|\arg z| \le \frac{3}{2}\pi - \delta, |z| > R_0\}, \quad \Gamma_2 := \{|\arg(-z)| \le \frac{3}{2}\pi - \delta, |z| > R_0\}.$$

Here, $U(a,c,z)$ is called the Tricomi function and it turns out that U has an integral representation.

Proposition 3.20. *For* $|\arg z| < \frac{1}{2}\pi$, $\mathrm{Re} a > 0$,

$$U(a,c,z) = \frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}(1+t)^{c-a-1}e^{-zt}\,dt. \tag{3.27}$$

Proof. First, we check that the right hand side of (3.27) is a solution to (3.25). We denote the right hand side by $u(z)$ and compute

$$u'(z) = -\int_0^\infty t^a(1+t)^{c-a-1}e^{-zt}\,dt, \quad u''(z) = \int_0^\infty t^{a+1}(1+t)^{c-a-1}e^{-zt}\,dt$$

and hence

$$\begin{aligned} zu'' + cu' =& (a+1-c)\int_0^\infty t^a(1+t)^{c-a-1}e^{-zt}\,dt + (c-a-1)\int_0^\infty t^{a+1}(1+t)^{c-a-2}e^{-zt}\,dt \\ =& -(c-a-1)\int_0^\infty t^a(1+t)^{c-a-2}e^{-zt}\,dt = zu' + au. \end{aligned}$$

Moreover, one can check that it has the same asymptotic series by applying the Watson's lemma. Then we only need to compute the constant coefficients. □

Actually, what is more important than the proof itself is that this formula for U can be guessed by taking the formal limit.

Now we present the connection formula.

Theorem 3.21. *For* $c \notin \mathbb{Z}$, *we have*

$$\begin{aligned} M(a,c,z) &= \frac{\Gamma(c)}{\Gamma(c-a)}e^{a\pi i}U(a,c,z) + \frac{\Gamma(c)e^{(a-c)\pi i}}{\Gamma(a)}V(a,c,z), \\ N(a,c,z) &= -\frac{\Gamma(2-c)e^{(a-c)\pi i}}{\Gamma(1-a)}U(a,c,z) + \frac{\Gamma(2-c)e^{(a-c)\pi i}}{\Gamma(1+a-c)}V(a,c,z), \end{aligned}$$

where we use $\{-\pi < \arg z \le \pi\}$ *for* U *and* N *while we use* $\{-\pi \le \arg(-z) < \pi\}$ *for* V.

Proof. We focus on the first formula and suppose

$$M(a,c,z) = AU(a,c,z) + BV(a,c,z).$$

By applying the ratio test to (3.15), we know M is an entire function. Then we would like to derive A and B by using a similar argument as what we did in the last step of the proof for Theorem 3.16. Since U dominates V along $\arg z = \pi$, we let z tend to infinity along this ray to compute A and let z tend to infinity along $\arg z = 0$ to compute B.

Recall from Proposition 3.18 that

$$M(a,c,z) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{tz}\, dt.$$

To compute B, let us understand the asymptotics of M as $z \to \infty$ along $\arg z = 0$ by the so-called Laplace's method. We write

$$\int_0^1 t^{a-1}(1-t)^{c-a-1}e^{tz}\,dt = e^z \int_0^1 (1-(1-t))^{a-1}(1-t)^{c-a-1}e^{-(1-t)z}\,dt = e^z \int_0^1 (1-s)^{a-1}s^{c-a-1}e^{-sz}\,dt.$$

Then by expanding

$$(1-s)^{a-1}s^{c-a-1} = s^{c-a-1}(1+(a-1)s+\dots)$$

in terms of s, and noticing that

$$\int_0^1 s^{\alpha-1}e^{-sz}\,ds = z^{-\alpha}\int_0^z \tau^{\alpha-1}e^{-\tau}\,d\tau \to z^{-\alpha}\Gamma(\alpha) + O(e^{-(1-\delta)z}),$$

we get

$$\begin{aligned}\int_0^1 t^{a-1}(1-t)^{c-a-1}e^{tz}\,dt =& e^z\left(\int_0^1 s^{c-a-1}e^{-sz}\,dt + O(|z|^{-(c-a+1)})\right)\\ =& e^z\left(z^{-(c-a)}\Gamma(c-a) + O(|z|^{-(c-a+1)})\right)\end{aligned}$$

as $z \to \infty$ along $\arg z = 0$. Hence, we know that $M(a,c,z) \sim \dfrac{\Gamma(c)}{\Gamma(a)} e^z z^{-(c-a)}$ as $z \to \infty$ along $\arg z = 0$.

On the other hand, thanks to (3.26), we know the asymptotics of $V(a,c,z)$ along $\arg(-z) = -\pi$. By rewriting it to be the ray $\arg(z) = 0$, we get

$$V(a,c,z) \sim e^z(|z|e^{-i\pi})^{a-c} = e^z z^{a-c} e^{-i(a-c)\pi}$$

as $z \to \infty$ along $\arg(z) = 0$. Then

$$Be^z z^{a-c}e^{-i(a-c)\pi} = \frac{\Gamma(c)}{\Gamma(a)} e^z z^{-(c-a)},$$

which implies $B = \dfrac{\Gamma(c)}{\Gamma(a)} e^{i(a-c)\pi}$.

To evaluate A, we need to consider along the ray $\arg z = \pi$ and we write

$$\begin{aligned} M(a,c,z) =& \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} e^{-t(-z)}\, dt \\ =& \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 t^{a-1} e^{-t(-z)}\, dt + O(|z|^{-a-1}) = \frac{\Gamma(c)}{\Gamma(c-a)}(-z)^{-a} + O(|z|^{-a-1}). \end{aligned}$$

From (3.26), we can derive the asymptotic of $U(a,c,z)$ along $\arg z = \pi$ as

$$U(a,c,z) \sim (|z|e^{i\pi})^{-a} = (-z)^{-a} e^{-a\pi i}.$$

By equating

$$\frac{\Gamma(c)}{\Gamma(c-a)}(-z)^{-a} = A(-z)^{-a} e^{-a\pi i},$$

we have $A = \dfrac{\Gamma(c)}{\Gamma(c-a)} e^{a\pi i}$, which completes the proof. □

4. Nonlinear problems

We would like to study nonlinear ODEs from a dynamical point of view. For an autonomous system

$$y' = F(y)$$

on $\mathbb{R}^n$, it can also be viewed as the integral curves of the vector field $F(y)$ on $\mathbb{R}^n$.

4.1. **Stability and instability of equilibria and the stable manifold theorem.**

Set $\vec{x} = (x^1, \dots, x^n)^T$. Let

$$\vec{x}' = \vec{f}(\vec{x}), \tag{4.1}$$

where $\vec{f} : D \to \mathbb{R}^n$ is a vector filed on $D \subset \mathbb{R}^n$ and $\vec{f}$ is C^1 on D. Then it follows from the Picard-Lindelof theorem that the initial value problem (4.1) has local wellposedness.

Definition 4.1. *We say $\vec{x}_0$ is a critical point or equilibrium if $\vec{f}(\vec{x}_0) = 0$. In other words, $\vec{x}(t) = \vec{x}_0$ is a constant solution to (4.1).*

Definition 4.2. *We denote $\vec{x}(t, s; \vec{\eta})$ is the unique (possibly local) solution to (4.1) satisfying $\vec{x}(s, s; \vec{\eta}) = \vec{\eta}$. Moreover, we write $\vec{x}(t; \vec{\eta}) = \vec{x}(t, 0; \vec{\eta})$.*

To study the global-in-time behavior of $\vec{x}(t)$, we use the equilibria of (4.1) as building blocks, which have largest possibilities to be candidates of $\lim_{t\to\infty} \vec{x}(t)$.

Without loss of generality, we assume that $\vec{0}$ is a critical point. By linearization, let us write

$$\vec{x}' = A\vec{x} + \vec{g}(\vec{x}), \quad A = D\vec{f}(\vec{0}), \tag{4.2}$$

where A is a constant real-valued $n \times n$ matrix and $|\vec{g}(\vec{x})| = o(|\vec{x}|)$ as $\vec{x} \to 0$.

For the linearized equation

$$\vec{y}' = A\vec{y}, \tag{4.3}$$

the stability of (4.3) is determined by the spectrum of A. Suppose $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of A with associated generalized eigenspace $E_1, \dots, E_k$ such that $\dim E_j = n_j$, where n_j is the algebraic multiplicity of λ_j. Then $\mathbb{R}^n = E_1 \oplus \cdots \oplus E_k$, and E_j is invariant under A and $\exp(tA)$. Moreover,

$$\exp(tA)|_{E_j} = e^{t\lambda_j} P_j(A - \lambda_j I),$$

where P_j is a finite degree polynomial. Therefore, the stability of the linear flow $\exp(tA)$ on each E_j is determined by $|e^{t\lambda_j}| = e^{t\mathrm{Re}\lambda_j}$, which motivates the following definition. See [10, Section 2.5].

Definition 4.3. *Set*

$$E_s^{\mathbb{C}} = \oplus\{E_j : \mathrm{Re}\lambda_j < 0\}$$

to be the complex stable linear subspace, which means that the solution to the linearized system shall be stable forward in time. On the other hand, the complex unstable linear subspace is given by

$$E_u^{\mathbb{C}} = \oplus\{E_j : \mathrm{Re}\lambda_j > 0\}$$

and the complex center linear subspace is given by

$$E_c^{\mathbb{C}} = \oplus\{E_j : \mathrm{Re}\lambda_j = 0\}.$$

Definition 4.4. *Note that* $\mathbb{C}^n = E_s^{\mathbb{C}} \oplus E_u^{\mathbb{C}} \oplus E_c^{\mathbb{C}}$ *naturally induce three linear projections* P_s, P_u *and* P_c.

One can check that $P_s\bar{x} = \overline{P_s x}$ and then set $E_s := P_s\mathbb{R}^n$.

Definition 4.5. *Set*

$$E_s := P_s\mathbb{R}^n, \quad E_u := P_u\mathbb{R}^n, \quad E_c := P_c\mathbb{R}^n$$

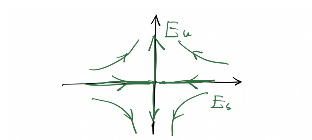
to be the stable linear subspace, unstable linear subspace and center linear subspace, respectively.

Definition 4.6. *We say* A *is hyperbolic at* $\vec{0}$ *if* $\mathrm{Re}\lambda_j \neq 0$ *for all* j.

From now on, we assume that $\vec{0}$ is indeed hyperbolic, so

$$\mathbb{R}^n = T_{\vec{0}}\mathbb{R}^n = E_s \oplus E_u.$$

For $n = 2$, the picture for the linearized system is roughly as follows.



By the stable manifold theorem, we can extend this picture for linearized case to the nonlinear system. We need to introduce some notions at first.

Definition 4.7. *The global stable manifold is given by*

$$\Sigma_s(\vec{0}) := \left\{\vec{\eta} \in D : \vec{x}(t;\vec{\eta}) \text{ exists for all time } t \geq 0, \text{ and } \lim_{t\to\infty} \vec{x}(t,\vec{\eta}) = \vec{0}\right\},$$

which is the domain of attraction for $\vec{0}$.

Let U *be a neighborhood of* $\vec{0}$*, then a local stable manifold of* $\vec{0}$ *relative to* U *is given by*

$$\Sigma_s^U(\vec{0}) := \left\{\vec{\eta} \in \Sigma_s(\vec{0}) : \vec{x}(t;\vec{\eta}) \in U \text{ for all } t \geq 0\right\}.$$

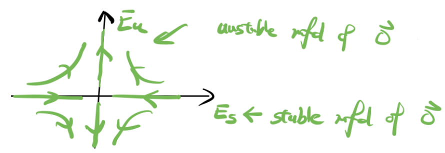
On the other hand, the global unstable manifold is given by

$$\Sigma_u(\vec{0}) := \left\{\vec{\eta} \in D : \vec{x}(t;\vec{\eta}) \text{ exists for all time } t \leq 0, \text{ and } \lim_{t\to-\infty} \vec{x}(t;\vec{\eta}) = \vec{0}\right\}$$

and a local unstable manifold of $\vec{0}$ *relative to* U *is given by*

$$\Sigma_s^U(\vec{0}) := \left\{\vec{\eta} \in \Sigma_u(\vec{0}) : \vec{x}(t;\vec{\eta}) \in U \text{ for all } t \leq 0\right\}.$$

Then the picutre for the linearized system above serves as an example for Σ_s and Σ_u.



Now we view (4.2) as a perturbative system of the linearized one (4.3) and we have the following local stable manifold theorem.

Theorem 4.8 (Local Stable Manifold Theorem)**.** *For (4.2), we assume that $\vec{f}$ is C^1, $\vec{0}$ is a critical point and $DF(\vec{0})$ is hyperbolic. Then there exist $\lambda > 0, r > 0$, an open set $U \subset E_s$ which contains $\vec{0}$ and a C^1 function $\gamma : U \to E_u$ with the following properties :*

(1) $\gamma(\vec{0}) = \vec{0}$ *and* $D\gamma(\vec{0}) = 0$*;*

(2) set the graph of γ by

$$\Gamma = \{\vec{\eta} \in D : P_s\vec{\eta} \in U, \quad P_u\vec{\eta} = \gamma(P_s\vec{\eta})\}.$$

If $\vec{\eta} \in \Gamma$, then $\vec{x}(t, \vec{\eta})$ is defined for all $t \geq 0$ and

$$\|\vec{x}(t, \vec{\eta})\| \leq C\|P_s\vec{\eta}\|e^{-\lambda t}.$$

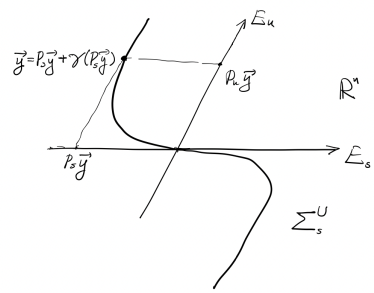
In particular, Γ is a local stable manifold of $\vec{0}$ relative to $B_r(\vec{0})$.

(3) The stable manifold is given by

$$\Gamma_s(\vec{0}) = \bigcup_{t \leq 0} x(t, \Gamma).$$

Remark 4.9. From the statement, it follows that Γ is a C^1 submanifold tangent to E_s at $\vec{0}$. Moreover, $\Gamma_s(\vec{0})$ is an immersed C^1 manifold. See [10, Theorem 7.7].

Remark 4.10. By reversing time, one can also construct unstable manifold Σ_u by reversing the time, which is also called the unstable manifold theorem.



We only sketch the proof for this theorem. See [10, Section 7.3] for a proof in detail. We use Perron's method (Lyapunov-Perron's method), which requires using Duhamel's principle to derive the key integral formula.

We begin by writing

$$\vec{x}' = A\vec{x} + \vec{g}(\vec{x}).$$

For $\vec{x}(t) = \vec{x}(t; \vec{\eta})$, by Duhamel's principle, we have

$$\vec{x}(t) = e^{tA}\vec{\eta} + \int_0^t e^{(t-s)A}\vec{g}(\vec{x}(s))\, ds.$$

A key observation is that if $\vec{x}(t)$ stays bounded for all $t \geq 0$, then

$$\vec{x}(t) = e^{tA}\vec{\eta} + \int_0^t e^{(t-s)A}P_s\vec{g}(\vec{x}(s))\, ds - \int_t^\infty e^{(t-s)A}P_u\vec{g}(\vec{x}(s))\, ds. \tag{4.4}$$

The proof of (4.4) is as follows. We write $e^{tA} = e^{tA}P_s + e^{tA}P_u$, then

$$\begin{aligned}\vec{x}(t) =& e^{tA}P_s\vec{\eta} + \int_0^t e^{(t-s)A}P_s\vec{g}(\vec{x}(s))\, ds + e^{tA}P_u\vec{\eta} + \int_0^t e^{(t-s)A}P_u\vec{g}(\vec{x}(s))\, ds\\ =& e^{tA}P_s\vec{\eta} + \int_0^t e^{(t-s)A}P_s\vec{g}(\vec{x}(s))\, ds - \int_t^\infty e^{(t-s)A}P_u\vec{g}(\vec{x}(s))\, ds + e^{tA}P_u\vec{\eta} + \int_0^\infty e^{(t-s)A}P_u\vec{g}(\vec{x}(s))\, ds,\end{aligned}$$

where the first three terms are bounded if $\vec{g}(\vec{x}(s))$ is bounded. However, the last two terms

$$e^{tA}P_u\vec{\eta} + \int_0^\infty e^{(t-s)A}P_u\vec{g}(\vec{x}(s))\, ds = e^{tA}P_u\left(\vec{\eta} + \int_0^\infty e^{-sA}P_u\vec{g}(\vec{x}(s))\, ds\right)$$

would go to infinity as $t \to \infty$ unless the terms in the parentheses cancel to be zero. Then the idea is to start from (4.4) to show existence, uniqueness by Picard iteration.

4.2. **Application of stable manifold theorem : shooting method.** We will construct a travelling wave solution to

$$\partial_t w - \partial_x^2 w + f(w) = 0, \tag{4.5}$$

which is a reaction-diffusion equation. Here, a travelling wave solution means that $w(t, x) = u(x + ct)$, that is, a wave travelling to the left with speed c. Set $f(w) = w(w-a)(w-1)$ and $0 < a < \frac{1}{2}$. If we plug $w = u(x + ct)$ into (4.5), we get

$$cu' - u'' + f(u) = 0, \tag{4.6}$$

which is our new ODE, which can also be expressed in the first order formulation

$$\frac{d}{dx}\begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} \dot{u} \\ c\dot{u} + f(u) \end{pmatrix}. \tag{4.7}$$

By Definition 4.1, the critical points of (4.7) are $(0, 0)$, $(a, 0)$ and $(1, 0)$. The energy functional given by

$$E(u, \dot{u}) = \frac{1}{2}\dot{u}^2 + F(u), \quad F'(u) = -f(u)$$

satisfies

$$\frac{d}{dx}E(u, \dot{u}) = \dot{u}\ddot{u} + (-f(u))\dot{u} = c\dot{u}^2 + \dot{u}f(u) - f(u)\dot{u} = c\dot{u}^2. \tag{4.8}$$

Hence, we require $c \geq 0$, which makes E nondecreasing along trajectories of (4.7).

We are looking for a solution $(u, \dot{u})(x)$ to (4.7) such that

$$\lim_{x\to-\infty}\begin{pmatrix} u \\ \dot{u} \end{pmatrix}(x) \text{ and } \lim_{x\to+\infty}\begin{pmatrix} u \\ \dot{u} \end{pmatrix}(x)$$

both converge to critical points. In particular, we want

$$\lim_{x\to-\infty}(u,\dot{u})(x)=(0,0).$$

Observe that if we normalize $E(0,0)=0$, then

$$E(a,0)=F(a)=-\int_0^a f(v)\,dv<0$$

and

$$E(1,0)=-\int_0^1 f(v)\,dv=-\int_0^1 v(v-a)(v-1)\,dv>0,$$

thanks to the assumption $a<\frac{1}{2}$. So $\lim_{x\to+\infty}(u,\dot{u})(x)$ cannot be $(a,0)$ by the energy identity (4.8) and $E(0,0)=0$, $E(1,0)>0$. We look for

$$\lim_{x\to+\infty}(u,\dot{u})(x)=(1,0).$$

Theorem 4.11. *Given $0<a<\frac{1}{2}$, there exists $c\in[0,\infty)$ such that (4.7) admits a solution $\begin{pmatrix}u\\ \dot{u}\end{pmatrix}$ defined for all $x\in(-\infty,\infty)$ and*

$$\lim_{x\to-\infty}\begin{pmatrix}u\\ \dot{u}\end{pmatrix}(x)=\begin{pmatrix}0\\ 0\end{pmatrix},\quad \lim_{x\to+\infty}\begin{pmatrix}u\\ \dot{u}\end{pmatrix}(x)=\begin{pmatrix}1\\ 0\end{pmatrix}.$$

In the language of dynamical systems, it is equivalent to say that there exists a heteroclinic orbit from $\begin{pmatrix}0\\ 0\end{pmatrix}$ to $\begin{pmatrix}1\\ 0\end{pmatrix}$ for some c.

Proof. Though the statment we need to prove is a global property, we start by computing the linearization of (4.6) near each equilibrium:

$$\vec{F}=\begin{pmatrix}\dot{u}\\ c\dot{u}+u(u-a)(u-1)\end{pmatrix},\quad D\vec{F}(0,0)=\begin{pmatrix}0&1\\ a&c\end{pmatrix}.$$

It is easy to notice that $(0,0)$ is hyperbolic and it is a saddle point, where the eigenvalues are $\lambda_1=\frac{c+\sqrt{c^2+4a}}{2}>0$ and $\lambda_2=\frac{c-\sqrt{c^2+4a}}{2}<0$. The corresponding eigenvectors are

$$\vec{v}_1=\begin{pmatrix}1\\ \lambda_1\end{pmatrix},\quad \vec{v}_2=\begin{pmatrix}1\\ \lambda_2\end{pmatrix}.$$

By the (un)stable manifold theorem and Remark 4.9 corresponded, there exists an unstable manifold $\Sigma_u(\vec{0};c)$ tangent to $\vec{v}_1$. Moreover, by the Picard iteration method used for the proof of the unstable manifold theorem, we can prove that

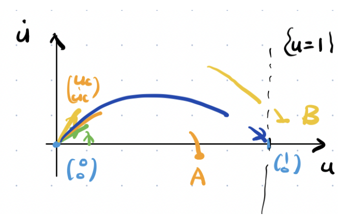
- $\forall c\geq 0$, there exists a unique $\begin{pmatrix}u_c\\ \dot{u}_c\end{pmatrix}$ on $\Sigma_u(\vec{0};c)$ such that

$$\lim_{x\to-\infty}\frac{u_c(x)}{e^{\lambda_1 x}}=1,$$

- $c \mapsto (u_c, \dot{u}_c)$ is continuous in the following sense: If $(u_c, \dot{u}_c)(t)$ exists on $(-\infty, T]$, then for c' close c, $(u_{c'}, \dot{u}_{c'})(t)$ exists on $(-\infty, T]$ and
$$\lim_{c \to c'} \sup_{(-\infty, T]} |(u_c, \dot{u}_c)(x) - (u_{c'}, \dot{u}_{c'})(x)| = 0.$$

We omit the proof for these two properties.

To prove the existence of c such that $\begin{pmatrix} u_c \\ \dot{u}_c \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as $x \to +\infty$, we perform a shooting argument. Define two bad sets
$$A := \{c > 0 : \exists x_1, \dot{u}_c(x_1) < 0, u_c(x) < 1, \forall x \in (-\infty, x_1)\},$$
$$B := \{c > 0 : \exists x_1, u_c(x_1) > 1, \dot{u}_c(x) > 0, \forall x \in (-\infty, x_1)\}.$$



Our goal is to show that for $A, B \subset (0, \infty)$

(1) $A \cap B = \varnothing$,
(2) A, B are open,
(3) A, B nonempty,

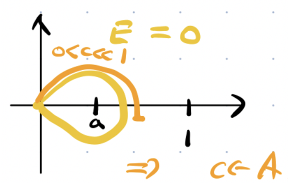
then by connectedness of $(0, \infty)$, we know $(0, \infty) \setminus (A \cup B) \neq \varnothing$. Let $c \in (0, \infty) \setminus (A \cup B)$, then
$$u_c(x) \leq 1 \text{ and } \dot{u}_c(x) \geq 0 \tag{4.9}$$
for all x's. In particular, u_c stay bounded and $u_c(\infty)$ exists. Moreover, combining this result with (4.8), we know u_c and $\dot{u}_c$ both stay bounded, which in turn shows that u_c exists for $x \in (-\infty, \infty)$. On the other hand, from (4.8), E is monotonic. In particular, $\lim_{x \to +\infty} E(u_c, \dot{u}_c)$ exists, which implies the existence of $\dot{u}_c(\infty)$. Now, the existence of $u_c(\infty)$ and $\dot{u}_c(\infty)$ implies that the trajectory $(u_c, \dot{u}_c)$ has a limit as $x \to \infty$, which must be the critical point $(1, 0)$. Otherwise, suppose it converges to (α_0, β_0). If $\beta_0 > 0$, then for sufficiently large x, $\dot{u}_c(x) > \frac{1}{2}\beta_0$, which means that $u_c(x)$ cannot converge to a finite value $\alpha_0 < 1$. This implies $\beta_0 = 0$ and then we should require $E(\alpha_0, 0) = F(\alpha_0) > 0$. In particular, $\alpha_0 > a$ and hence $f(\alpha_0) < 0$ unless $\alpha_0 = 1$, which implies $\ddot{u}_c(x) < -\frac{1}{2} f(\alpha_0)$ for sufficiently large x and hence $\dot{u}_c$ cannot have a finite limit α_0. Therefore, $\alpha_0 = 1$. (Actually, this can be explained in an easier and general way. For $X' = F(X)$, if $X(t)$ converges to X_∞ as $t \to \infty$, then there exists a sequence of t_j such that $X'(t_j) = 0$, which implies $F(X_\infty) = \lim_{t \to \infty} F(X(t_j)) = 0$, that is, the limit point X_∞ is a critical point.)

Now we prove the three properties for A and B. The first two are obvious from the definition. Now we show the third one by proving the folowing claim :

- if $0 < c \ll 1$ is small enough, then $c \in A$;
- if $c \gg 1$, then $c \in B$.

The proof is as follows.

- If $c = 0$, then $(u_0, \dot{u}_0)$ stays on the closed curve $E = 0$, which is the yellow curve in the picture. Note that all the points on $\{E = 0\}$ are ordinary (not critical) except $(0, 0)$. So $(u_0, \dot{u}_0)$ is a homoclinic orbit associated with $(0, 0)$ and there exists x_1 such that $\dot{u}_0(x_1) < 0$ yet $u_0(x) < 1$ for all $x \in (-\infty, x_1)$, that is, $0 \in A$. By continuity in c, $c \in A$ for $0 < c \ll 1$.



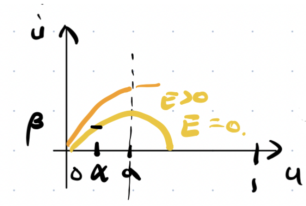
- Now we want to show the second claim.
  - Claim 1: any $(u_c, \dot{u}_c)$ must cross the line $\{u = a\}$. Since $\frac{d}{dx}E \geq 0$, $(u_c, \dot{u}_c)$ must stay outside of $\{E = 0\}$. Thus, for any x_0 such that the solution exists in $(-\infty, x_0)$, if

$$u_c(x) < a, \qquad \text{for all } x \in (-\infty, x_0), \tag{4.10}$$

then $\dot{u}_c(x) > 0$ for all $x \in (-\infty, x_0)$. Fix $0 < \alpha < a$, where $(u_c, \dot{u}_c)$ crosses $\{u = \alpha\}$. Let $\beta > 0$ be such that $(\alpha, \beta) \in \{E = 0\}$. Now we claim that

$$\text{if (4.10) holds and } x_0 < \infty, \text{ then } E(x_0) := E(u_c(x_0), \dot{u}_c(x_0)) < \infty. \tag{4.11}$$

This implies that $(u_c, \dot{u}_c)$ stay bounded in $(-\infty, x_0)$. By monotonicity, $(u_c, \dot{u}_c)$ can be extended to any finite interval. Furthermore, since $u_c(x)$ is increasing, if (4.10) holds, then $\dot{u}_c(x) \geq \beta$ for all x such that $u_c(x) \geq \alpha$, which implies $u_c(x) \geq a$ for x sufficiently large.



Hence, it suffices to prove the claim above. If $E(x) < \max_{0 \leq u \leq a} |F(u)|$ for all $x \in (-\infty, x_0)$, then we are done. Otherwise, there exists $x < x_0$ such that $E(x) = \max_{0 \leq u \leq a} |F(u)|$ and hence $E(x') \geq \max_{0 \leq u \leq a} |F(u)|$ for all $x' \in [x, x_0)$. Therefore,

$$E(x') \geq \frac{1}{2}E(x') + \frac{1}{2}E(x') \geq \frac{1}{4}\dot{u}_c^2(x') + \frac{1}{2}F(u) + \frac{1}{2}\max_{0 \leq u \leq a} |F(u)| \geq \frac{1}{4}\dot{u}_c^2(x') \geq \frac{1}{4c}\frac{d}{dx}E(x')$$

for all $x' \in [x, x_0)$. By Gronwall's inequality, we get $\sup_{x \leq x' \leq x_0} E(x') < +\infty$. Since $\frac{1}{2}\dot{u}_c^2 \leq E + |F(u_c)|$, it follows that $(u_c, \dot{u}_c)(x) < +\infty$ for all $x < x_0$, so the solution exists for all $x \in (-\infty, \infty)$ and by previous argument, we derived a contradiction. Thus, given c, we set

$$x_a := \inf\{x : u_c(x) = a\}, \quad x_{\frac{a}{2}} := \sup\{x : u_c(x) = \frac{a}{2}\},$$

which are well-defined.

– Claim 2 : $\lim_{c\to\infty} \dot{u}_c(x_a) = \infty$. Since
$$\frac{d\dot{u}_c}{du_c} = \frac{c\dot{u}_c + f(u_c)}{\dot{u}_c} \geq c$$
for $x < x_a$, we have
$$\dot{u}_c(x_a) = \dot{u}_c(x_{\frac{a}{2}}) + \int_{a/2}^{a} \frac{d\dot{u}_c}{du_c}(u)\, du \geq \frac{a}{2}c \to \infty$$
as $c \to \infty$. If we arrange by taking $c \gg 1$ so that
$$E(u_c, \dot{u}_c)(x_a) > \max_{0\leq u\leq 1} F(u) + 1,$$
then for all $x \geq x_a$,
$$\frac{1}{2}\dot{u}_c(x)^2 + F(u_c(x)) = E(u_c, \dot{u}_c)(x) \geq E(u_c, \dot{u}_c)(x_a) > \max_{0\leq u\leq 1} F(u) + 1.$$
In particular, $\frac{1}{2}\dot{u}_c(x)^2 > 1$. By an argument similar to Claim 1, we know $(u_c, \dot{u}_c)$ must cross $\{u = 1\}$, which implies $c \in B$. This completes the proof.

□

This example above is taken from [4, Chapter 2].

4.3. **Hamiltonian mechanics and completely integrable systems.**

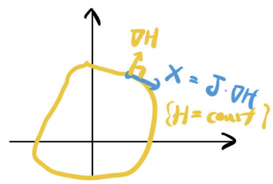
4.3.1. *Hamiltonian mechanics.* We start from classical mechanics. The Newton's equation is given by
$$\ddot{x} = F, \quad x \in \mathbb{R}^n, \quad F \in \mathbb{R}^n. \tag{4.12}$$
We assume F is of the special form (conservative force), that is, $F = -\nabla U(x)$ with the potential energy $U : \mathbb{R}^n \to \mathbb{R}$.

Definition 4.12. *The total energy or the Hamiltonian of the system is defined to be*
$$H(x, \dot{x}) = \frac{1}{2}|\dot{x}|^2 + U(x).$$

It is easy to see that H is conserved along any trajectory of (4.12). For instance, multiply (4.12) by $\dot{x}$ and use the product rule. But one may ask whether there is a way to write (4.12) so that the conservation of H is clear.

We write
$$\begin{pmatrix} x \\ \xi \end{pmatrix}' = X(x, \xi),$$
where $X : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a vector field on $\mathbb{R}^{2n}$, $\xi = x'$. Indeed,
$$X = \begin{pmatrix} \xi \\ -\nabla U(x) \end{pmatrix}, \quad H(x, \xi) = \frac{1}{2}|\xi|^2 + U(x).$$
Then H is conserved if and only if X is tangential to the level surface of H, which is equivalent to $X \cdot \nabla H = 0$.

Geometrically, if we have a Hamiltonian H and we can plot the level surface of H for $n = 1$ case as follows.



Indeed, $\nabla H = \begin{pmatrix} \nabla U(x) \\ \xi \end{pmatrix}$ and $X = J\nabla H$, where $J = \begin{pmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & 0 \end{pmatrix}$. When $n = 1$, J is the rotation matrix.

Even in the general case $n \geq 1$, $X \cdot \nabla H = J\nabla H \cdot \nabla H = 0$ since J is anti-symmetric. This suggests writing (4.12) as

$$\begin{pmatrix} x \\ \xi \end{pmatrix}' = J\nabla H(x, \xi) \tag{4.13}$$

on $\mathbb{R}^{2n}$, which is the Hamiltonian system associated with H. The advantage of this form is that the ODE is directly related to the conservative quantity H.

Note that for any $N \times N$ anti-symmetric matrix A,

$$z' = A\nabla H(z), \quad z \in \mathbb{R}^N \tag{4.14}$$

would also conserve H. But then it turns out that if A is invertible, this system is equivalent to the case $A = J$, which is stated in detail in the following proposition.

Proposition 4.13. *Suppose A is an $N \times N$ anti-symmetric, real-valued, invertible matrix, then necessarily $N = 2n$ and there exists a matrix $P : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ a linear invertible transformation such that $P^{-1}AP = J$.*

Proof. We do Gram-Schmidt process for $(Az) \cdot w$ to prove. This is a simple linear algebra exercise. □

This implies J, or equivalently the associated bilinear map

$$\omega_{\mathbb{R}^{2n}}(z, w) = (Jz) \cdot w$$

is the canonical form for (2), where $\omega_{\mathbb{R}^{2n}}$ is called the symplectic product on $\mathbb{R}^{2n}$. Write $z = (x, \xi)$, $w = (y, \eta)$, then $\omega_{\mathbb{R}^{2n}}(z, w) = \xi \cdot y - x \cdot \eta$. The discussion above gives the motivation of introducing such J and the symplectic product.

Though x, ξ here are tied with the physical meaning position and generalized momentum, it turns out that what is important in the ODE (4.13) is the matrix J and the Hamiltonian H. We can make use of this form to make a flexible change of variable as long as it preserves the bilinear map associated with J. To facilitate the discussion of possibly nonlinear change of coordinates and to allow for possible variation of ω at different points, we introduce a differential geometric generalization of the above. This leads to the notion of symplectic manifolds. From now on, set M to be an even dimensional differential manifold with $\dim M = 2n$.

Definition 4.14. *Let ω_p be a bilinear form on T_pM (covariant 2 tensor field) is a symplectic form if*

(1) ω_p is anti-symmetric, $\omega(X, Y) = -\omega(Y, X)$ for all $X, Y \in T_pM$;

(2) ω_p is non-degenerate in the sense that if for all $Y \in T_pM$, $\omega(X, Y) = 0$, then $X = 0$;
(3) $d\omega = 0$, that is, ω is closed.

Later we would see that the assumption that ω is a closed form is crucial for the conservative property.

Example 4.15. *The example we start with, $M = \mathbb{R}^n \times \mathbb{R}^n$ with the coordinate system $(x^1, \dots, x^n, \xi_1, \dots, \xi_n)$ is a symplectic manifold with the 2-form ω can be written as the differential form*

$$\omega = \sum_{j=1}^{n} d\xi_j \wedge dx^j.$$

If we identify $z \in \mathbb{R}^{2n}$ with $z^1\partial_{x^1} + \cdots + z^n\partial_{x^n} + z^{n+1}\partial_{\xi_1} + \cdots + z^{2n}\partial_{\xi_n}$, then ω can be identified with the standard one $\omega_{\mathbb{R}^{2n}}$ in the following sense

$$\omega_{\mathbb{R}^{2n}}(z, w) = \omega(z^1\partial_{x^1} + \cdots + z^n\partial_{x^n} + z^{n+1}\partial_{\xi_1} + \cdots + z^{2n}\partial_{\xi_n}, w^1\partial_{x^1} + \cdots + w^n\partial_{x^n} + w^{n+1}\partial_{\xi_1} + \cdots + w^{2n}\partial_{\xi_n}).$$

Note that $\omega = d\theta$ is exact, where $\theta = \sum_{j=1}^{n} \xi_j dx_j$, which is called the canonical 1-form.

Now we view this example from a geometric perspective.

Example 4.16. *The previous example is geometrically, the cotangent bundle $T^*\mathbb{R}^n$,. Let $(x^1, \dots, x^n)$ be the rectangular coordinates on $\mathbb{R}^n$. For all $q \in \mathbb{R}^n$, $dx^1, \dots, dx^n$ form a basis for $T^*\mathbb{R}^n$ and $\partial_{x^1}, \dots, \partial_{x^n}$ form a dual basis on $T\mathbb{R}^n$. Any $\eta \in T_q^*M$ can be written as*

$$\eta = \xi_1 dx^1 + \cdots + \xi_n dx^n,$$

which is equivalent to $\xi_j = \eta(\partial_{x^j})$. Due to this formula for ξ_j, we can easily check that $\theta = \sum_{j=1}^{n} \xi_j dx^j$ is invariant under the change of variables of x. Hence, this is a well-defined geometric object and the canonical form θ gives rise to the symplectic form $\omega = d\theta$ on $T^*\mathbb{R}^n$.

Definition 4.17. *For (M, ω_M), (N, ω_N), a symplectomorphism is defined as a diffeomorphism $\Phi : M \to N$ such that $\Phi^*\omega_N = \omega_M$. If such map exists, then M and N are said to be isomorphic.*

Suppose (Φ_t) is a 1-parameter family of symplectomorphism on $M \to M$.

Definition 4.18. *We define X to be the infinitesimal generator of each trajectory*

$$\frac{d}{dt}\Phi_t = (\Phi_t)_* X.$$

We say X is the symplectic vector field associated with (Φ_t).

Since for each t, $(\Phi_t)^*\omega = \omega$. By differentiating this, $\frac{d}{dt}(\Phi_t)^*\omega = 0$. Thanks to the definition for the Lie derivative in [14],

$$\frac{d}{dt}(\Phi_t)^*\omega = 0 \iff L_X\omega = 0$$

since $\frac{d}{dt}(\Phi_t)^*\omega = \Phi_t^*(L_X\omega)$. Moreover, by Cartan's formula and the closedness of ω, we have

$$L_X\omega = d\iota_X\omega + \iota_X d\omega = d(\iota_X\omega),$$

where ι_X is the interior product. Hence, X is a symplectic vector field if and only if $\iota_X\omega$ is closed. So the invertibility of the relation

$$X \mapsto \iota_X \omega = \eta$$

is equivalent to the non-degeneracy of ω as defined in Definition 4.14 (2).

Definition 4.19. *Given a function F on M, the Hamiltonian vector field X_F associated with F is defined by $\iota_{X_F}\omega = -dF$.*

Lemma 4.20. *X_F is a symplectic vector field.*

Proof. By the definition for X_F, $d(\iota_{X_F}\omega) = 0$. Moreover, thanks to Definition 4.14, ω is closed, we know $L_{X_F}\omega = 0$ by Cartan's formula. This implies X_F is a symplectic vector field by reversing the previous discussion above. $\square$

Definition 4.21. *Given two functions F, G on M, the Poisson bracket of F and G is defined to be*

$$\{F, G\} = \omega(X_F, X_G).$$

Equivalently,

$$\{F, G\} = -\iota_{X_F}\iota_{X_G}\omega = \iota_{X_F} dG = X_F(G) = -X_G(F).$$

Example 4.22. *On $T^*\mathbb{R}^n$ with coordinates (x, ξ), we have*

$$\{F, G\} = \sum_{j=1}^{n} \partial_{\xi_j} F \partial_{x^j} G - \partial_{x^j} F \partial_{\xi_j} G.$$

Proposition 4.23. *For all $F, G, H \in C^\infty(M)$, the following properties hold:*

(1) anti-symmetry: $\{F, G\} = -\{G, F\}$;
(2) Jacobi identity: $\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0$.

In particular, $\{F, F\} = 0$.

Corollary 4.24. *$\{\cdot, \cdot\}$ is conjugate to $[\cdot, \cdot]$ via $F \mapsto X_F$, that is, $X_{\{F,G\}} = [X_F, X_G]$.*

Proof. It suffices to prove that for all $H \in C^\infty(M)$, $X_{\{F,G\}}H = [X_F, X_G]H$. By applying the equivalent definition for Poisson bracket in Definition 4.21, one would find that it follows from the Jacobi identity. $\square$

Example 4.25. *For $M = \mathbb{R}^{2n}$, $X_H = J\nabla H$. So the Hamiltonian vector field is a geometric generalization of what we discussed at the beginnning of this subsection.*

4.3.2. *Canonical coordinates and Darboux theorem.*

Definition 4.26. *A coordinate system $(x^1, \cdots, x^n, \xi_1, \cdots, \xi_n)$ on $U \subset M$ is called a canonical coordinate system if*

$$\{x^j, x^k\} = 0, \quad \{\xi_j, \xi_k\} = 0, \quad \{\xi_j, x^k\} = \delta_j^k.$$

This is motivated by the properties of the coordinates (x, ξ) on $\mathbb{R}^{2n}$.

Lemma 4.27. *Alternatively, (x, ξ) is canonical if and only if in this coordinates, $\omega = \sum_{j=1}^n d\xi_j \wedge dx^j$.*

Theorem 4.28. *Suppose $A, B \subset \{1, \cdots, n\}$ are index sets. Given smooth functions $q^j, j \in A$ and $p_k, k \in B$ defined in a neighborhood U of $z \in M$ satisfying*

- *$(dq^j)_{j\in A} \cup \{dp_k\}_{k\in B}$ are linearly independent at z;*
- *$\{q^j, q^{j'}\} = 0$, $\{p_k, p_{k'}\} = 0$, $\{p_k, q^j\} = \delta_k^j$ for $j, j' \in A$ and $k, k' \in B$,*

which is a partial canonical coordinate system near z. Then there exists a canonical coordinate system (x, ξ) in a neighborhood of z such that $x^j = q^j$ for all $j \in A$ and $\xi_k = p_k$ for all $k \in B$.

Proof. See [5, Theorem 21.1.6]. □

Corollary 4.29 (Darboux's theorem)**.** *For any $z \in M$, there exists a local canonical coordinate system. Equivalently, for any $z \in M$, there exists a neighborhood U which is isomorphic to a neighborhood of $(\mathbb{R}^{2n}, \omega)$. The second version is a more geometric characterization in the sense of Definition 4.17.*

Proof. By choosing $A = B = \varnothing$, we complete the proof. □

Corollary 4.30. *Given any $H \in C^\infty(M)$, $dH|_z \neq 0$, there exists a local canonical coordinate (x, ξ) near z such that $H = \xi_1$*

Proof. By choosing $A = \varnothing$, $B = \{1\}$ and $p_1 = H$, we complete the proof. □

4.3.3. *Completely integrable Hamiltonian systems.* By a Hamiltonian system, we mean an ODE system associated with a Hamiltonian vector field.

Definition 4.31. *Given $H \in C^\infty(M)$. We say $F \in C^2(M)$ is an integral of H if $\{H, F\} = 0$ and $dF \neq 0$.*

Note that $\{H, F\} = X_H(F) = 0$ implies F is conserved under the flow of H. Having an integral helps us to solve Hamiltonian systems.

Example 4.32. *Let $H = \frac{1}{2}\xi^2 + U(x)$ in dimension 1. The Hamiltonian flow is given by*

$$\begin{cases} \dot{x} = \frac{\partial}{\partial \xi} H = \xi, \\ \dot{\xi} = -\frac{\partial}{\partial x} H = -U'(x) \end{cases}$$

can be solved by using the property of the conservation of H. If (x, ξ) solves the equation, then $H = \frac{1}{2}(\dot{x})^2 + U(x)$ is a constant, which means that we only need to solve

$$\dot{x} = \pm\sqrt{2(H - U(x))}$$

by separation of variables.

Definition 4.33. *We say $H \in C^\infty(M)$ is completely integrable if there exists $F_1, \cdots, F_n \in C^2(M)$ integrals of H such that*

(1) $dF_1, \cdots, dF_n$ are linearly independent everywhere;

(2) $\{H, F_j\} = 0$;
(3) $\{F_j, F_k\} = 0$ *for all* j, k,
where the third relation is called F_j*'s are in involution.*

Note that the notion of complete integrability is global. Recall that for all $H \in C^\infty(M)$, $z \in M$, there exists a local canonical coordinate system (x, ξ) near z such that $\xi_1 = H$. But $\{\xi_k, \xi_j\} = 0$, so all the three conditions are satisfied by $F_j = \xi_j$ on U. So this is a trivial notion locally, but it is useful globally.

Example 4.34 (A trivial example for complete integrability)**.** *Suppose* $H = H(\xi_1, \cdots, \xi_n)$ *only depends on the generalized momentum variables in the canonical coordinates in* $\mathbb{R}^{2n}$*. Then* $F_j = \xi_j$ *define* n *integrals of* H*.*

Example 4.35 (Harmonic Oscillator - A physics related example)**.** *Suppose* $H(x, \xi) = \frac{1}{2}\sum_{j=1}^n \left(\xi_j^2 + \omega_j^2 (x^j)^2\right)$*. By considering* $F_j(x, \xi) = \frac{1}{2}(\xi_j)^2 + \frac{1}{2}\omega_j^2(x^j)^2$*, we know that* H *is completely integrable in*

$$M = \{(x, \xi) \in \mathbb{R}^{2n} : F_1, \cdots, F_n \neq 0\},$$

where dF_j*'s are linearly independent.*

Example 4.36 (Kepler's problem)**.** *The general two-body problem can be reduced by considering the center of mass frame to the case* $H = \frac{1}{2}|\xi|^2 + \frac{1}{|x|}$ *with* $x, \xi \in \mathbb{R}^3$*, which corresponds to the model for a single particle associated to the potential* $1/r$*.*

This is a system with three degrees of freedom. One natural integral is H *itself. Basically, you need two more except for* H*, namely,* $|J|^2$ *and* J_3*, which comes from symmetries. Here,* J_j *is the angular momentum with respect to* x^j*-axis. Using these conserved quantities, Kepler's problem is explicitly solvable.*

Theorem 4.37 (Existence of action-angle variable)**.** *Let* H *be completely integrals with* $F_1, \ldots, F_n$ *as in Definition 4.33. Assume also that the zero set*

$$N = \{z \in M : F_1(z) = \cdots = F_n(z) = 0\} = F^{-1}(0)$$

is compact, where $F(z) = (F_1(z), \ldots, F_n(z))$*. Then*

- N *is an embedded torus* $\mathbb{T}^n$*;*
- *there exists an open neighborhood* $U(N)$ *of* N *on which there exists a canonical coordinate system* (θ, I)*, where* $\theta \in \mathbb{T}^n$*,* $I = I(F)$ *and* $H = H(I)$*.*

Such a canonical coordinate system (θ, I) *is called the action-angle variables.*

Proof. See [7, Section 3.1]. □

Remark 4.38. One can show that $\dot{I} = 0$ and $\dot{\theta} = \frac{\partial}{\partial I}H(I) = \frac{\partial}{\partial I}H(I_0) =: \Omega(I_0)$. Thus, $I = I_0$ and $\theta = \theta_0 + \Omega(I_0)\theta$.

Unfortunately, an N-body system with $N > 2$ is not completely integrable. So it is not explicitly solvable.

One direction to generalize this is to consider the perturbation

$$H = H_0(I) + \varepsilon H_1(I, \theta, \varepsilon).$$

A really remarkable theorem shows that most invariant tori survive under small enough perturbations, which is called the KAM theorem. Refer to [9] for a proof in detail.

Another direction to generalize this is the notion of Lagrange submanifolds. In fact, the zero set N in the preceding theorem is a Lagrangian manifold. And actually the preceding theorem can be generalized to the case for Lagrangian manifolds, which is called Weinstein Lagrangian tabular neighborhood theorem. See [1] for more details.

References

1. Ana Cannas da Silva, *Lectures on symplectic geometry*, Lecture Notes in Mathematics, vol. 1764, Springer-Verlag, Berlin, 2001. MR 1853077
2. Roland Donninger, Min Huang, Joachim Krieger, and Wilhelm Schlag, *Exotic blowup solutions for the u^5 focusing wave equation in $\mathbb{R}^3$*, Michigan Math. J. **63** (2014), no. 3, 451–501. MR 3255688
3. Brian C. Hall, *Quantum theory for mathematicians*, Graduate Texts in Mathematics, vol. 267, Springer, New York, 2013. MR 3112817
4. Stuart P. Hastings and J. Bryce McLeod, *Classical methods in ordinary differential equations*, Graduate Studies in Mathematics, vol. 129, American Mathematical Society, Providence, RI, 2012, With applications to boundary value problems. MR 2865597
5. Lars Hörmander, *The analysis of linear partial differential operators. III*, Classics in Mathematics, Springer, Berlin, 2007, Pseudo-differential operators, Reprint of the 1994 edition. MR 2304165
6. Joachim Krieger, Wilhelm Schlag, and Daniel Tataru, *Slow blow-up solutions for the $H^1(\mathbb{R}^3)$ critical focusing semilinear wave equation*, Duke Math. J. **147** (2009), no. 1, 1–53. MR 2494455
7. Jürgen Moser and Eduard J. Zehnder, *Notes on dynamical systems*, Courant Lecture Notes in Mathematics, vol. 12, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2005. MR 2189486
8. F. W. J. Olver, *Asymptotics and special functions*, Computer Science and Applied Mathematics, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1974. MR 0435697
9. Jürgen Pöschel, *A lecture on the classical KAM theorem*, Smooth ergodic theory and its applications (Seattle, WA, 1999), Proc. Sympos. Pure Math., vol. 69, Amer. Math. Soc., Providence, RI, 2001, pp. 707–732. MR 1858551
10. Thomas C. Sideris, *Ordinary differential equations and dynamical systems*, Atlantis Studies in Differential Equations, vol. 2, Atlantis Press, Paris, 2013. MR 3154864
11. Terrence Tao, *Notes 1: Zeroes, poles, and factorisation of meromorphic functions*, `https://terrytao.wordpress.com/2020/12/23/246b-notes-1-zeroes-poles-and-factorisation-of-meromorphic-functions/#more-12187`.
12. Gerald Teschl, *Ordinary differential equations and dynamical systems*, Graduate Studies in Mathematics, vol. 140, American Mathematical Society, Providence, RI, 2012. MR 2961944
13. ———, *Mathematical methods in quantum mechanics*, second ed., Graduate Studies in Mathematics, vol. 157, American Mathematical Society, Providence, RI, 2014, With applications to Schrödinger operators. MR 3243083
14. Maciej Zworski, *Basic symplectic geometry*, `https://math.berkeley.edu/~zworski/symple.pdf`.

Department of Mathematics, University of California, Berkeley, CA 94720, USA
Email address: math_tn@berkeley.edu