NOTES FOR PARTIAL DIFFERENTIAL EQUATIONS

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CONTENTS

1 Sobolev spaces in domains $\Omega(\subset \mathbb{R}^n)$ 1
1.1 Recap of Sobolev spaces in $\mathbb{R}^n$ 1
1.2 Sobolev spaces in domains Ω 1
1.3 Trace inequality 3
2 Additional topics of Sobolev spaces 5
2.1 A universal way to do extension 5
2.2 Extension operator if the boundary is not flat 5
2.3 $W_0^{k,p}(\Omega)$ - Approximation by $\mathcal{D}(\Omega)$ 6
2.4 Homogeneous Sobolev spaces and Poincaré inequality 6
2.5 Compact Sobolev embeddings 8
3 Compact Sobolev embeddings (Continued) 10
3.1 Non-examples of compact embeddings 10
3.2 Rellich-Kondrachov compactness theorem 10
4 Elliptic Equations 13
4.1 Ellipticity of a differential operator P 13
4.2 L^2 theory of the Laplace equation 14
5 Local properties of solutions to Laplace equation 17
5.1 A continuation from last time - L^p theory of Laplace equation 17
5.2 Local properties - Elliptic regularity 17
5.3 Local properties - Weak maximum principle for $-\Delta u \leq 0$ (subharmonic) 20
6 Strong maximum principle, mean value property 21
6.1 Weak maximum principle for general second order elliptic operator 21
6.2 Mean value property for $-\Delta$ implies strong maximum principle in any compact domain 21
6.3 Harmonic functions in unbounded domains : Liouville's theorem, general type maximum principle 22
6.4 Comparison principle 23
7 Elliptic boundary value problem 1 - Adjoint method, Lax-Milgram 24
7.1 Dirichlet boundary condition - uniqueness theory 24
7.2 Existence of solutions - L^2 theory, duality argument 26
7.3 Lax-Milgram theorem 27

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8 Elliptic boundary value problem 2 - Maximum principle approach, Variational method 29
8.1 Maximum principle approach to prove existence - Perron's method 29
8.2 Variational methods 32
9 Elliptic boundary value problem 3 - Variational method, Higher regularity 34
9.1 Variational methods (continued) 34
9.2 Elliptic regularity 36
9.3 More general boundary conditions 39
10 Elliptic boundary value problem 4 - Fredholm theory 40
10.1 Recap of functional analysis 40
10.2 Application of Fredholm theory - Solvability of second order elliptic operators 42
10.3 Application of Fredholm theory - Eigenvalues and eigenfunctions 43
11 Eigenvalues and eigenfunctions (continued) 44
11.1 Visualization of eigenvalues in the self-adjoint and non-self-adjoint case 44
11.2 Variational characterization of eigenvalues and eigenfunctions 45
12 Examples of eigenvalue problem 47
12.1 Basic examples with Dirichlet or periodic boundary condition 47
12.2 Laplacian on n-sphere 48
12.3 Hermite operator (Harmonic Oscillator) 50
13 Green functions and an intro to unique continuation 53
13.1 Green functions : an analogue of fundamental solutions 53
13.2 Introduction to potential theory 54
13.3 Introduction to unique continuation, Cauchy-Kowalevski theorem 55
14 Unique continuation property and Carleman estimates 57
14.1 Unique continuation without analyticity condition, Carleman estimates 57
14.2 Proof of Carleman estimate (Theorem 14.2) 58
15 Strong unique continuation principle 61
15.1 Carleman estimates implies strong unique continuation property 62
15.2 Proof of the Carleman estimate (Proposition 15.3) 62
16 Hopf's lemma, Harnack principle and regularity for Perron's method 65
16.1 Hopf's lemma for $-\Delta u = 0$ 65
16.2 Harnack's inequality 66
16.3 Application of Harnack's inequality - Improving regularity from L^∞ to C^ε 67
17 Nonlinear elliptic PDEs 69
17.1 Fully nonlinear equations 69
17.2 Semilinear equations with the favorable sign 69
17.3 Semilinear equations with the unfavorable sign 70
17.4 Mountain pass theorem 72
18 Second order parabolic equations - Introduction and L^2 theory 73
18.1 Recap of the properties of heat equation 73
18.2 L^2 theory for parabolic equations - energy estimates 74
18.3 L^2 theory for parabolic equations - existence 76

19 Second order parabolic equations - L^2 regularity · · · 79
19.1 Higher regularity - energy estimates and existence · · · 79
19.2 Parabolic regularity · · · 81
20 Second order parabolic equations - L^∞ theory · · · 83
20.1 Weak maximum principle (in parabolic cylinders) for subsolutions, comparison principle · · · 83
20.2 Strong maximum principle (in parabolic cylinders) and a mean value property for heat equation · · · 84
21 Second order parabolic equations - L^∞ theory · · · 87
21.1 Harnack's inequality - Statements · · · 87
21.2 Regularity of solutions for the heat equation · · · 87
21.3 Behavior at infinity in space - Maximum principle in $[0, T] \times \mathbb{R}^n$ for the heat equation 88
21.4 Unique continuation · · · 89
22 Hyperbolic equations · · · 91
22.1 Transport equation · · · 91
22.2 Recap of the wave equation · · · 93
23 Conserved quantities for wave equations - 1 · · · 95
23.1 Energy-momentum tensor · · · 95
23.2 Finite speed of propagation · · · 97
24 Conserved quantities for wave equations - 2 · · · 99
24.1 Deformation tensor of the wave equation · · · 99
24.2 Variable coefficient wave equations · · · 101
25 Wrap-up for second order hyperbolic equation and set-up for hyperbolic systems · · · 104
25.1 Local theory for second order variable coefficient hyperbolic equations · · · 104
25.2 Comments on boundary value problems · · · 104
25.3 Differential geometry setup · · · 105
25.4 Nonlinear wave equations · · · 105
25.5 First order hyperbolic systems · · · 106
26 First order hyperbolic systems - Finite speed of propagation, Quasilinear systems, Hadamard type well-posedness · · · 109
26.1 Finite speed of propagation for symmetric hyperbolic systems · · · 109
26.2 Examples of quasilinear hyperbolic system · · · 110
26.3 Hadamard type well-posedness · · · 111
27 Hadamard type well-posedness for first order quasilinear hyperbolic systems · · · 113
28 Burgers equation · · · 116
28.1 Solving Burgers equation using characteristics method · · · 116
28.2 Examples of intersecting characteristics, shocks, Rankine–Hugoniot conditions · · · 117
28.3 Riemann-Hilbert problem · · · 119

Sobolev spaces in domains $\Omega(\subset \mathbb{R}^n)$

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1.1 Recap of Sobolev spaces in $\mathbb{R}^n$

For any $s \in \mathbb{R}$, $\|u\|_{W^{s,p}} = \|\langle D\rangle^s u\|_{L^p}$. The general form of Sobolev embeddings what we ask for is

$$W^{k,p} \subset W^{j,\tau}, \quad k > j, q \geq p,$$

which is equivalent to

$$W^{k-j,p} \subset L^q.$$

Morally speaking, you trade derivatives for integrability. Obviously, we need some restriction relation between exponents as follows.

For $p \leq q \leq p^{*k}$, we have $W^{k,p} \subset L^q$, where p^{*k} is given by a scaling law $x \mapsto \lambda x$ which is asked by the Gagliardo-Norenberg-Sobolev inequality

$$\|u_\lambda\|_{L^q} \lesssim \|u_\lambda\|_{\dot{W}^{k,p}}.$$

Explicitly, $\frac{n}{p} - k = \frac{n}{q}$ with $q = p^{*k}$, where the left hand side is the scaling index.

If $\frac{n}{p} - k < 0$, then for $s > 0$ satisfying $\frac{n}{p} - k = \frac{n}{\infty} - s$, we have the Morrey's inequality

$$W^{k,p} \subset C^s,$$

where s is a non-integer and C^s is the Holder space. If s is an integer, then $W^{k,p} \subset C^{s-1,\gamma}$ for any $\gamma \in [0,1)$. For Holder spaces, $C^1 \subset Lip \subset C^\gamma$ for $\gamma \in (0,1)$.

1.2 Sobolev spaces in domains Ω

Before, we only discuss Sobolev spaces in $\mathbb{R}^n$ and we now extend it to domains in $\mathbb{R}^n$. For any open set $\Omega \subset \mathbb{R}^n$, $\partial\Omega$ is the boundary of Ω.

The simplest case is that Ω is half space, a more complicated one is fractal boundaries. The most common set-up is that the boundary is locally a graph.

Definition 1.1. We say $\partial\Omega$ is C^k for $k \geq 1$ if $\partial\Omega$ is a finite union of C^k graphs.

Definition 1.2. We define $W^{k,p}(\Omega)$ as a space of functions satisfying $u \in L^p(\Omega)$ and $\partial^\alpha u \in L^p$ for all $|\alpha| \leq k$.

Note that a priori, $u \in \mathcal{D}(\Omega)$ implies $\partial^\alpha u \in \mathcal{D}(\Omega)$.

Another natural definition for Sobolev spaces in Ω is

Definition 1.3. We say $u \in W^{k,p}(\Omega)$ if there exists $\bar{u} \in W^{k,p}(\mathbb{R}^n)$ such that $u = \bar{u}$ in Ω.

It turns out that these two definitions are equivalent. Suppose $\bar{u} \in W^{k,p}(\mathbb{R}^n)$, then it is obvious that $\|\partial^\alpha \bar{u}\|_{L^p(\Omega)} \leq \|\partial^\alpha u\|_{L^p(\mathbb{R}^n)}$ and hence the second definition implies the first one. For the other implication, it is equivalent to the following question "Given $u \in W^{k,p}(\Omega)$, can we find an extension $\bar{u} \in W^{k,p}(\mathbb{R}^n)$?"

First, we consider the simplest case when Ω is the half space H.

1.2.1 How to extend $C^k(H)$ functions to $C^k(\mathbb{R}^n)$ - motivation of trace inequality

For $u \in L^p(H)$, we make a trivial reflection to extend it to $\mathbb{R}^n$ by defining

$$\bar{u}(x) = \begin{cases} u(x), & x \in H, \\ u(x^*), & x \notin H. \end{cases}$$

If $u \in C^0(\bar{H})$, then this extension gives a function $\bar{u} \in C^0(\mathbb{R}^n)$. However, this extension does not map $C^1(\bar{H})$ to $C^1(\mathbb{R}^n)$. The strategy to invent a nice extension is that we can do a unbalanced reflection

$$x = (x', x_n) \mapsto (x', -kx_n) = x^*.$$

Moreover, an affine combination for different k's would not affect the mapping property $C^0(\bar{H}) \to C^0(\mathbb{R}^n)$. We define

$$\bar{u}(x) = \begin{cases} u(x), & x_n > 0 \\ c_1 u(x', -x_n) + c_2 u(x', -2x_n), & x_n < 0. \end{cases}$$

In order to match u and $\partial_n u$ at $x_n = 0$, we need $c_1 + c_2 = 1$ and $-c_1 - 2c_2 = 1$, respectively. More generally, we can find an extension $\bar{u} \in C^k(\mathbb{R}^n)$ by extending $C^k(\bar{H})$ with an affine combination of k coefficients, which is solvable since the coefficients is corresponded to a Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^k & (-2)^k & \cdots & (-k)^k \end{pmatrix}.$$

In order to make an extension of $W^{k,p}$ functions, we need to match at least the first $k-1$ derivatives so that they do not have any jumps at the boundary in the sense that when we differentiate any derivative of order $\le k-1$, it does not produce delta functions or other exotic distributions so that its derivative no longer belongs to L^p. For now, this idea is just heuristic, but we will see this is rigorous if we can make sense of the trace operator.

1.2.2 Trace inequality will suffice to show the equivalence of definitions of $W^{k,p}(\Omega)$

We want to know whether the trace of $W^{k,p}$ functions (restriction on the boundary) is well-defined. For any $u \in W^{k,p}(\mathbb{R}^n)$, we find a Cauchy sequence $u^n \in W^{k,p} \cap \mathcal{D}$. We can define $u|_{\partial H} = \lim u^n|_{\partial H}$ in L^p if we can prove the following trace inequality

$$\|u\|_{L^p(\partial H)} \le \|u\|_{W^{k,p}(H)}.$$

If this trace inequality holds, then we make a claim that if $u \in W^{k,p}(H)$, $v \in W^{k,p}(\mathbb{R}^n \setminus H)$ and $Tu = Tv$, where T is the trace operator, then the function $w^k := \begin{cases} u(x), & x \in H, \\ v(x), & x \notin H \end{cases}$ is combined to be a function in $W^{k,p}(\mathbb{R}^n)$. The reason why this is true is that we can prove that the divergence theorem holds for functions in $u \in W^{1,p}(H)$ and $\phi \in C_c^\infty(\overline{\Omega})$ since we can prove this by approximating u by smooth functions. Then we can justify $w \in W^{k,p}(\mathbb{R}^n)$

by showing that $w^l := \begin{cases} \partial^l u(x), & x \in H, \\ \partial^l v(x), & x \notin H \end{cases}$ is the weak derivative (by pairing with $\mathcal{D}(\mathbb{R}^n)$) of w by using divergence theorem in H and $\mathbb{R}^n \setminus H$ for u, v, respectively.

Therefore, to prove the equivalence of the two definitions of $W^{k,p}(\Omega)$ for Sobolev spaces in domains Ω in $\mathbb{R}^n$, it suffices to show the trace inequality holds.

1.3 Trace inequality

1.3.1 A revisit of a simple L^2 case of trace inequality

We start with a revisit of a homework problem from last semester's course. For an hyperplane $V \subset \mathbb{R}^n$, we proved

$$\|u\|_{L^2(V)}^2 \lesssim \|u\|_{L^2(\mathbb{R}^n)} \|\partial u\|_{L^2(\mathbb{R}^n)},$$

for all $u \in \mathcal{D}$. This holds for all $u \in H^1(\mathbb{R}^n)$ after noticing that we can use this inequality to extend the definition for trace from $\mathcal{D}$ to H^1. This problem is quite simple since it is just a one dimensional problem. Note that $\partial_x u^2 = 2uu_x$, where x is the normal direction of the hyperplane V. This implies

$$u^2(0) = 2 \int_{x<0} uu_x \leq 2\|u\|_{L^2(\mathbb{R}^n)} \|\partial u\|_{L^2(\mathbb{R}^n)},$$

which completes the proof. This problem is just an introduction to the trace inequality we would like to prove for now.

1.3.2 Nonsharp L^p case by introducing a cutoff (breaking the scaling)

The simplest trace inequality we want to prove is

$$\|u\|_{L^p(\partial H)} \leq \|u\|_{W^{1,p}(H)}.$$

The same strategy

$$u(x', 0) = \int_{-\infty}^0 \partial_n u(x', x_n)\, dx_n$$

does not work anymore since the integrand is not integrable unless $p = 1$. A trick is to replace $\partial_n u$ by $\partial_n(\chi u)$ with $\chi = \chi(x_n) \in C_c^\infty$ such that $\chi \equiv 1$ near $x_n = 0$.

We compute

$$u(x', 0) = \chi(0)u(x', 0) = \int_{-\infty}^0 \chi' u(x', x_n)\, dx_n + \int_{-\infty}^0 \chi \partial_n u(x', x_n)\, dx_n.$$

By triangle inequality in L^p and Minkowski inequality,

$$\begin{aligned} \|Tu\|_{L^p(\partial H)} \leq & \int_{-\infty}^0 |\chi'(x_n)| \|u(\cdot, x_n)\|_{L^p(H)}\, dx_n + \int_{-\infty}^0 |\chi| \cdot \|\partial_n u(\cdot, x_n)\|_{L^p}\, dx_n \\ \leq & \|\chi'\|_{L^{p'}} \|u\|_{L^p_{x_n} L^p_{x'}} + \|\chi\|_{L^{p'}} \|\partial_n u\|_{L^p_{x_n} L^p_{x'}}, \end{aligned}$$

where we use the support property and Holder's inequality in the last step. Therefore,

$$\|Tu\|_{L^p(\partial H)} \leq c_1 \|u\|_{L^p(H)} + c_2 \|\partial u\|_{L^p(H)}.$$

This inequality is obviously weaker than the one we obtain compared to the one in our homework. The reason is that we introduce an artificial scaling by introducing the fixed χ.

1.3.3 A better way to use the cutoff - introducing parameters to be optimized at the end

To make c_1, c_2 two moving targets, we replace χ by $\chi(\lambda x_n)$ with a scaling parameter λ. Then

$$c_2 = \left(\int \chi^{p'}(\lambda x_n)\, dx_n \right)^{1/p'} = c\lambda^{-1/p'}, \quad c_1 = \lambda c \lambda^{-1/p'} = c\lambda^{1/p}.$$

By minimizing the right hand side, we get

$$\|Tu\|_{L^p} \lesssim \|\partial u\|_{L^p}^{1/p} \|u\|_{L^p}^{1/p'}.$$

Recall that for $p = 1$, we do not need to introduce χ to make the integrand integrable. Hence, we can derive $\|Tu\|_{L^1} \leq \|\partial u\|_{L^1}$, which is the sharp case in the scaling sense.

Note that $\|u\|_{L^\infty(\partial H)} \leq \|u\|_{L^\infty(H)}$ fails but it holds for continuous functions

$$\|u\|_{C(\partial H)} \leq \|u\|_{C(H)}.$$

1.3.4 L^p results optimal in the sense of scaling

It is easy to observe that

$$T : W^{1,p} \to L^p$$

is not optimal unless $p = 1$ in the sense of scaling. From a scaling perspective, the optimal s for

$$T : W^{s,p}(\mathbb{R}^n) \to L^p(\partial H)$$

would be $s = 1/p$ since $\partial H = \mathbb{R}^{n-1}$, and $\frac{n}{p} - s = \frac{n-1}{p}$ gives $s = 1/p$. However, it does not hold for $s = 1/p$ but fortunately, it almost holds and the correct statements are

$$T : W^{1/p+s,p} \to W^{s,p}, \quad \forall s > 0.$$

Additional topics of Sobolev spaces

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2.1 A universal way to do extension

Last time, we noticed that the trace operator doe not work for L^p functions, but we proved that

$$T : W^{k,p}(H) \to W^{k-1,p}(\partial H).$$

In the proof, we need to match the first k derivatives on the boundary. As a result, it is enough to take a simple symmetry

$$\bar{u}(x) = \begin{cases} u(x), & x_n > 0 \\ u(x', -x_n), & x_n < 0 \end{cases}$$

to extend $W^{1,p}$ functions. On the other hand, to extend $W^{k,p}$, we take

$$\bar{u}(x) = \begin{cases} u(x), & x_n > 0 \\ \sum_{j=1}^{L} c_j u(x', -jx_n), & x_n < 0. \end{cases}$$

for $L \geq k$. In order to get a systematic/uniform $\bar{u}$, we change it to

$$\bar{u}(x) = \begin{cases} u(x), & x_n > 0 \\ \sum_{j=1}^{L} c_j u(x', -\alpha_j x_n), & x_n < 0. \end{cases}$$

with $\alpha_j \in (1, 2)$ and by taking the limit and viewing it as sort of the Riemann sum, we define

$$\bar{u}(x) = \begin{cases} u(x), & x_n > 0 \\ \int c(\alpha) u(x', -\alpha x_n)\, d\alpha, & x_n < 0, \end{cases}$$

where $c(\alpha)$ satisfies

$$\int c(\alpha)\, d\alpha = 1, \qquad \int c(\alpha)\alpha^j\, d\alpha = 0, \forall j.$$

If $c \in \mathcal{D}$, then $\widehat{c}$ is analytic at 0 and the relation tells us $\widehat{c}(0) = 1, \partial^k \widehat{c}(0) = i^k$, which implies the convergence of $\widehat{c}$ but it might not be Schwartz. However, we only require c has sufficient decay at infinity so that we can interchange the differentiation and the integral. And this would not be a problem since we can simply choose $\widehat{c}$ as a function which satisfies the required properties at 0 and compactly supported.

2.2 Extension operator if the boundary is not flat

In the setting of $\Omega = \{x_n > f(x')\}$, we make a simple argument to flatten the boundary by considering

$$(x', x_n) \mapsto (x', z), \quad z = x_n - f(x'),$$

which maps Ω to H. With a slight abuse of notation, we use x for x', y for x_n. In order to compute

$$\partial^\alpha u(x, z) = \partial^\alpha \left(u(x, y - f(x)) \right),$$

we need to require $f \in C^k$ (i.e. $\partial\Omega \in C^k$) to make the extension for $W^{k,p}$. However, we do not need all this boundary regularity.

Theorem 2.1 (Stein's extension theorem)**.** *Suppose $\partial\Omega$ is Lipschitz. Then there exists a universal extension operator, that is, for any $k \in \mathbb{Z}$, $1 \le p < \infty$, there exists a bounded extension operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$. If Ω is bounded, then Eu has compact support.*

We only sketch the proof for Stein's extension theorem. Our starting point is also

$$\bar{u}(x,y) = \begin{cases} u(x,y), & y > f(x), \\ \int c(\alpha) u(x, y + C\alpha(f(x) - y))\, d\alpha. \end{cases}$$

However, we run into the same problem by differentiating using the chain rule. The idea is to change $f(x) - y$ into $d((x,y), \partial\Omega)$ and regularize it in a little ball around (x,y).

2.3 $W_0^{k,p}(\Omega)$ - Approximation by $\mathcal{D}(\Omega)$

Recall that one can approximate any function in $W^{k,p}(\mathbb{R}^n)$ by functions $\mathcal{D}(\mathbb{R}^n)$ when $1 \le p < \infty$. Now we care about whether this would be true for domains in $\mathbb{R}^n$. Unfortunately, the closure of $\mathcal{D}(\Omega)$ in $W^{k,p}(\Omega)$ is not $W^{k,p}(\Omega)$ provided $k \ge 1$. One way to see this is that for $u \in W^{k,p}(\Omega)$, we have $T\partial^\alpha u \in L^p$, $|\alpha| \le k-1$. Suppose $u = \lim u_n$ with $u_n \in \mathcal{D}(\Omega)$. Then we need to require $\partial^\alpha u = 0$ on $\partial\Omega$ for $|\alpha| \le k-1$ since u_n vanishes near $\partial\Omega$.

Definition 2.2.

$$W_0^{k,p} := \text{ closure of } \mathcal{D}(\Omega) \text{ in } W^{k,p}.$$

Proposition 2.3. *When $\partial\Omega$ is C^k, $u \in W^{k,p}(\Omega)$ is in $W_0^{k,p}(\Omega)$ if and only if $\partial^\alpha u = 0$ in $\partial\Omega$ for $|\alpha| \le k-1$.*

For the proposition above, we proved one direction and the proof of the converse direction can be found on [7]. As a corollary of the proposition above, we have the following

Proposition 2.4. *$u \in W_0^{k,p}(\Omega)$ if and only if its extension by 0 is in $W^{k,p}(\mathbb{R}^n)$.*

2.4 Homogeneous Sobolev spaces and Poincaré inequality

In order to have a better scaling property, we introduced

$$\dot{W}^{k,p}(\mathbb{R}^n) = \{u \in \mathcal{D}' : \partial^\alpha u \in L^p, |\alpha| = k\}$$

with $\|u\|_{\dot{W}^{k,p}} = \sum_{|\alpha|=k} \|\partial^\alpha u\|_{L^p}$. Since $\|\cdot\|_{\dot{W}^{k,p}} = 0$ holds for any polynomial of order $\le k-1$, we need to consider the quotient space $\dot{W}^{k,p}/P_{\le k-1}$ with the same norm to make it a Banach space.

To rectify this, we take the closure of $\mathcal{D}(\mathbb{R}^n)$ in $\dot{W}^{k,p}$. Thanks to G-N-S inequality, we have $\|u\|_{L^q} \le \|u\|_{\dot{W}^{k,p}}$ with $\frac{n}{q} = \frac{n}{p} - k$ provided $\frac{n}{p} - k > 0$.

Proposition 2.5. *This closure is a Banach space provided $\frac{n}{p} - k > 0$ ($p < \frac{n}{k}$).*

When $\frac{n}{p} - k < 0$, then this approximation strategy does not eliminate all polynomials, but it reduces some. So we cannot use this strategy to define the homogeneous space. Instead, we can take the quotient space as a definition. One can also use the completion of $\mathcal{D}$ as a definition, which benefits the computations for nice functions. However, when we define it, we need to say if $u_n \to u$ in H^s, then u is not in $\mathscr{S}'$. Instead, u is defined module polynomials. In other words, we artificially make these with the same idea.

For negative exponents, we expect $\mathcal{D} \subset H^s$ for $s < 0$. Thus,

$$\dot{H}^s := \{u \in \mathscr{S}' : |\xi|^s \widehat{u}(\xi) \in L^2\}, \qquad -\frac{d}{2} < s < 0,$$

where the requirement $s > -\frac{d}{2}$ is to make $\mathcal{D} \subset \dot{H}^s$ since $|\xi|^s \in L^2_{loc}$. Moreover, one should note that this space is not a quotient space.

For any bounded domain Ω,

$$W^{1,p}(\Omega) := \{u \in L^p, \partial u \in L^p\}, \quad \dot{W}^{1,p}(\Omega) := \{u \in \mathcal{D}' : \partial u \in L^p\}$$

and hence $\dot{W}^{1,p}$ is a quotient space modulo constants. Since ∂u only determines u modulo constants, $\|u\|_{L^p} \leq \|\nabla u\|_{L^p}$ is obviously false. How should we modify this to make it true? One way to modify this is to subtract the average to eliminate the constants.

2.4.1 Proof by using an estimate obtained as a byproduct of Morrey's inequality

Theorem 2.6 (Poincaré inequality)**.** *For $u \in W^{1,p}$,*

$$\|u - \fint_\Omega u(x)\,dx\|_{L^p} \leq \|\nabla u\|_{L^p}.$$

Proof. In the proof of Morrey's inequality, we came up with an estimate

$$\fint_{B_r(x)} |u(x) - u(y)|\,dy \leq \int_{B_r(x)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy$$

for any ball B centered at x. Recall that we also use this inequality to prove the endpoint Sobolev embedding $W^{1,n} \subset BMO$.

First, we prove this for the case where Ω are balls. Set $\bar{u}_B := \fint_B u(y)\,dy$. Then

$$|u(x) - \bar{u}_{B_r(x)}| \lesssim \fint_{B_r(x)} |u(x) - u(y)|\,dy \leq \int_{B_r(x)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy.$$

Note that we can also prove this inequality for any $z \in B_r(x)$ instead of only the center of $B_r(x)$, namely x. We connect z with each point on ∂B and consider each layer $S(t) = t\partial B$ if we do a translation such that $z = 0$ when proving

$$\fint_B |u(x) - u(y)|\,dy \lesssim_B \int_B \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy,$$

where the constant depends on $\operatorname{diam} B$. Note that originally, the layers $S(t)$ is just concentric balls with different radius, this time it becomes eclipse when $t \neq 1$. However, we can still estimate $|u(y) - u(0)|$ by integrating over the line we draw for $y \in S(r) = rS(1)$. (Note that the distance from z to each point on the boundary is not the same now, but we still can estimate the distance by $\operatorname{diam} B$, so the same idea of proof still applies.) Therefore, by taking the L^p norm on both sides of

$$|u(z) - \bar{u}_B| \lesssim \int_B \frac{|\nabla u(y)|}{|z-y|^{n-1}}\,dy, \quad \forall z \in B,$$

the result follows from the Young's convolution inequality.

Note that the same strategy can be applied to any convex domain with smooth boundary. □

2.4.2 Proof by contradiction and size of the constant

One can refer to [7] for another proof by contradiction. This contradiction method can be applied to many estimates of this kind. Note that this requires the use of Rellich-Kondrachov compactness theorem, which we will discuss next lecture.

However, our preceding proof directly shows that the constant is of size $\operatorname{diam}\Omega$. For the sake of determining the size of the constant directly from the inequality itself, we still need the scaling argument. Since we have

$$\|u_\lambda(x) - \overline{(u_\lambda)_\Omega}\|_{L^p(\Omega)} \leq C(\Omega)\|\nabla u_\lambda\|_{L^p(\Omega)}, \quad \|u(x) - \bar{u}_\Omega\|_{L^p(\lambda\Omega)} \leq C(\lambda\Omega)\|\nabla u\|_{L^p(\lambda\Omega)}$$

for $u_\lambda = u(\lambda x)$, one can expand the first one to see that $\lambda C(\Omega) = C(\lambda\Omega)$, which means that the constant is of size $\operatorname{diam}\Omega$.

2.4.3 Other Poincaré-type inequalities

Other ways to fix our constants are as follows. We may ask "Is $\|u\|_{L^p(\Omega)} \leq \|\nabla u\|_{L^p(\Omega)}$ if $u(x_0) = 0$?" By Morrey's inequality, $W^{1,p} \subset C^0$ provided $p > n$, so our question will be meaningful and the answer is yes provided $p > n$. For simplicity, we assume $x \in \Omega$ can connect with x_0 using a simple line $x = x_0 + tw$ with $w \in \mathbb{S}^{n-1}$. Take $w = (0, x_n/|x_n|)$ as an example. We write by Cauchy-Schwarz inequality that

$$|u(x)|^p \leq \left(\int_{y_0}^{y_1} |\partial_{x_n} u(x', x_n)|\, dx_n\right)^p \lesssim \int_{y_0}^{y_1} |\partial_{x_n} u(x', x_n)|^p\, dx_n$$

and then integration both sides along x'-direction, which implies $\|u\|_{L^p(R)} \lesssim_\Omega \|Du\|_{L^p(R)}$, where $R \subset \Omega$ is a rectangle containing x_0 with sides parallel to axes. We can alter w to get infinitely many R_w, covering the support of u (if assuming $u \in \mathcal{D}$), then by choosing a finite sub-covering, we complete the proof. This is roughly the main idea of the proof.

Moreover, for Ω bounded, $\|u\|_{L^p(\Omega)} \leq \|\nabla u\|_{L^p(\Omega)}$ also holds for $u \in W_0^{1,p}(\Omega)$, $\forall 1 \leq p \leq \infty$. The proof can be found at [7, Chapter 5.6.1], which is basically an easy application of G-N-S inequality, extension theorem and the fact that $L^r(\Omega) \subset L^s(\Omega)$ if $r > s$.

One can also expect a Poincaré-type lemma for higher derivatives.

2.5 Compact Sobolev embeddings

Definition 2.7. Given two Banach spaces X, Y such that $X \subset Y$, that is, for any $u \in X$, $\|u\|_Y \leq \|u\|_X$. The embedding $X \subset Y$ is compact ($X \subset\subset Y$) if any bounded sequence in X has a compact subsequence in Y.

Note that $X \subset\subset X$ if and only if X is finite dimensional.

2.5.1 Nonexistence of compact Sobolev embeddings in $\mathbb{R}^n$ - translations

We would like to know when the Sobolev embeddings $W^{k,p} \subset L^q$ is compact. Unfortunately, there are no compact Sobolev embeddings for Sobolev spaces in $\mathbb{R}^n$. Our first enemy is the translations. Set $u_n(x) := u(x+n)$ for some $u \in \mathcal{D} \subset W^{k,p}(\mathbb{R}^n)$. Since $u_n \to 0$ in $\mathcal{D}'$, if u_n converges to some v in L^q, then $v = 0$. Also, $\|v\|_{L^q} = \lim \|u_n\|_{L^q} = \|u\|_{L^q}$, which implies that such a compact Sobolev embedding does not exist.

Compact Sobolev embeddings (Continued)

Date: January 24, 2023

For homogeneous spaces, the embeddings $\dot{W}^{k,p} \subset L^q$ only works for $\frac{n}{p} - k = \frac{n}{q}$, $q = p^{*_k}$, where p^{*_k} is called the sharp exponent. For inhomogeneous spaces, the embeddings $W^{k,p} \subset L^q$ works for all $p \leq q \leq p^{*_k}$ since we naturally have $W^{k,p} \subset L^p$.

We would like to know when the embedding $W^{k,p} \subset L^q$ is compact and look for enemies to compactness, coming from symmetries.

3.1 Non-examples of compact embeddings

3.1.1 Nonexistence of compact Sobolev embeddings in $\mathbb{R}^n$ - scalings

Last time we discussed the first kind of enemies - translations and reached the conclusion that no compact Sobolev embedding exists in $\mathbb{R}^n$. Now we discuss the second type of enemies - scalings. We start with the homogeneous case. For $u \in \dot{W}^{k,p}$, we define $u_\lambda(x) = \lambda^\sigma u(\lambda x)$ and we would like to find σ such that $\|u_\lambda\|_{\dot{W}^{k,p}} = \|u\|_{\dot{W}^{k,p}}$. It turns out that $\sigma = \frac{n}{p} - k$. Note that σ is positive so if we let $\lambda_n \to 0$, then the graph of $u^n := u_{\lambda_n}$ squashes. Note that

$$\|u^n\|_{\dot{W}^{k,p}} = \|u\|_{\dot{W}^{k,p}}, \quad \|u^n\|_{L^q} = \|u\|_{L^q}$$

On the other hand, $u^n \to 0$ uniformly if we choose $u \in \mathcal{D}$, so no compact Sobolev embedding exists if we can spread the graph out. In other words, we again show that no compact Sobolev embedding exists in $\mathbb{R}^n$.

3.1.2 Sharp Sobolev embeddings not compact - scalings as well

Another attempt is to let $\lambda \to \infty$ and then the graph squeezes. It is easy to see $u^n \to 0$ uniformly everywhere away from 0. Suppose $u^n \to \tilde{u}$ in L^q, we also know $u^n \to \tilde{u}$ in $\mathcal{D}'$. On the other hand, the uniformly convergence of u^n tells us $\operatorname{supp}\tilde{u} \subset \{0\}$ provided $u \in \mathcal{D}$. However, $\tilde{u} \in L^q$, then we know $\tilde{u} = 0$, which implies $\|u^n\|_{L^q} \to 0$, which is a contradiction.

To draw the conclusion here, the sharp homogeneous Sobolev embeddings are not compact even in bounded domains.

Now we consider the sharp inhomogeneous case. Instead of considering the $\dot{W}^{k,p}$ norm, we also consider the L^p norm. It turns out that

$$\|u_{\lambda_n}\|_{L^p} = \lambda_n^{-k}\|u\|_{L^p} \to 0$$

as $\lambda_n \to \infty$. However, $\|u_{\lambda_n}\|_{L^q} = \|u\|_{L^q}$ still holds, which is a contradiction. So the sharp (in)homogeneous Sobolev embeddings are not compact even in bounded domains. This strategy can also help us to show that the sharp Morrey's embeddings are not compact.

3.2 Rellich-Kondrachov compactness theorem

Theorem 3.1 (Rellich - Kondrachov). *Inhomogeneous non-sharp Sobolev embeddings in a bounded domain are compact.*

Proof. We only prove for the G-N-S case and one can find references for Morrey's case.

Step 1: Suppose $u^n \in W^{k,p}$, $\|u^n\|_{W^{k,p}} \leq 1$. We want a convergent subsequence in L^q for $p \leq q < p^{*_k}$. We know $\{u^n\}$ is bounded in $L^{p^{*_k}}$. To find a subsequence, we try to use the Arzela-Ascoli, which says that if $\{u^n\}$ is equi-bounded and equi-continuous, then there exists a uniform convergent subsequence.

Step 2: Extension to $\mathbb{R}^n$

First, we replace u^n by extensions, still denoted by u^n, such that $\|u^n\|_{W^{k,p}}$ is still bounded and supp$u^n \in B$ for a fixed large ball B. This saves us from worrying about the boundaries.

Step 3: $u^n_\varepsilon \to v_\varepsilon$ in L^∞ by passing to a subsequence (Arzela-Ascoli)

Second, to apply Arzela-Ascoli, we need to regularize the functions u_n. For $\varphi \in \mathcal{D}$, $\int \varphi = 1$, we denote $\varphi_\varepsilon = \varepsilon^{-n}\varphi(x/\varepsilon)$. Set $u^n_\varepsilon := u^n * \varphi_\varepsilon$.

Then we compute

$$\|u^n_\varepsilon\|_{L^\infty} \le \|u^n\|_{L^p}\|\varphi_\varepsilon\|_{L^{p'}} \lesssim \varepsilon^{d/p'-d}\|u^n\|_{L^p},$$

which is uniformly bounded in n. Similarly,

$$\|\partial u^n_\varepsilon\|_{L^\infty} \le \|u^n\|_{L^p}\|\partial\varphi_\varepsilon\|_{L^{p'}} \lesssim \varepsilon^{d/p'-d-1}\|u^n\|_{L^p},$$

which implies $\{u^n_\varepsilon\}_{n=1}^\infty$ is equi-continuous. Therefore, by Arzela-Ascoli theorem, there exists a uniformly convergent subsequence of $\{u^n_\varepsilon\}_{n=1}^\infty$ such that $u^n_\varepsilon \to v_\varepsilon$ in L^∞ or more specifically, C^0. In particular, it converges in L^q since the domain is bounded.

Step 4: $u^n_\varepsilon \to u^n$ in L^q (uniformly in n) by interpolation

For the convergence $u^n_\varepsilon \to u^n$, we have nonuniform convergence in $W^{k,p}$. To look for uniform convergence, we need to look at L^q. For $p \le q < p^{*_k}$, since

$$\|u^n_\varepsilon - u^n\|_{L^q} \le \|u^n_\varepsilon - u^n\|_{L^1}^h \|u^n_\varepsilon - u^n\|_{L^{p^{*_k}}}^{1-h},$$

it is enough to show that $u^n_\varepsilon - u^n \to 0$ in L^1 uniformly in n since $\|u^n_\varepsilon - u^n\|_{L^{p^{*_k}}}$ is uniformly bounded. (This is where we use the fact that $q \neq p^{*k}$.) We compute

$$\begin{aligned} u^n_\varepsilon(x) - u^n(x) &= \int (u^n(y) - u^n(x))\,\varphi_\varepsilon(x-y)\,dy = \int (u^n(x+\varepsilon z) - u^n(x))\,\varphi(z)\,dz \\ &= \int_B \int_0^1 \partial u^n(x + h\varepsilon z)\cdot \varepsilon z\,dh\,\varphi(z)\,dz, \end{aligned}$$

where supp$\varphi \subset B$. Then

$$\|u^n_\varepsilon - u^n\|_{L^1} \le \varepsilon \int\int_B\int_0^1 |\partial u^n(x+h\varepsilon z)||z|\,dh\,\varphi(z)\,dz\,dx \lesssim \varepsilon\|\partial u^n\|_{L^1} \le \varepsilon\|\partial u^n\|_{L^p} \to 0$$

uniformly. Therefore, $\|u^n_\varepsilon - u^n\|_{L^q} = O(\varepsilon)$ uniformly in n.

Step 5: A diagonal argument to extract a subsequence for u^n to converge in L^q

For any $\delta > 0$, we choose ε small enough such that $\|u^n_\varepsilon - u^n\|_{L^q} \le \delta$ for all n thanks to Step 4. Then it suffices to show that for this fixed ε, we can extract a subsequence $u^{n_k}_\varepsilon$ such that it converges in L^q.

This is already done in Step 3. That means, there exists N such that $\forall j,k > N$, $\|u^{n_j}_\varepsilon - u^{n_k}_\varepsilon\|_{L^q} \le \delta$ we have

$$\|u^{n_j} - u^{n_k}\|_{L^q} \le \|u^{n_j}_\varepsilon - u^{n_k}_\varepsilon\|_{L^q} + \|u^{n_j}_\varepsilon - u^{n_j}\|_{L^q} + \|u^{n_k} - u^{n_k}_\varepsilon\|_{L^q} \le 2\delta + \|u^{n_j}_\varepsilon - u^{n_k}_\varepsilon\|_{L^q} \le 3\delta.$$

Moreover, one should notice that the subsequence now depends on the ε we choose. So we need to employ a diagonal argument to conclude. To be precise, for $\varepsilon = 2^{-k}$, we need to ensure that the subsequence for $\varepsilon = 2^{-k}$ is a subsequence of the one we chose for $\varepsilon = 2^{-(k-1)}$.

Then we just need to extract the things in the diagonal (corresponding to ε and n) and this completes the proof. $\square$

Remark 3.2. In particular, $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$ for all $1 < p < \infty$ thanks to the theorem above.

Elliptic Equations

Date: January 26, 2023

For a differential operator $P(D) = \sum c_\alpha D^\alpha$ with constant coefficients, $D_j = \frac{1}{i}\partial_j$, its symbol is given by $P(\xi) = \sum c_\alpha \xi^\alpha$ and we have $P(\xi)\widehat{u} = \widehat{f}$.

4.1 Ellipticity of a differential operator P

A naive definition of ellipticity is that $P(\xi)$ has no real zeros.

If we consider the Fourier transform $\widehat{u}(\xi)$, then those ξ's in a bounded set correspond to the smooth component of our functions because the inverse Fourier transform of an L^1 function with compact support is analytic thanks to the Paley-Wiener theorem. On the other hand, the behavior of $\xi \to \infty$ tells us the singularities of u. So one should give priority to the singularities of the symbol u to make it have better behavior.

This motivates a better definition for ellipticity : We say P is elliptic if $P(\xi)$ does not have real zeroes for large ξ.

Furthermore, when ξ is large, the highest order terms dominate in $P(\xi)$, so we define the principal symbol as follows :

Definition 4.1. For a symbol of order m, $P(\xi) = \sum_{|\alpha|\leq m} c_\alpha \xi^\alpha$, its principal symbol is defined as

$$P_m(\xi) := \sum_{|\alpha|=m} c_\alpha \xi^\alpha,$$

where the subscript m is just for principal symbol not denoting the order.

A even better definition for ellipticity is as follows and this would be our primary notion of ellipticity.

Definition 4.2. We say P is elliptic if $P_m(\xi) \neq 0$ for $\xi \neq 0$. Equivalently,

$$|P_m(\xi)| \geq c|\xi|^m \tag{4.1}$$

for some constant c.

Remark 4.3. The criterion (4.1) also works for variable coefficients $P_m(x,\xi) = \sum c_m(x)\xi^\alpha$.

When the order $m = 1$, you will notice that there is no choice for a real symbol to be elliptic unless in 1 dimension. If we focus on 2 dimensions, the operator

$$\bar{\partial} = \partial_1 + i\partial_2$$

with complex principal symbol takes a fundamental role in complex analysis. The idea to consider in 2 dimensions is that the real part and imaginary part of the operator vanish on a codimension 1 set, respectively, so the operator vanishes on a codimension 2 set. We require this codimension 2 set to be the origin, so we consider 2 dimensional case.

If $m = 2$, we are allowed to have real-valued polynomials in higher dimensions which do not vanish anywhere except the origin. Our main object is

$$-\Delta = \sum_j D_j^2.$$

Solving $\bar{\partial} u = f$ is equivalent to solve the Laplace equation $\Delta u = (\partial_1 - i\partial_2) f$. To study holomorphic functions ($\bar{\partial} u = 0$), it suffices to study harmonic functions (real part/ imaginary part of holomorphic functions).

This observation also helps us to know a bunch of harmonic polynomials by considering $\mathrm{Re} z^k$ for any k.

There are also elliptic operators of second order with complex symbols, which is not of so much interest since some theories for real symbols may not apply to complex ones.

For $m = 3$, the polynomial is odd and if you look for real symbols, then you end up with some restrictions on dimension. When dimension $d \geq 3$, one need to consider complex symbols again.

For $m = 4$, an important real operator is the bilaplacian Δ^2, which comes from the plate equation.

Henceforth, we consider second order elliptic equations with real principal symbols. One example is the Laplacian equation $P = -\Delta$ and the variable coefficient analogue is

$$P = -\sum_{i,j=1}^{n} a^{ij}(x) \partial_i \partial_j.$$

If a^{ij} is constant, we can assume (a_{ij}) is symmetric and hence it can be diagonalized, so it suffices to consider $-\Delta$ for all constant coefficients operator. However, when $a_{ij}(x)$ are not constants, we can only diagonalize it at one point, so the second case is of great interest. Moreover, we can put lower order terms without affecting the principal symbol,

$$P = a^{ij} \partial_i \partial_j + b^j \partial_j + c.$$

For nonlinear elliptic equations, we may consider semilinear equations

$$-\Delta u = f(u), \qquad -\Delta u = f(u, \nabla u),$$

and quasilinear equations

$$-a^{ij}(u) \partial_i \partial_j u = f(u, \nabla u)$$

and fully nonlinear equations

$$F(u, Du, D^2 u) = 0.$$

4.2 L^2 theory of the Laplace equation

We start from the inhomogeneous Laplace equation

$$-\Delta u = f$$

to study the existence and uniqueness and continuous dependence in the nonlinear term.

4.2.1 Revisit of fundamental solutions of Laplacian equation

Let us recall what we know for a Laplace equation from last semester. The fundamental solution in dimension 2 is $K(x) = \frac{1}{2\pi} \ln |x|$ and $K(x) = c_n |x|^{2-d}$ in dimension $d \geq 3$. Suppose

$$-\Delta u = f$$

in $\mathbb{R}^n$, then $u = K * f$ is a solution. If $\mathrm{supp} f$ is compact, then u makes sense as a distribution even if f is merely a distribution in $\mathcal{E}'$ since $K \in L^1_{loc} \subset \mathcal{D}'$ for $d \geq 3$. Obviously, this is not the unique solution in distributions. Suppose $-\Delta\tilde{u} = f$, then $\xi^2\widehat{(u-\tilde{u})} = 0$. Hence, $\widehat{(u-\tilde{u})} = \sum c_\alpha \delta^{(\alpha)}$ with a finite sum. Therefore, $u - \tilde{u}$ is a polynomial. In other words, the solution is unique modulo polynomials. More precisely, it is unique up to harmonic polynomials.

Now let's start with the L^2 theory for the Laplace equations.

4.2.2 Existence of $u \in \dot{H}^2$ such that $-\Delta u = f \in L^2$ for dimension $d > 4$

For $f \in L^2$, we study $-\Delta u = f$. By taking the Fourier transform, we get

$$|\xi|^2\widehat{u} = \widehat{f}.$$

If $d > 4$, then $\frac{1}{|\xi|^2} \in L^2_{loc}$ and hence

$$\widehat{u} = \frac{1}{|\xi|^2}\widehat{f}$$

is well-defined and therefore u is in $\dot{H}^2$. An intuition for the reason why we get the dimension restriction is that for $0 \le s < \frac{d}{2}$, $\dot{H}^s$ consists of functions instead of pure distributions. This can be seen from the fact that $\mathcal{F}(1/|\xi|^s) = c/|x|^{d-s} \in L^1 + L^2$ when $s \in [0, d/2)$.

4.2.3 Cutting off low frequencies to discuss lower dimensions

Now we discuss lower dimensions.

$$\widehat{u} = \frac{1}{|\xi|^2}\widehat{f}$$

also works if $\widehat{f}$ vanishes near 0. For arbitrary f, we cut off the low frequencies by

$$\widehat{f_\varepsilon}(\xi) := \widehat{f} \cdot \chi_{|\xi| \ge \varepsilon}.$$

Then we can solve $-\Delta u_\varepsilon = f_\varepsilon$ and find a solution $u_\varepsilon \in \dot{H}^2(\mathbb{R}^n)$ by defining $\widehat{u_\varepsilon} := \frac{1}{|\xi|^2}\widehat{f_\varepsilon}$. In fact, $u_\varepsilon \in H^2(\mathbb{R}^n)$ since $\|(1+|\xi|^2)\widehat{u}_\varepsilon(\xi)\|_{L^2} \le (1+\varepsilon^{-2})\|f\|_{L^2}$, so we can apply the Poincaré inequality in the following discussion.

4.2.4 Arguments for dimension $d = 4$ in detail

Let us keep things simple first and we discuss in dimension 4. We want to look for a compact subsequence. Since $\dot{H}^1 \subset L^4$, we have $\dot{H}^2 \subset \dot{W}^{1,4} \subset BMO$. This already tells us that when we look at the sequence u_ε, we would like to define a convergence modulo constants. In general, it does not have convergent subsequence even if $u_\varepsilon = u_0 + c_\varepsilon$ for some fixed u_0 with a blow-up constant c_ε. Therefore, to define a convergence, we should take away this constant first.

We want to choose c_ε such that $u_\varepsilon - c_\varepsilon$ converges. Set $c_\varepsilon := \bar{u}_\varepsilon|_B$. By Poincaré's inequality with $p = 4$, we get

$$\|u_\varepsilon - \bar{u}_{\varepsilon,B}\|_{L^4(B)} \le \|u_\varepsilon\|_{\dot{W}^{1,4}} \le \|u_\varepsilon\|_{\dot{H}^2} = \|f_\varepsilon\|_{L^2} \le \|f\|_{L^2}.$$

and hence in particular, $u_\varepsilon - \bar{u}_{\varepsilon,B}$ is uniformly bounded in $L^2(B)$. Moreover, $u_\varepsilon - \bar{u}_{\varepsilon,B}$ is uniformly bounded in $H^2(B)$. Therefore, by the compactness theorem, there exists some $u \in L^2$ such that $\{u_\varepsilon - \bar{u}_{\varepsilon,B}\} \to u$ in $L^2(B)$ by passing to a subsequence. (Now we consider this for B_R of any radius R.) Furthermore, the convergence also holds in $\mathcal{D}'$ since $\mathcal{D}$ is dense in L^2. Moreover, since $\{u_\varepsilon - \bar{u}_{\varepsilon,B_R}\}$ is uniformly bounded in $H^2(B_R)$, it implies a weak-* convergence to u in $H^2(B_R)$ by the Banach-Alaoglu theorem. (This is a problem in last semester's HW.) Therefore, we know $u \in H^2(B_R)$. So as we can see, we reached a stronger conclusion than we expected.

4.2.5 Sketch for dimension $d = 2$ or 3

In dimension 3, we have $\dot{H}^1 \subset L^6$ and $\dot{H}^2 \subset \dot{W}^{1,6}$, which still allows us to apply Poincaré's inequality and use the same argument to conclude.

However, when we come to 2 dimensions, $\dot{H}^1 \subset BMO$, which do not allow us to have Poincaré's inequality anymore. We should modify the normalization $u_\varepsilon - \bar{u}_{\varepsilon,B_R}$ so that we also eliminate the contribution of first order polynomials since we lose control of first order derivatives as well.

Another generalization is that for $f \in \dot{H}^{-1}$, we can reach the conclusion that $u \in \dot{H}^1$.

Local properties of solutions to Laplace equation

Date: January 31, 2023

Last time, we studied $-\Delta u = f \in L^2$ and proved $f \in \dot{H}^s$ would imply $u \in \dot{H}^{s+2}$ for $s < \frac{n}{2}$ (This requirement make sure that $\frac{1}{|\xi|^2}\widehat{f}(\xi)$ defines a tempered distribution.)

5.1 A continuation from last time - L^p theory of Laplace equation

Theorem 5.1. *If $-\Delta u = f \in L^p$ for some $u \in \mathscr{S}'$, then $u \in \dot{W}^{2,p}$ for $1 < p < \infty$.*

Proof. Though there is no Plancherel theorem for $p \neq 2$, but we still can use Fourier transform to proceed. Since $-\widehat{\Delta u} = |\xi|^2 \widehat{u} = \widehat{f}$ and $-\widehat{\partial_i \partial_j u} = \xi_i \xi_j \widehat{u}$, it suffices to show the symbol $m(\xi) = \mathcal{F}^{-1}\left(\frac{\xi_i \xi_j}{|\xi|^2}\right)$ maps L^p to L^p. Obviously, $m(\xi) \in L^\infty$, so $m(\xi) : L^2 \to L^2$. To prove it $L^p \to L^p$, we examine the Hormander-Mikhlin condition

$$|\partial_\xi^\alpha m(\xi)| \leq C_\alpha |\xi|^{-|\alpha|},$$

which is a sufficient condition for $m : L^p \to L^p$, $1 < p < \infty$. Note that $m(\xi) = \frac{\xi_i}{|\xi|}\frac{\xi_j}{|\xi|}$, where each factor is called the Riesz transform $R_j := \frac{D_j}{|D|}$. $\square$

Remark 5.2. Remember that $p = 1, \infty$ are disallowed, $-\Delta u = f \in L^\infty$ does not imply $u \in C^{1,1}$. The idea from Daniel is that this can be seen from the fundamental solutions and you need to choose some nice f such that $K * f$ is an integration with some cancellation when varying between positive part and negative part. You only need to do the estimate instead of computing the integral explicitly.

Note that $u(x,y) = (x^2 - y^2)\ln(x^2 + y^2)$ in $\mathbb{R}^2$ satisfies $\Delta u = 8\frac{x^2 - y^2}{x^2 + y^2} \in L^\infty$ while

$$\partial_x^2 u = 2\ln(x^2 + y^2) + 2\frac{3x^4 + 6x^2y^2 - y^4}{(x^2 + y^2)^2}$$

is unbounded. So it means that $u \notin \dot{W}^{2,\infty}$ with $\Delta u \in L^\infty$. From a discussion with Ryan, the idea behind is as follows. We would like to find in $\mathbb{R}^2$ to construct a counterexample. And $\ln|x| \in BMO$ but not bounded, so if we want $\partial_x \partial_y u = \ln\sqrt{x^2 + y^2}$, then by integrating in polar coordinates, we get $u = \frac{1}{2}(x^2 + y^2)\ln(x^2 + y^2)$, which is still bad. However, everything works well if we change the plus sign to a minus sign.

In [8, Section 2.2], the authors introduce $u(x,y) = (x^2 - y^2)\ln|\ln(x^2 + y^2)|$ as an example such that Δu is continuous but $u \notin C^{1,1}$. Moreover, $u(x,y) = \ln\ln\frac{1}{x^2 + y^2}$ is given as an example for $\Delta u \in L^1_{loc}$ with $\partial_x^2 u \notin L^1_{loc}$.

5.2 Local properties - Elliptic regularity

We talk about elliptic regularity first. If a solution is given, we ask how regular it is.

5.2.1 Starting with $u \in L^2_{loc}$(or H^s_{loc}, $s < 0$) and $f \in L^2_{loc}$ gives $u \in H^2_{loc}$

Suppose

$$-\Delta u = f, \quad f \in L^2_{loc}, u \in L^2_{loc}.$$

Replace $u \in L^2_{loc}$ by $v = \chi u$, $\chi \in \mathcal{D}$ such that $\chi \equiv 1$ near x_0. Of course, $v \in L^2$. An application of Lebniz rule gives

$$-\Delta v = -\Delta(\chi u) = -\chi\Delta u - u\Delta\chi - 2\nabla u\nabla\chi = \chi f - u\Delta\chi - 2\nabla\chi\nabla u,$$

where the first two terms are in L^2 and the last term is in H^{-1}. Hence, $-\Delta v = g \in H^{-1}$. Moreover, $v - \Delta v = g_1 \in H^{-1}$. By doing a Fourier transform, we get

$$\widehat{v} = \widehat{g}_1 \cdot \frac{1}{1 + |\xi|^2}.$$

Hence, $v \in H^1$. In other words, $u \in H^1_{loc}$ and we get an increase by one for the order of legitimate derivatives.

If we do the argument again, then we will end up with $g \in L^2$, $g_1 \in L^2$ and $v \in H^2$. This proves the following theorem.

Theorem 5.3 (Elliptic regularity). *For $-\Delta u = f$, $u \in L^2_{loc}$, $f \in L^2_{loc}$, then $u \in H^2_{loc}$.*

Corollary 5.4. *The theorem also works if we start with $u \in H^s_{loc}$ and $f \in L^2_{loc}$ with $s < 0$.*

Proof. We just iterate the same proof and note that $u \in H^s_{loc}$ implies $g \in H^{s-1}$ and then $v \in H^{s+1}$ as long as $s < 2$. By performing an iteration, we can also conclude the same result as above. □

5.2.2 Starting with $u \in \mathcal{D}'$ and $f \in H^s_{loc}$ (resp. C^∞) gives $u \in H^{s+2}_{loc}$ (resp. C^∞)

Now we try to make another extension. First, we consider the following problem. Suppose $u \in \mathcal{D}'$, can we conclude that $u \in H^{s_0}_{loc}$ for some s_0? This is a subtle question. The answer is no since

$$\delta_0 + \delta_1' + \delta_2'' + \cdots + \delta_n^{(n)} + \cdots$$

will be an enemy. However, $\chi u \in H^{s_0}$ for some s_0 since for any distribution $u \in \mathcal{D}'$, we can write $\chi u \in \mathcal{E}'$ in the form of

$$\chi u = \sum_{|\alpha| \le k} \partial^\alpha g$$

for some $g \in C^0$ and an integer k which depends on χu. This is the so-called structure theorem for compactly supported distributions, which can be found in [10, Corollary 5.4.1]. So, $\chi u \in H^{s_0}$ for some s_0 since $g \in L^2_{loc}$ but keep in mind that we cannot conclude that $u \in H^{s_0}$. The reason why $u \notin H^{s_0}_{loc}$ is that for different cut-off functions χ, we will get a different exponent s_0. This can be easily read from the counterexample above. (This argument above has nothing to do with the Laplacian so far.)

Finally, the discussion above tells us we can extend the elliptic regularity theorem to :

Theorem 5.5. *Suppose $u \in \mathcal{D}'$ and $f \in H^s_{loc}$ with $-\Delta u = f$, then $u \in H^{s+2}_{loc}$.*

Proof. For any $\chi \in \mathcal{D}$, we consider $\chi u \in H^{s_0}$ and do the argument as in the proof of the basic version of elliptic regularity. This leads us to $\chi u \in H^{s+2}$. By choosing different χ, we finally conclude $u \in H^{s+2}_{loc}$. □

Corollary 5.6. *Suppose $-\Delta u = f$ with $f \in C^\infty$, then we have $u \in C^\infty$.*

Proof. This follows from the Sobolev embedding theorem and the generalized elliptic regularity theorem above. $\square$

In particular, for any harmonic function u in D, we know $u \in \mathcal{E}(D)$. Here the notation $v \in \mathcal{E}(D)$ means that $v \in C^\infty(D)$ and v can have any kind of growth near the boundary ∂D.

5.2.3 Harmonic functions are analytic

Theorem 5.7. *If u is harmonic, then u is analytic.*

Proof. We use the fundamental solutions to prove that u is analytic in B provided that u is harmonic in $2B$.

Step 1 : Localiztion

Choose $\chi \in \mathcal{D}$ such that $\chi \equiv 1$ in B and $\chi \equiv 0$ in $(2B)^c$. Set $v = \chi u$. For $-\Delta v = f := -u\Delta\chi - 2\nabla u \cdot \nabla \chi$, we have supp$f \in (2B) \setminus B$. From the preceding corollary, we know $v \in \mathcal{D}$.

Step 2 : Show $v = K * f$ (Nontrivial!)

The second step is to show that we can solve this using the fundamental solution. We claim $v = K * f$. Note that the solution is not unique, so this claim is not trivial. Since $f \in \mathcal{D}$ and $K \in L^1_{loc}$ except for dimension 2, so $K * f \in \mathcal{E}$. (For dimension 2, one can use complex analysis to prove the theorem directly. So we can assume without loss of generality that $n \geq 3$.)

Moreover, the criterion for a smooth functions to be a tempered distribution is that the function has at most polynomial growth near infinity. (This is trivial to check by definition.) For $|x|$ sufficiently large and for $y \in \text{supp}f$, $|x - y|$ is away from 0 so everything is nice and it follows directly that

$$\left| \int \frac{1}{|x-y|^{n-2}} f(y)\, dy \right| \leq 1/|x|, \quad d(x, \text{supp} f) \gg 1$$

thanks to the fact that f is compactly supported.

Hence, $v = K * f$ on grounds the uniqueness of smooth solutions with decay at ∞. This uniqueness is easy to see since $K * f \in \mathscr{S}', v \in \mathcal{D}$, then $-\Delta(K * f - v) = 0$ implies $K * f - v$ are polynomials. Moreover, since $(K * f - v)(x) \to 0$ as $x \to \infty$, we know $v = K * f$.

Step 3 : Prove analyticity by noting that K is analytic away from 0

Since for $x \in B$, when $|x - y| > 0$, K is analytic, so $v(x) = \int_{2B\setminus B} f(y) K(x-y)\, dy$ is an integral of a family of functions which is analytic in x and hence the integral is analytic. $\square$

Remark 5.8. We compare $P_0 = -\Delta$, $P_+ = 1 - \Delta$ and $P_- = -1 - \Delta$ to see the effect of lower order terms.

- Global solvability :
  - P_0 : use homogeneous Sobolev spaces;
  - P_+ : use inhomegeneous Sobolev spaces;
  - P_- : no more naive solvability, but it can be studied using more advanced theory called Sommerfeld radiation condition.
- Elliptic regularity : Nothing changes.

5.3 Local properties - Weak maximum principle for $-\Delta u \leq 0$ (subharmonic)

Now we study the solution to the Laplacian equation to see how big it is pointwisely. Suppose $\Omega \subset \mathbb{R}^n$ and $-\Delta u = 0$ in Ω. It follows from the discussion above that $u \in C^\infty(\Omega)$. We need to make a stronger assumption to initiate the discussion.

Suppose $u \in C(\overline{\Omega})$, we look at the maximum points for u. Set x_0 to be a local maximum for u. Then $\nabla u(x_0) = 0$. Since $u(x) = u(x_0) + \nabla u(x_0)(x - x_0) + \frac{1}{2}\nabla^2 u(x_0)(x - x_0) \cdot (x - x_0) + o(|x - x_0|^3)$, we know $\nabla^2 u(x_0) \leq 0$. Thus $\Delta u(x_0) = \mathrm{tr}\nabla^2 u(x_0) \leq 0$.

Now if we change the hypothesis to $-\Delta u < 0$, then we know that there is no maximum points inside and $\max_\Omega u = \max_{\partial\Omega} u$. However, we can prove the same result by only assuming $-\Delta u \leq 0$.

Theorem 5.9 (Weak Maximum Principle)**.** *Suppose $u \in C(\overline{\Omega})$ and $-\Delta u \leq 0$ in a compact domain Ω, then $\max_\Omega u = \max_{\partial\Omega} u$.*

Proof. Set $u_\varepsilon := u + \varepsilon|x|^2$, and then $-\Delta u_\varepsilon < 0$. Since $|x|^2$ is bounded in a compact domain, $u_\varepsilon \to u$ uniformly in $\overline{\Omega}$. By passing to the limit in u_ε, we know $\max_\Omega u = \max_{\partial\Omega} u$. More precisely, we consider

$$\max_\Omega u \leq \max_\Omega u_\varepsilon = \max_{\partial\Omega} u_\varepsilon \leq \max_{\partial\Omega} u + \varepsilon \max_{\partial\Omega} |x|^2$$

and let $\varepsilon \to 0$. $\square$

Strong maximum principle, mean value property

Date: February 2, 2023

We say $u \in C(\overline{\Omega})$ such that $-\Delta u \le 0$ is a subharmonic functions. By replacing u by $-u$, an easy corollary for superharmonic functions ($-\Delta u \ge 0$), we have $\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$. Later, we discuss why we need Ω to be bounded.

6.1 Weak maximum principle for general second order elliptic operator

6.1.1 Variable coefficients with $c = 0$

The same proof of the maximum principle applied for variable coefficient problems

$$-a^{ij}(x)\partial_i\partial_j u + b^i \partial_i u \le 0,$$

where the real matrix $(a^{ij}(x))$ is symmetric and positive definite. At a maximum point, $Hess(u)(x_0) = \nabla^2 u(x_0) \le 0$, which implies $a^{ij}(x_0)\partial_i\partial_j u(x_0) \le 0$, since it is the trace of the product of a positive definite matrix (a^{ij}) and a semi-negative definite matrix $\nabla^2 u$, which is semi-negative definite. One can see this from diagonalizing (a^{ij}) using an orthogonal matrix and hence we know the sum is non-positive.

6.1.2 Only non-negative maximum taken into account when $c \ge 0$

If one want to apply the same method for

$$-a^{ij}\partial_i\partial_j u + b^i \partial_i u + cu \le 0,$$

where we need extra conditions that $c \ge 0$ and we only consider positive maximum so that the last term is positive at our maximum.

6.2 Mean value property for $-\Delta$ implies strong maximum principle in any compact domain

Now we discuss the strong maximum principle. Given a connected compact domain Ω. Suppose $-\Delta u \le 0$ and $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$, the maximum can be achieved inside if and only if it is constant.

The proof needs the mean value property.

6.2.1 Mean value property for harmonic or subharmonic functions

Suppose $-\Delta u = 0$, then

$$u(x_0) = \fint_{B(x_0,r)} u\,dx, \qquad u(x_0) = \fint_{\partial B(x_0,r)} u\,dx.$$

For subharmonic functions, we have

$$u(x_0) \le \fint_{B(x_0,r)} u\,dx, \qquad u(x_0) \le \fint_{\partial B(x_0,r)} u\,dx.$$

We prove the case for subharmonic functions by applying Green's theorem. Set $\Omega = B(0,r) = B$. The Green's theorem gives

$$\int_B -\Delta u \cdot v \, dx = \int_B u \cdot (-\Delta v) \, dx + \int_{\partial B} \frac{\partial u}{\partial \nu} v - u \frac{\partial v}{\partial \nu} \, d\sigma.$$

To prove the mean value property, we need a good choice for v. So we need

- $-\Delta v = \delta_0$,
- $v \geq 0$ in B,
- $v|_{\partial B} = 0$, we don't want the appearance of $\frac{\partial u}{\partial \nu} v$ in the boundary terms.

Set

$$v(x) = K(|x|) - K(r), \qquad K(x) = c_n |x|^{2-n}$$

and then $\frac{\partial}{\partial \nu} K(|x|) = c(n,r)$. We write

$$u(0) = \int_B -\Delta u \cdot v + c \int_{\partial B} u \, d\sigma.$$

By setting $u \equiv 1$, we know $c = |\partial B|$ and hence

$$u(0) \leq \fint_{\partial B} u \, d\sigma.$$

By a linear change of coordinates, you can prove the mean value property for $a^{ij}\partial_i \partial_j$ in some eclipse since the linear change of a ball is a eclipse.

6.2.2 Strong maximum principle for $-\Delta$ in compact domains Ω

Now we prove the strong maximum principle. Set $M = \max_{\overline{\Omega}} u = \max_{\partial \Omega} u$. Suppose $u(x_0) = \max u$. We choose r such that $B(x_0, r) \subset \Omega$. By applying mean value property,

$$M = u(x_0) \leq \fint_B u \, dx \leq M.$$

This implies that $u \equiv M$ in $B(x_0, r)$. Then

$$D = \{x \in \Omega : u(x) = M\}$$

is open and closed. By connectedness, $D = \Omega$, that is, $u \equiv M$ in Ω.

6.3 Harmonic functions in unbounded domains : Liouville's theorem, general type maximum principle

6.3.1 Two different proofs for Liouville's theorem

For harmonic functions, $\Delta u = 0$ implies $\Delta \partial_j u = 0$. Hence, we can also apply the mean value property to derivatives as

$$\partial_j u(x_0) = \fint_{B(x_0,r)} \partial_j u \, dx = \frac{1}{|B|} \int_{\partial B(x_0,r)} \nu_j u \, dx.$$

We use this fact to prove the Liouville's theorem.

Theorem 6.1. *Any bounded harmonic function in $\mathbb{R}^n$ is constant.*

Proof. Suppose $|u| \leq M$ and we write

$$|\partial u(x_0)| \leq M \frac{1}{|B|} \cdot |\partial B| = \frac{Mn}{r} \to 0$$

as $r \to \infty$.

Alternative proof : Alternative proof is to use the distribution theory that we discussed before. Since bounded functions are tempered distributions, we can apply Fourier transform and conclude that u is a polynomial. Then it is constant. □

6.3.2 A brief discussion on maximum principles in unbounded domains

What happens to the maximum principle if the domain is not compact. We consider a simplest unbounded domain - half space first. Fix $H = \{x_n \geq 0\}$. Then $u(x) = x_n$ does not satisfy

$$\max_H u \leq \max_{\partial H} u.$$

At least you need to impose some decay condition. One can refer to [3, Theorem 2.7, Theorem 2.9] for a different kind of maximum principle involving unbounded domains.

A second example is an angle and you can change it by using $z = z^\alpha$. For the stripe in two dimensions, we refer to the Phragmén–Lindelöf theorem in complex analysis for the condition such that the maximum principle holds in a strip.

6.4 Comparison principle

One can view the maximum principle as a comparison of (super, sub) solutions with constant functions. Note that the reason why we care about the constant functions is that they are solutions.

Corollary 6.2 (Comparison principle)**.** *Suppose* $-\Delta u \leq 0$ *and* $-\Delta v \geq 0$*. If* $u \leq v$ *on* $\partial\Omega$*, then* $u \leq v$ *in* Ω*.*

Proof. Note that $-\Delta(u - v) \leq 0$, we know

$$\max_{\overline{\Omega}}(u - v) \leq \max_{\partial\Omega}(u - v) \leq 0.$$

□

In fact, the comparison principle also holds for

$$-a^{ij}\partial_i\partial_j + b^j\partial_j + c,$$

where $c \geq 0$. We just need to modify the proof with the weak maximum principle for this general second order elliptic operator today. Note that this is in line with the different behavior of operators $P = -\Delta + 1$ and $P = -\Delta - 1$ we discussed in last lecture.

Such comparison principle can be also extended to lots of other equations.

Elliptic boundary value problem 1 - Adjoint method, Lax-Milgram

Date: February 7, 2023

Let $\Omega \subset \mathbb{R}^n$, we discuss $-\Delta u = f$ in Ω. The Dirichlet boundary condition is $u = 0$ on boundary and the Neumann boundary condition is $\frac{\partial u}{\partial \nu} = 0$, which means that you cut off all the heat transmissions through the boundary.

7.1 Dirichlet boundary condition - uniqueness theory

If instead we look at

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ u = g \text{ on } \partial\Omega. \end{cases} \tag{7.1}$$

If one can extend g to the interior of the region to obtain a v in Ω such that $v|_{\partial\Omega} = g$. Then, by letting $u = v + w$, we get

$$\begin{cases} -\Delta w = f + \Delta v \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega. \end{cases}$$

A good topic to discuss is the existence and uniqueness. There are two ways to develop the uniqueness theory. The first one is based on L^2 estimates. The other one is based on the maximum principle.

7.1.1 Uniqueness for $-\Delta$ - Performing an estimate by the source term

Suppose u solves (7.1) with $f = g = 0$, then we compute

$$0 = \int_\Omega (-\Delta u) u \, dx = \int_\Omega |\nabla u|^2 \, dx - \int_{\partial\Omega} u \cdot \frac{\partial u}{\partial \nu} \, dx = \int_\Omega |\nabla u|^2 \, dx,$$

which implies $\nabla u = 0$. Since $u|_{\partial\Omega} = 0$, we know $u \equiv 0$. Note that this argument works for Neumann boundary condition as well.

To make this computation rigorous, one needs $\nabla u \in L^2$. Since $u|_{\partial\Omega} = 0$, we require $u \in H_0^1(\Omega)$. Then one could regularize u by $u_\varepsilon \in \mathcal{D}(\Omega)$, which satisfies $u_\varepsilon \to u$ in H_0^1. If we redo the computation, we end up with

$$\int_\Omega |\nabla u_\varepsilon|^2 \, dx = \int_\Omega f_\varepsilon u_\varepsilon \, dx,$$

which implies

$$\|\nabla u_\varepsilon\|_{L^2}^2 \leq \|f_\varepsilon\|_{L^2} \|u_\varepsilon\|_{L^2}.$$

The it follows from Poincaré's inequality that

$$\|\nabla u_\varepsilon\|_{L^2} \leq \|f_\varepsilon\|_{L^2}.$$

However, we can achieve a better result than this one. In the Cauchy Schwartz above, instead of using L^2 for both, we do

$$\|\nabla u_\varepsilon\|_{L^2}^2 \leq \|f_\varepsilon\|_{H^{-1}} \|u_\varepsilon\|_{H_0^1},$$

which implies

$$\|\nabla u_\varepsilon\|_{L^2} \leq \|f_\varepsilon\|_{H^{-1}}. \tag{7.2}$$

Thus we end up with a smaller norm on the right hand side. Note that $-\Delta : H_0^1 \to H^{-1}$, so here we get the right space.

Since $u_\varepsilon \to u$ in H_0^1, we know $f_\varepsilon \to f$ in H^{-1} and hence we can pass to the limit in (7.2), which proves the uniqueness. When we become familiar with these, one usually omits the justification by smooth functions. We summarize as the following proposition.

Proposition 7.1. *If $u \in H_0^1(\Omega)$, $-\Delta u = f \in H^{-1}(\Omega)$, then*

$$\|u\|_{H_0^1(\Omega)} \leq \|f\|_{H^{-1}(\Omega)}.$$

In particular, the estimates of the solution by the source term implies uniqueness.

7.1.2 Uniqueness for $-\Delta$ - Applying maximum principle

Another way to prove uniqueness is to use maximum principle. Suppose u satisfies (7.1) with $f = g = 0$. By maximum principle, $\max_\Omega u = \max_{\partial\Omega} u = 0$. If we consider $-u$ instead of u, by minimum principle, we get $\min_\Omega u = 0$. Hence, $u = 0$. This argument works if $u \in C(\overline{\Omega})$.

If u satisfies (7.1) with $g = 0$. Suppose $f \in L^\infty$ with $|f| \leq M$. We penalize u by $v := u + \frac{M}{2n}|x - x_0|^2$, then $-\Delta v \leq 0$, that is, v is sub-harmonic. Since $v \leq \max_{\partial\Omega} v \leq MR^2/2n$ provided $\Omega \subset B(x_0, R)$. Then we get

$$\max_\Omega u \leq \|f\|_{L^\infty} R^2.$$

Compared to the L^2-based estimates we obtained above, this is imperfect in the following sense : if one takes $u \in L^\infty$ and then takes two derivatives, then it will not end up being in L^∞, which means that the spaces for both sides of the inequality do not perfectly match with each other. A more subtle observation is that the term R^2 on the right hand side match with the two derivatives we need to take, so it is in some sense scaling invariant.

7.1.3 Uniqueness for variable coefficient operators in divergence form - Energy estimates

Before we go further, we replace $-\Delta$ by a variable coefficient operator

$$\begin{cases} -\partial_i a^{ij}(x)\partial_j u = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Since $a^{ij}(x)\partial_j$ is a vector field, we call $-\partial_i a^{ij}(x)\partial_j$ a divergence form of operators. Our convention of the order of computations for $-\partial_i a^{ij}(x)\partial_j$ is $-\partial_i\left(a^{ij}(x)\partial_j\right)$. If we try to reproduce the arguments above, then we note that the only requirements for a_{ij} are $a_{ij} \in L^\infty$ and (a_{ij}) is uniformly elliptic (strictly positive definite). Note that we do not need further regularity on a_{ij} since the first thing we do in L^2-based estimates is to integration by parts.

Remark 7.2. However, note that one cannot put first order terms into the equation if we want to use this method to perform L^2-based estimates.

7.1.4 Uniqueness for variable coefficient operators in non-divergence form - Maximum principle

Since the weak maximum principle also applies to operators of the form

$$\begin{cases} -a^{ij}\partial_i\partial_j u + b^i\partial_i u + cu = f, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Note that $-a^{ij}\partial_i\partial_j$ is in non-divergence form. The requirements for the maximum principle to hold is to require $a \in C, b \in C, c \in C$ with $c \geq 0$. So, we don't need $a \in C^1$ compared to the divergence form above.

Remark 7.3. In Nash-Moser theory, you still can treat in two different ways by using divergence form and L^2-based estimates or non-divergence form and maximum principles.

7.2 Existence of solutions - L^2 theory, duality argument

One kind of arguments, which manifests the idea in numerics, is to consider the difference quotients $\Delta_h u := \frac{u(x+h)-u(x)}{h}$ to produce an approximate solution. By refining your grid, one may get an exact solution. The arguments in [7] is largely based on this notion.

We introduce a duality argument, which can be adapted to many other problems. By replacing $-\Delta$ by P,

$$\int -\Delta u \cdot v\, dx = \int u \cdot (-\Delta v)\, dx$$

is written as

$$\int_\Omega Pu \cdot v\, dx = \int_\Omega u \cdot P^* v\, dx.$$

For $P = a^{ij}\partial_i\partial_j$, we have $P^* = -\partial_i\partial_j a^{ij}$ and $P = -\partial_i a^{ij}\partial_j$, we have $P^* = -\partial_i a^{ij}\partial_j$. If $P = P^*$, we say P is self-adjoint (as a bounded operator $H_0^1 \to H^{-1}$).

The adjoint equation becomes

$$\begin{cases} P^* v = g, \\ v|_{\partial\Omega} = 0 \end{cases}$$

while the original equation is

$$\begin{cases} Pu = f, \\ u|_{\partial\Omega} = 0. \end{cases}$$

The duality relation can be written as

$$\int u \cdot g = \int v \cdot f.$$

Note that the energy estimates for the adjoint equation $\|v\|_{H_0^1} \leq \|g\|_{H^{-1}}$ implies

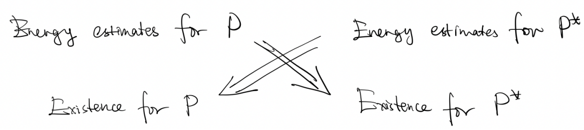
$$\left|\int ug\right| \leq \|v\|_{H_0^1}\|f\|_{H^{-1}} \leq \|g\|_{H^{-1}}\|f\|_{H^{-1}}.$$

If we know the quantity $\int u \cdot g$ for all $g \in H^{-1}$, then this uniquely determines $u \in H_0^1$. For now, $g \in Ran(-\Delta)$, a subspace of H^{-1}. Moreover, we have

$$u : Ran(-\Delta) \to \mathbb{R}, \qquad g \mapsto \int ug$$

and thus we are allowed to use Hahn-Banach theorem to extend it to $u : H^{-1} \to \mathbb{R}$. However, keep in mind that Hahn-Banach theorem does not give uniqueness. We only obtain uniqueness if $Ran(-\Delta)$ is dense in H^{-1}.

The discussion above can be summarized into the following diagram.



Energy estimates for P^* implies the existence for P and the energy estimates for P also implies the existence for P^*. Also, we have the other way around.

To prove the non-existence of a solution to P, one can prove by claiming that there are no nice energy estimates for P^*. This is the idea of [25] in showing that not all differential operators are locally solvable.

7.3 Lax-Milgram theorem

If you have a Riemannian manifold, there is a corresponding operator called the Laplace-Beltrami operator. It is self-adjoint with respect to the Riemannian metric. In other words, whether the operator is self-adjoint with respect to a weight function, if we write everything in local coordinates. Now the question is whether we can prove estimates even if our operator is not self-adjoint.

For an operator in divergence form, a formal computation leads to

$$\int Pu \cdot u \, dx = \int (-\partial_i a^{ij} \partial_j + b^j \partial_j + c) u \cdot u \, dx = \int a^{ij} \partial_j u \partial_i u + b^j u \partial_j u + cu^2 \, dx,$$

which result in a corresponding quadratic form $B : H_0^1 \times H_0^1 \to \mathbb{R}$ given by

$$B(u,v) := \int a^{ij} \partial_j u \partial_i v + b^j v \partial_j u + cuv \, dx.$$

The important property is whether we have

$$B(u,u) \geq c \|u\|_{H_0^1}^2,$$

which is called the coercivity property. A good feature is that the coercivity property would be more robust when we try to introduce some nice weights. However, an operator is self-adjoint or not usually depends on specific choice of weights.

The key ingredients of Lax-Milgram theorem are the coercivity assumption plus a duality argument, where the duality stuff is hidden in the proof of Lax-Milgram thoerem. Though [7] does not call the argument before as a duality argument, the idea is essentially the same.

By combining these ingredients, one can prove solvability by Lax-Milgram theorem.

Remark 7.4. An advantage of Lax-Milgram theorem is that it can handle second order elliptic PDEs with first order and zeroth order terms.

For the sake of completeness, we record the results in [7, Chapter 6.2] here.

Theorem 7.5 (Lax-Milgram Theorem). *Assume that $B : H \times H \to \mathbb{R}$ is a bilinear mapping, for which there exist constants $\alpha, \beta > 0$ such that*

$$|B(u,v)| \leq \alpha \|u\| \|v\|, \quad |B(u,u)| \geq \beta \|u\|^2.$$

Finally, let $f : H \to \mathbb{R}$ be a bounded linear functional on H. Then there exists a unique element $u \in H$ such that $B(u,v) = f(v)$ for all $v \in H$.

Proof. The proof is sketched as follows.

Step 1 : Application of Riesz representation theorem to obtain a unique element $w \in H$ for each u such that $B(u,v) = \langle w, v \rangle$ and denote $w = Au$.
Step 2 : A is linear and bounded : a direct estimate by definition.
Step 3 : A is one-to-one and $R(A)$ is closed : it suffices to prove $\beta \|u\|^2 \leq B(u,u) \leq \langle Au, u \rangle \leq \|Au\| \|u\|$.
Step 4 : $R(A) = H$: proof by contradiction.
Step 5 : Riesz representation theorem applied again to obtain $w \in H$ such that $\langle w, v \rangle = f(v)$ for all v and hence by Step 4, $Au = w$ for some u.
Step 6 : Uniqueness of u.

□

Remark 7.6. If B is symmetric, then one can show $B(\cdot, \cdot)$ is an inner product on H. With symmetry of B, one can apply Riesz representation theorem to prove this directly. So the importance of Lax-Milgram theorem is that it can apply to PDEs with first order terms as we can see in a second.

We discuss the specific $B : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$, where

$$B(u,v) = \int_\Omega a^{ij} \partial_i u \partial_j u + b^i \partial_i u v + cuv \, dx.$$

Then one can check the following energy estimates :

$$|B(u,v)| \leq \alpha \|u\|_{H_0^1} \|v\|_{H_0^1}, \quad \beta \|u\|_{H_0^1} \leq \gamma \|u\|_{L^2}^2 + |B(u,u)|$$

for some $\alpha, \beta > 0$, $\gamma \geq 0$. One can note from the computation that if $b^i = 0$, then one can take $\gamma = 0$. In general, we obtain a unique weak solution $u \in H_0^1(\Omega)$ for the boundary value problem

$$\begin{cases} Lu + \mu u = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

for any $f \in L^2$.

Elliptic boundary value problem 2 - Maximum principle approach, Variational method

Date: February 9, 2023

Last time, we set $P = -\partial_i a^{ij}\partial_j + b^i\partial_i + c$ with a corresponding bilinear form

$$B(u,v) = \int_\Omega Pu \cdot v = \int u \cdot P^* v = \int a^{ij}\partial_i v \partial_j u + b^i \partial_i uv + cuv.$$

If it satisfies the coercivity condition $B(u,u) \geq C\|u\|^2_{H^1_0}$, then we have the L^2 solvability.

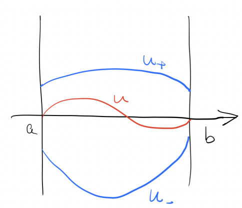
An important case is when P is self-adjoint. Technically, one can add a weight $\omega(x) > 0$ to make an operator P self-adjoint. In other words, we change the measure from dx to $\omega(x)\,dx$.

8.1 Maximum principle approach to prove existence - Perron's method

For maximum principle, one can think of it as a L^∞ theory if one wants to compare it with the L^2 theory. The setting is to write the operator in the non-divergence form : $P = -a^{ij}\partial_i\partial_j + b^j\partial_j + c$. We assume that maximum principle holds for P. (Assume a_{ij} symmetric, positive definite and $c \geq 0$.) In the following, we would like to find solutions to

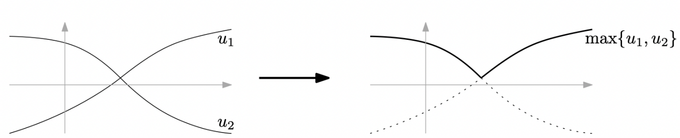
$$\begin{cases} Pu = f \\ u = 0 \end{cases}.$$

We can find a sub-solution u_- and a super-solution u_+. A priori, a sub-solution (in the setting of a boundary value problem) means that $Pu_- \leq f$ and $u_-|_{\partial\Omega} \leq 0$. Similarly, a super-solution means that $Pu_+ \geq f$ and $u_+|_{\partial\Omega} \geq 0$.



As shown in the graph (for a one-dimensional case), if we make the function convex enough, then by the positivity of (a_{ij}), the Hessian will dominate the negativeness and we get $Pu_- \leq f$. Then it follows from the maximum principle that if a solution u exists, then it is in between any sub-solution and any super-solution. Moreover, it is unique by the maximum principle provided the existence. Therefore, it would be both the largest sub-solution and the smallest super-solution.

Suppose u^1_-, u^2_- are two sub-solutions, then we claim that $\max\{u^1_-, u^2_-\}$ is still a sub-solution. We provide a heuristic argument. If we consider the one-dimensional case, then the only place we need to take care of is the intersection point x_0 of the two sub-solutions.



At x_0 where where u_-^1, u_-^2 meet with each other, since the coefficients of $\partial_i\partial_j$ is negative, we know that $(P\max\{u_-^1, u_-^2\})(x_0) = ... + c\delta_{x_0}$ with $c < 0$. Therefore, heuristically speaking, $(P\max\{u_-^1, u_-^2\})$ can be sufficiently negative at x_0. By this philosophy, the max of all sub-solutions should be still a sub-solution. We also want to show it is a super-solution. The idea here is to increase a sub-solution a little bit would still be a sub-solution.

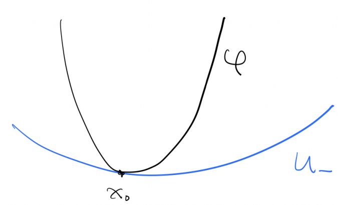
By implementing this, one actually needs the solution to be at least C^2. To get rid of the regularity issue, we reinvent the notion of sub-solutions and super-solutions in a way that resembles the proof of the maximum principle.

Definition 8.1. We say u_- (resp. u_+) $\in C(\Omega)$ is a viscosity sub-solution (resp. super-solution) to $Pu = f \in C(\Omega)$ if the following property holds : if for any $x_0 \in \Omega$ and any function $\varphi \in C^2(\Omega)$ such that $u_- - \varphi$ (resp. $u_+ - \varphi$) has a local max (resp. min) at x_0, then

$$P\varphi(x_0) \leq f(x_0) \quad (\text{resp. } P\varphi(x_0) \geq f(x_0)).$$

If u is a viscosity sub-solution and a viscosity super-solution at the same time, then we say u is a viscosity solution.

Remark 8.2. One can also add boundary condition to it as what we did in the baby version of the definition of sub-solutions and super-solutions to take the boundary value into account.

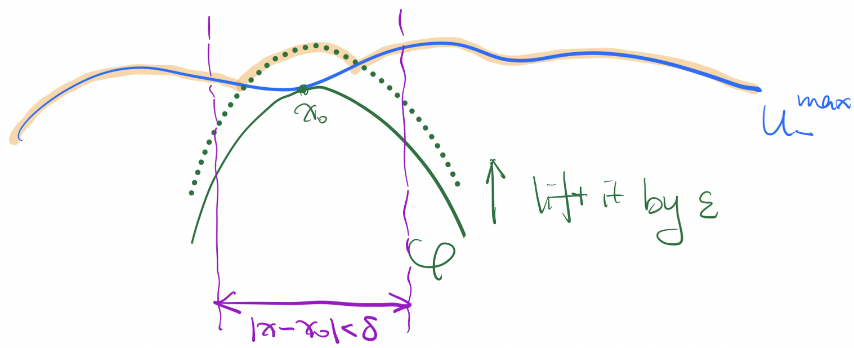


Motivation of the definition : Suppose $u_- \in C^2(\Omega)$ is a sub-solution and φ touches u_- from above at a single point x_0, then in the smooth setting,

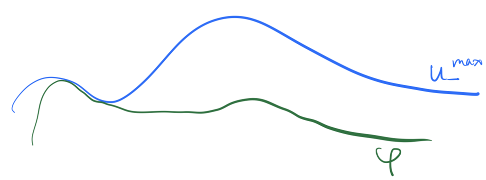
$$u_-(x_0) = \varphi(x_0), \quad Du_-(x_0) = D\varphi(x_0), \quad D^2u_-(x_0) \leq D^2\varphi(x_0),$$

which implies $Pu_-(x_0) \geq P\varphi(x_0)$. Therefore, if we adopt the baby version of the definition for sub-solutions, then one needs $Pu_-(x_0) \leq f$. Therefore, it is natural to ask that $P\varphi(x_0) \leq f$ for all $\varphi \in C^2$ satisfying some "touching" property from above. Note that touching from above at a single point implies that $u_- - \varphi$ has a local maximum at this point, which coincides our definition. This definition does not rely on higher regularity.

Sketch of the proof for the existence : Now in this sense, it follows directly from the definition that the maximum of two sub-solutions is also a sub-solution. The next part is to show that the largest sub-solution is also a super-solution. We prove by contradiction. Suppose not, then there exists a $\varphi \in C^2$ touching from below at x_0, which satisfies $P\varphi(x_0) < f(x_0)$. By continuity, we know $P\varphi(x) < f(x)$ in $|x - x_0| < \eta$ for some small $\eta > 0$. Then by lifting φ by a sufficiently small distance $\varepsilon > 0$, we note that the yellow line is a sub-solution since it is the maximum of two sub-solutions, which leads to a contradiction with the assumption that u_-^{max} is the largest sub-solution.



However, this argument may not be as rigorous as we want since lifting by a small number still can ruin the property that $\varphi + \varepsilon$ and u_-^{max} only intersect near x (i.e. $|x - x_0| < \eta$). To compensate this, we need to bend $\varphi+\varepsilon$ down away from x_0 to make sure the contact point is always localized. To be specific, we use $\varphi+\varepsilon-\delta(x-x_0)^2$ instead. This further modification helps us to eliminate the counterexamples as shown in the following picture.



Remark 8.3. There are some downsides of this argument. In the proof, we need to take the maximum of a bunch of functions. However, even if we suppose $\varphi_n \in C^\infty$, $\varphi(x) := \sup_n \varphi_n(x)$ is still probably not continuous. A counterexample is that one can approximate the Heaviside function by smooth functions.

On the other hand, notice that in the proof, we only use the continuity implies that we can obtain a maximum in a compact set for sub-solutions, so it is natural to work with upper-semi-continuous functions for sub-solutions (resp. lower-semi-continuous functions for super-solutions). However, if we adopt this definition, then a viscosity solution is continuous again since it is not only a sub-solution but also a super-solution. We need a further modification.

For a locally bounded function, we define $u^* = \limsup_{y\to x} u(y)$, which change an L^∞ function to an upper-semi-continuous function. Similarly, $u_*(x) = \liminf_{y\to x} u(y)$ can change an L^∞ function to a lower-semi-continuous function. This implies the following modification of our definition.

Definition 8.4. We say u is a sub-solution if u^* is a sub-solution in the previous sense.

Then we work through the previous argument with the new definition for sub-solutions (super-solutions), which would still work well and gives a solution in viscosity sense. This completes the sketch of the proof.

Remark 8.5. Note that if u is the viscosity solution, this means that the upper-semi-continuous function u^* is a sub-solution and the lower-semi-continuous function u_* is a super-solution. However, sub-solution is below super-solution, which implies $u^* \le u_*$ at each point, and therefore u is continuous.

Remark 8.6. Note that $\Delta u = 0$ in the viscosity sense. does not imply $\Delta u = 0$ in $\mathcal{D}'$ for free. We need some regularity theory. So the construction of a solution using maximum principle is not the easiest way we can do.

The reason why we care about this approach is that this also works for nonlinear equations with maximum principles, such as the fully nonlinear equation

$$\det(D^2 u) = f.$$

See [11] for further discussions.

All the discussions here for viscosity solutions can be found in [6].

8.2 Variational methods

Suppose we are in the self-adjoint case :

$$P = -\partial_i a^{ij} \partial_j + c$$

with a corresponding bilinear form B.

The idea behind the variational methods is to look for the solution u as a minimum point for some functional. However, in order to ensure the uniqueness of the solution, we may want a stronger assumption that our functional only has a unique minimum. A simple observation is that a strictly convex function has a unique minimum.

In calculus, in order to solve $Ax = b$ with $A > 0$ via numerical methods, we consider

$$\min_x \frac{1}{2} Ax \cdot x - b \cdot x := \varphi(x),$$

where a critical point x_0 satisfies

$$D\varphi(x) = Ax - b.$$

So by trying to minimize the functional, you find a way to invert this matrix, which is faster than computing the inverse of a matrix in numerical methods.

By replacing A with P, we define

$$\varphi(u) = \int \frac{1}{2} Pu \cdot u - f \cdot u \, dx = \int \frac{1}{2} b(u,u) - f \cdot u \, dx,$$

where $b(u,u)$ is the integrand in $B(u,u)$. From the previous discussion, it is natural to claim that if P is coercive, then the solution u is the unique minimum point for φ. To make this precise, we bring in the Sobolev spaces. In view of the appearance of ∇u in $b(u,u)$, we set $X = H_0^1$. Then $\varphi : X \to \mathbb{R}$ is strictly convex, that is,

$$\varphi\left(\frac{u+v}{2}\right) < \frac{\varphi(u) + \varphi(v)}{2}, \tag{8.1}$$

which follows from completing the squares

$$\frac{\varphi(u) + \varphi(v)}{2} - \varphi\left(\frac{u+v}{2}\right) = \frac{1}{4} B(u-v, u-v) \geq 0.$$

The equality holds if and only if $u = v$. The strict convexity guarantees the uniqueness of minimum.

Moreover, we show that φ coercive away from 0 since

$$\lim_{\|u\|_X \to \infty} \varphi(u) = \infty.$$

In fact, one can prove by Poincare inequality that

$$|\varphi(u)| \geq \alpha \|Du\|_{L^2}^2 - \beta \tag{8.2}$$

for some $\alpha, \beta > 0$, which is sufficient for the existence of minimizers.

In $\mathbb{R}^n$, one can prove that convex functions are continuous in $\mathbb{R}^n$ so we have a minimum. However, in the Hilbert space, we don't know whether we have a minimum.

Luckily, we also have a notion of weak convergence in Hilbert space, that is, $u_n \rightharpoonup u$ in X is equivalent to say $u_n \cdot v \to u \cdot v$ for all $v \in H$. But convex functions are in general not weakly continuous. We work on weakly semi-continuous functions to ensure that there exists a minimum as in the modification in Perron's method.

We claim that φ is weakly lower semi-continuous in the sense that $u_n \rightharpoonup u$ implies $\liminf \varphi(u_n) \geq \varphi(u)$.

Elliptic boundary value problem 3 - Variational method, Higher regularity

Date: February 14, 2023

9.1 Variational methods (continued)

For the boundary value problem

$$\begin{cases} -\partial_i a^{ij}\partial_j u = f \text{ in } \Omega, & f \in H^{-1}(\Omega), \\ u = 0 \text{ in } \partial\Omega \end{cases},$$

we have the Lagrangian $\mathcal{L}(u) = \int_\Omega \frac{1}{2}a^{ij}\partial_i u \partial_j u - fu\,dx$. We look for $\min_{u\in H_0^1(\Omega)} \mathcal{L}(u)$.

Theorem 9.1. *The Lagrangian $\mathcal{L}$ has a unique minimum u, which satisfies the boundary value problem.*

Proof. We start with a minimizing sequence $u_n \in H_0^1$ such that

$$\lim_{n\to\infty} \mathcal{L}(u_n) = \inf_{u\in H_0^1} \mathcal{L}(u).$$

Step 1 : Extracting a weak convergent subsequence thanks to coercivity
Then by coercivity (8.2), we know that u_n is bounded in H_0^1. Now we can find a weakly convergent subsequence $u_n \rightharpoonup u$ in H_0^1, which by definition implies $u_n \cdot v \to u \cdot v$ for all $v \in H_0^1$. One can also interpret this as a convergence in $\mathcal{D}'$. By compactness, $u_n \to u$ in L^2 as well.

Step 2 : Convexity of $\mathcal{L}$ in p implies lower semi-continuity and hence existence and uniqueness of the minimizer
We claim that $\mathcal{L}$ is convex implies $\mathcal{L}(u) \le \liminf \mathcal{L}(u_n)$. In fact, we only need the convexity of L, where $L(Du, u, x) = \frac{1}{2}a^{ij}\partial_i u \partial_j u - f(x)u$. We also write $L(p, z, x) = \frac{1}{2}a^{ij}p_i p_j - f(x)z$ and $\mathcal{L}(u) = \int L(Du, u, x)\,dx$.

Since L is convex in p ((8.1)),

$$L(hp_1 + (1-h)p_2, z, x) \le hL(p_1, z, x) + (1-h)L(p_2, z, x), \quad h \in [0,1],$$

where we omit z, x in the following computation for simplicity. Then we have

$$(1-h)\left(L(hp_1 + (1-h)p_2) - L(p_2)\right) \le h\left(L(p_1) - L(hp_1 + (1-h)p_2)\right)$$

and hence

$$\frac{L(hp_1 + (1-h)p_2) - L(p_2)}{h} \le \frac{L(p_1) - L(hp_1 + (1-h)p_2)}{1-h}.$$

Let $h \to 0$, we get

$$D_p L(p_2, z, x) \cdot (p_1 - p_2) \le L(p_1, z, x) - L(p_2, z, x).$$

In other words, this says that the tangent line $L(p_2, z, x) + D_p L(p_2, z, x) \cdot (p_1 - p_2)$ is below $L(p_1, z, x)$. This can be also viewed as a definition of convexity provided that L is differentiable.

Now we replace p_2 by Du, p_1 by Du_n, z by u_n then

$$L(Du, u_n, x) + D_p L(Du, u_n, x) \cdot (Du - Du_n) \le L(Du_n, u_n, x).$$

By weak convergence, $D_pL(Du,u,x)\cdot(Du-Du_n)\to 0$ thanks to the uniform boundedness of $\|u-u_n\|_{H^1}$ and dominated convergence theorem. Moreover, a direct computation shows

$$|(D_pL(Du,u_n,x)-D_pL(Du,u,x))\cdot(Du-Du_n)|\le |f(x)||u_n-u||Du_n-Du|,$$

which seems harder to estimate. Then we need to apply Egorov theorem to extract a uniform convergent subsequence u_n on G_ε for any ε such that $m(\Omega\setminus G_\varepsilon)<\varepsilon$. (Note that $u_n\to u$ in L^2 implies $u_n\to u$ almost everywhere by passing to a subsequence.) Therefore,

$$\begin{aligned}\mathcal{L}(u_n)=\int_\Omega L(Du_n,u_n,x)\ge\int_\Omega L(Du,u_n,x)+D_pL(Du,u,x)\cdot(Du-Du_n)\,dx\\ +\int_{G_\varepsilon}(D_pL(Du,u_n,x)-D_pL(Du,u,x))\cdot(Du-Du_n)\,dx.\end{aligned}$$

Set $n\to 0$,

$$\lim_n \mathcal{L}(u_n)\ge\mathcal{L}(u).$$

(In full generality version, one then needs to let $\varepsilon\to 0$.) Therefore, there exists a unique minimizer of $\mathcal{L}$, where the uniqueness follows from the proof last time by strict convexity.

Step 3 : The unique minimizer u solves the equation indeed

Now we show that the minimizer solves the equation. Since $\mathcal{L}(u+hv)\ge\mathcal{L}(u)$ for all $v\in\mathcal{D}(\Omega)$,

$$0=\frac{d}{dh}\mathcal{L}(u+hv)|_{h=0}=\int_\Omega a^{ij}\partial_i u\partial_j v-f\cdot v\,dx=\int_\Omega(-\partial_i a^{ij}\partial_j u-f)v\,dx,$$

where in the last line,

$$\int_\Omega(-\partial_i a^{ij}\partial_j u-f)v\,dx=\langle-\partial_i a^{ij}\partial_j u,v\rangle_{H^{-1},H_0^1}-\langle f,v\rangle$$

if we want to write in a rigorous way. This implies that

$$-\partial_i a^{ij}\partial_j u=f$$

in $\mathcal{D}'$. □

Remark 9.2.
- The method applies to nonlinear problems.
- Convexity can be weakened : there exists a Palais-Smale condition for a minimizing sequence to be compact.
- If $\mathcal{L}$ is not differentiable, we can introduce the subdifferential $\partial L(u)$, which is all the slope for which a line is under the graph : we say $p\in\partial\mathcal{L}(v)$ if for all u, $\mathcal{L}(u)\ge\mathcal{L}(v)+p\cdot(u-v)$. This is also connected to Legendre transform.
- When we discuss the zero Dirichlet boundary condition, it is inherited in the function space H_0^1. If we assume that there exists a minimizer in H^1, then what is the equation solved by the minimizer?

  For $v\in\mathcal{D}(\Omega)$, same computation applies $-\partial_i a^{ij}\partial_j u=f$. Now boundary condition asks us to use $v\in C^\infty(\overline{\Omega})$,

$$0=\int_\Omega a^{ij}\partial_i u\partial_j v-f\cdot v\,dx=\int_\Omega(-\partial_i a^{ij}\partial_j u-f)v\,dx+\int_{\partial\Omega}\nu_j a^{ij}\partial_i u\cdot v\,d\sigma=\int_{\partial\Omega}\nu_j a^{ij}\partial_i u\cdot v\,d\sigma,$$

which implies that

$$\nu_i a^{ij}\partial_j u = 0$$

on $\partial\Omega$, which is a conormal derivative. If $(a_{ij}) = I$ then it is just a normal derivative. This gives a solution to our Neumann boundary condition problem.

- For the Neumann condition,

$$\begin{cases} -\partial_i a^{ij}\partial_j u = f, \\ \nu_j a^{ij}\partial_i u = 0. \end{cases}$$

Solution is not unique since we can add any constant to a solution. For existence, we compute

$$\int f\,dx = \int -\partial_i a^{ij}\partial_j u\,dx = 0,$$

and hence solution does not exist unless $\int f = 0$. Later, we will make this sufficient and necessary. (We do not specify any regularity and only keep the argument above in a heuristic level at this time.)

Note that for Neumann boundary condition, we cannot say $u \in H_0^1$. Instead, the only regularity we have is $u \in H^1$. However, the dual space of H^1 is not a good space and it is just denoted by $(H^1)'$. Since the trace operator $T : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$, we know that for $\varphi \in (H^{\frac{1}{2}}(\Gamma))^* = H^{-\frac{1}{2}}(\Gamma)$,

$$\varphi \circ T : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma) \to \mathbb{R}$$

is a bounded linear functional and hence $\varphi \circ T \in (H^1(\Omega))'$. Therefore, one can view this as the inclusion

$$H^{-\frac{1}{2}}(\Gamma) \subset (H^1(\Omega))'.$$

The regularity here is subtle.

9.2 Elliptic regularity

From different kinds of methods, we reach the conclusion $u \in H_0^1$ when $f \in H^{-1}$.

Theorem 9.3. *Given a compact domain Ω. Suppose $f \in H^k(\Omega)$ and $a^{ij} \in C^k$, then $u \in H^{k+2}(\Omega)$ for $k \geq 0$.*

Remark 9.4. For k sufficiently large, f is smooth enough, then we at least need some regularity for a^{ij} for the equation to hold in classical sense even if u is smooth. Keep in mind that this has nothing to do with solvability. In other words, one can put lower order terms into it and invoke the argument that we are about to discuss as long as you know that there exists a solution.

Proof. As an example, we only prove a simple case : $f \in L^2$ implies $u \in H^2 \cap H_0^1$. It suffices to prove

$$\|u\|_{H^2} \leq \|u\|_{H^1} + \|f\|_{L^2}. \tag{9.1}$$

Step 1 : Localization will suffice

First note that it suffices to prove in sufficiently small region around each point. Since Ω is compact, we can find a finite covering to reduce the proof. Then we only need to consider two cases - a ball inside the interior or a region near the boundary. We provide an argument to precisely verify this localization argument.

Choose a cut-off function to V, a neighbourhood of x_0. Then $v = \chi u$ is well-defined and we compute

$$Pv = P(\chi u) = \chi Pu + C(\nabla\chi\nabla u + \nabla^2\chi \cdot u + \nabla\chi \cdot u).$$

Then we know

$$\|Pv\|_{L^2} \lesssim \|Pu\|_{L^2} + \|u\|_{H^1}.$$

Moreover, we know $\|v\|_{H^1} \leq \|u\|_{H^1}$. So it suffices to prove the inequality for v instead of u. Indeed, since the region is compact and hence there exists $\{\chi_k\}_{k=1}^K$ whose supports cover the region, then

$$\|u\|_{H^2(\Omega)} \lesssim \sum_{k=1}^K \|v_k\|_{H^2(\Omega)} \lesssim \sum_{k=1}^K \|v_k\|_{H^1} + \|Pv_k\|_{L^2}. \lesssim \|u\|_{H^1} + (\|Pu\|_{L^2} + \|u\|_{H^1}),$$

which completes the proof.

Therefore, we just need to do localization and prove the localized inequality, that is, proving (9.1) with u localized.

Step 2 : Proof of the localized version in the interior

For $x_0 \in \Omega$, we select a ball $B_R(x_0) \subset \Omega$ with R to be determined. By a linear transformation, we can assume without loss of generality that $A(x_0) = I$. With a slight abuse of notation, we denote $A = (a_{ij})$ and $A = -\partial_i a^{ij}\partial_j$. Then

$$\|\nabla^2 u\|_{L^2} \leq \|\Delta u\|_{L^2} \leq \|Au\|_{L^2} + \|(A - \Delta)u\|_{L^2} \leq \|Au\|_{L^2} + c(R)\|\nabla^2 u\|_{L^2},$$

where $c(R) \to 0$ as $R \to 0$. So we can select R small enough to absorb the last term to the left hand side and get the desired bound

$$\|u\|_{\dot{H}^2(B_R(x_0))} = \|\nabla^2 u\|_{L^2(B_R(x_0))} \lesssim \|Au\|_{L^2(B_R(x_0))} = \|f\|_{L^2(B_R(x_0))}.$$

It suffices to show the case when $x_0 \in \partial\Omega$. By introducing a cut-off function near x_0, we replace u by χu. Note that this does not kill the boundary condition.

Then we consider the boundary case.

Step 3 : Flatten the boundary (requiring some regularity assumptions of the boundary)

We find a change of coordinates to the half ball case. Obviously, the coefficients of the operator would change. When making the change of coordinates, we only ensure $(a^{ij})(x_0) = I$.

Remark 9.5. Can we flatten A at the same time? If $d = 1$, then we can flatten the real line with metric $ds^2 = a(x)\,dx^2$ by choosing the arc length parametrization. If $d = 2$, then it is overdetermined and we cannot flatten it. However, we can make it conformally to identity,

$$A(x) \mapsto c(x)I,$$

then this is relevant to complex analysis ($\bar{\partial}$). If $d \geq 3$, the answer is no.

Step 4 : Proof of the localized estimate near the boundary when $A = -\Delta$

We first prove for the Laplacian case as an instructive model case for the problem. The main idea is to distinguish the tangential derivatives and normal derivatives. With this idea in mind, one can recover the rigorous proof.

We only consider the half ball region in the following arguments without writing out explicitly. We have $\Delta u \in L^2$, we want $\partial^2 u \in L^2$. For tangential derivative $j \le n-1$ and any k, we compute

$$\int \Delta u \partial_j^2 u \, dx = \sum_k \int (\partial_k \partial_j u)^2 \, dx,$$

where the boundary condition is zero since at least one derivative is tangential. By denoting ∂' for any tangential derivative and ∂ for any derivative, we know from this integration by parts that

$$\|\partial' \partial u\|_{L^2}^2 \le \|\Delta u\|_{L^2} \|\partial' \partial' u\|_{L^2}.$$

Therefore,

$$\|\partial' \partial u\|_{L^2} \le \|\Delta u\|_{L^2}.$$

Then it follows from the original equation that the second order non-tangential (normal) derivative

$$\|\partial_n^2 u\|_{L^2}^2 \le \|\Delta u\|_{L^2} + \|\partial' \partial' u\|_{L^2},$$

which implies

$$\|\partial^2 u\|_{L^2} \le \|\Delta u\|_{L^2}.$$

Step 5 : The general case near the boundary in which $(a^{ij})(x_0) = I$

We write

$$-\Delta v = -\partial_i a^{ij}(x_0) \partial_j v = f - \partial_i (a^{ij}(x) - a^{ij}(x_0)) \partial_j v - b^j \partial_j v - cv,$$

then by denoting the first and zeroth order remainder by Rv, we can apply the bound for Δ in the previous step,

$$\|v\|_{\dot{H}^2} \le \|f\|_{L^2} + \|(a^{ij}(x) - a^{ij}(x_0)) \partial_i \partial_j v\|_{L^2} + \|Rv\|_{L^2},$$

where Rv

Note that

$$\|(a^{ij}(x) - a^{ij}(x_0)) \partial_i \partial_j v\|_{L^2} \le \|(a^{ij}(x) - a^{ij}(x_0))\|_{L^\infty} \|v\|_{H^2} \le \delta \|v\|_{H^2},$$

for sufficiently small δ, where we can make $x - x_0$ small enough in our first step. For lower order terms,

$$\|Rv\|_{L^2} = \|(\partial_i a^{ij}) \partial_j v + b^j \partial_j v + cv\|_{L^2} \le \|v\|_{H^1}.$$

Hence,

$$\|v\|_{H^2} \le \|v\|_{H^1} + \|f\|_{L^2},$$

which completes the proof.

Remark 9.6. In fact, we can always obtain a stronger estimate like what we have in Step 4 for $-\Delta$. The tool is the following generalized Poincaré-type inequality for $v \in H^2(\Omega), v|_{\partial\Omega} = 0$ in a domain of size 1,

$$\|v\|_{L^2} + \|\nabla v\|_{L^2} \le \|\partial^2 v\|_{L^2}.$$

Note that $v \in H^2(\Omega), v|_{\partial\Omega} = 0$ is equivalent to $v \in H_0^1 \cap H^2$. The proof follows from a simple contradiction as the usual Poincaré inequality. The only difference is that we would obtain $\partial^2 v = 0$ and $v|_{\partial\Omega} = 0$. This leads to a contradiction since $\partial_k v = C_k$ and hence $v(x) = \sum C_k x_k + D$, which is a hyperplane of which $\partial\Omega$ cannot be a subset.

Then By resizing it to a domain of radius r, we know

$$r^{-2}\|v\|_{L^2} + r^{-1}\|\nabla v\|_{L^2} \le \|\partial^2 v\|_{L^2},$$

where we gain smallness if r is small. The smallness helps us to absorb H^1 norms to the $\dot{H}^2$ norm. But this means that we have a stronger estimate when r is sufficiently small :

$$\|v\|_{\dot{H}^2} \lesssim \|f\|_{L^2}.$$

□

9.3 More general boundary conditions

What is a good boundary condition? We discussed Dirichlet and Neumann condition just now. We also have Robin condition $\frac{\partial u}{\partial \nu} = \lambda u$, which has different traces. The leading order of Robin boundary condition is Neumann.

Now, we discuss another type of boundary condition $\partial_n u = \sum_{j\le n-1} a_j \partial_j u$ in the half plane case.

On the boundary condition for half plane, $\Delta u = f$ can be written as

$$(\partial_n^2 + (\partial')^2)u = f.$$

If we only take the Fourier transform in the tangential direction and still use ξ to denote the $(n-1)$-vector, then we get a second order ordinary differential equation

$$(\partial_n^2 - \xi^2)\widehat{u} = \widehat{f}, \tag{9.2}$$

with two fundamental solutions $e^{x_n|\xi|}, e^{-x_n|\xi|}$ to the homogeneous equation. One grows exponentially as we move inside while the other decays exponentially. Set

$$\widehat{u}_1 = (\partial_n - |\xi|)\widehat{u}, \quad \widehat{u}_2 = (\partial_n + |\xi|)\widehat{u}, \tag{9.3}$$

then

$$(\partial_n + |\xi|)\widehat{u}_1 = \widehat{f}, \quad (\partial_n - |\xi|)\widehat{u}_2 = \widehat{f}.$$

For u_1, by starting from a vanishing condition at infinity and solving it towards the boundary, we obtain $u_1|_{\partial\Omega}$. For u_2, you want to solve from near the boundary towards the interior. To solve this, we want to use the boundary condition to give $u_2|_{\partial\Omega}$ from $u_1|_{\partial\Omega}$.

For the zero Dirichlet boundary condition, we have $u_1 = u_2$ on boundary thanks to (9.3). For the zero Neumann boundary condition implies $u_1 = -u_2$ on the boundary thanks to (9.3). For $\partial_n u = \sum a_j \partial_j u$,

$$\partial_n u = \sum a_j \partial_j u \quad (\partial_n - i a_j \xi_j)\widehat{u} = 0,$$

and we know $\widehat{u}_1 = (\partial_n - |\xi|)\widehat{u}$. Subtracting gives $(-i a_j \xi_j + |\xi|)\widehat{u} = \widehat{u}_1$. If the symbol does not vanish, then this gives $\widehat{u}|_{\partial\Omega}$. The boundary condition in this example is called the Lopatinsky boundary condition.

On the other hand, the way of doing Fourier transform in (9.2) can give the Poisson formula on the half space. See [17, Chapter 8.3].

Elliptic boundary value problem 4 - Fredholm theory

Date: February 16, 2023

Recall the following motivating example from last time. For $P = -\partial_i a^{ij}\partial_j$ is in divergence form, we consider the Neumann boundary value problem

$$\begin{cases} Pu = f \text{ in } \Omega \\ \nu_j a^{ij}\partial_i u = 0 \text{ on } \partial\Omega \end{cases}.$$

Heuristically speaking, the obstruction of its solvability is a one-dimensional condition on f, which is $\int f = 0$. On the other hand, the obstruction of its uniqueness is also one-dimensional since it would be unique up to a constant.

The summary we are going to make today is : Obstructions to solvability of the bounded linear operator $P : H_0^1(\Omega) \to H^{-1}(\Omega)$ are only finite dimensional. In this context, we also want to study the adjoint operator $P^* : H_0^1(\Omega) \to H^{-1}(\Omega)$. A more precise version is stated as Theorem 10.12. This brings us back to functional analysis.

10.1 Recap of functional analysis

10.1.1 Basics

Suppose X, Y are two Banach spaces with $T : X \to Y$ bounded and linear. Then for dual spaces X', Y', $T^* : Y' \to X'$ is also bounded and linear.

If

$$\ker T := \{x \in X : Tx = 0\}$$

is not empty, then the solution to $Tx = y$ will only be determined modulo elements in $\ker T$. The range of T is $R(T) = TX \subset Y$ while the range of T^* is $R(T^*) = T^*Y' \subset X'$. For

$$\langle Tx, y'\rangle = \langle x, T^*y'\rangle,$$

if $y' \in \ker T^*$, then $y' \perp Tx$ and hence $R(T) \subset \ker(T^*)^\perp$. Similarly, $R(T^*) \subset \ker(T)^\perp$. In general, one can prove that

$$\overline{R(T)} = \ker(T^*)^\perp, \qquad \overline{R(T^*)} = \ker(T)^\perp. \tag{10.1}$$

The closed range theorem in functional analysis asserts that the following conditions are equivalent for any closed operator T:

- $R(T)$ closed;
- $R(T^*)$ closed;
- $R(T) = \ker(T^*)^\perp$;
- $R(T^*) = \ker(T)^\perp$.

Going forward, we also use $N(T)$ to denote $\ker T$. Suppose $R(T)$ is closed, then

$$T : X/\ker T \to R(T) \subset Y$$

is bounded, injective and surjective. On the other hand, the open mapping theorem tells us if T is surjective, then it is an open mapping, that is, the image of an open set under T is an open set. An easy corollary is that $T : X/\ker T \to R(T)$ is invertible with a bounded inverse.

(Thanks to the open mapping theorem, the inverse is continuous and hence bounded.) Hence, $T^{-1} : N(T^*)^\perp \to X/\ker T$, which is related to our solvability.

10.1.2 Fredholm operator

Definition 10.1. We say T is a Fredholm operator if

- $N(T)$, $N(T^*)$ are finite dimensional;
- $R(T)$, $R(T^*)$ are closed.

Remark 10.2. In fact, the second line in the definition of Fredholm operator is redundant in view of the following fact that $\dim Y/R(T) = \dim N(T^*)$ is finite and hence closed. See [1, Section 4.4].

Moreover, since $R(T)$ is closed, we know $R(T) = N(T^*)^\perp$ and $Y/R(T)$ is well-defined. In general, for any closed subspace $K \subset H$, H/K is isometrically isomorphism $K^\perp$ since

$$H/K \to K^\perp, \quad h + K \mapsto h_2 \text{ with } h = h_1 + h_2 \in K + K^\perp$$

Therefore,

$$\dim N(T^*) = \dim Y/R(T).$$

The dimensions $\dim N(T), \dim N(T^*)$ tell us how many obstructions we have. The first one characterizes the obstructions for uniqueness while the second one characterizes the obstructions for existence.

The outcome of today's class is the main theorem - Theorem 10.12. Let us finish the introduction of tools in functional analysis before proving this theorem. One of the reasons why the Fredholm operators are introduced is that they are stable in the following sense. In particular, for second order elliptic operator, it is stable under first order perturbation.

Theorem 10.3. *Suppose T is Fredholm, then $T + S$ is also Fredholm if*

- *S is small ($\|S\|$ is small compared to $\|T\|$);*
- *or S is compact.*

Proof. See [1, Corollary 4.47, Theorem 4.48]. □

Remark 10.4. Note that the Fredholm theory in [7, Appendix D] is incomplete in the sense that they only consider the case $T = I$. However, in view of Atkinson's theorem, [1, Theorem 4.46], we can find an almost inverse, so they are equivalent.

Remark 10.5. One can think of a compact perturbation as follows. A compact perturbation might be large in at most finite dimensions. Note that even if S is small, it may change the dimension of $\ker T$ and $\ker T^*$. However, as long as S compact/small, the index of T does not change, which is defined as follows.

10.1.3 Index of an operator and invariance for Fredholm operators

Definition 10.6. The index of T is given by

$$ind(T) := \dim N(T) - \dim N(T^*).$$

For second order elliptic equations, lower order terms are compact. Suppose $P = -\partial_i a^{ij}\partial_j + b^i\partial_i + c$, Note that $b^i\partial_i + c : H_0^1 \to L^2$, and the inclusion $L^2 \subset H^{-1}$ is compact thanks to the compactness inclusion of its dual $H_0^1 \subset L^2$ by Rellich-Kondrachov. (The dual operator of a compact operator is compact.)

Moreover, note that P depends on x, if we choose another operator with the principal part $-\partial_i a^{ij}\partial_j$, then a linear transformation

$$P_t = tI + (1-t)A$$

can help us to turn $-\partial_i a^{ij}\partial_j$ into $-\Delta$, which is uniquely solvable in H_0^1 for $f \in H^{-1}$ and hence $ind(-\Delta) = 0$. So the dimension of kernel and cokernel are the same. Note that $ind(P_t)$ will keep the same because it is continuous with respect to t due to the fact that small perturbations will not change an Fredholm operator out of the Fredholm class and $ind(P_t) \in \mathbb{Z}$. Note that we cannot connect any two operators together. Let's say Δ and some arbitrary B, then $(t+\varepsilon)\Delta + (1-(t+\varepsilon))B - (t\Delta + (1-t)B) = \varepsilon(\Delta - B)$, which requires at least $\Delta - B$ to be bounded. However, this is natural for any two second order elliptic operators.

Remark 10.7. The index of $-\Delta$ is zero follows from the flatness of $\mathbb{R}^n$. However, this does not hold for general manifolds, vector bundles. The index of elliptic operators is a topological invariant.

Example 10.8. For a matrix $A : \mathbb{R}^n \to \mathbb{R}^m$, $indA = n - m$.

Example 10.9. We take $\ell^2(\mathbb{N})$, the following operator $T : \ell^2 \to \ell^2$ given by

$$T(x_1, \cdots, x_n, \cdots) = (0, x_1, x_2, \cdots).$$

We have $N(T) = \{0\}$ and $N(T^*) = span\{(1, 0, 0, \cdots)\}$. Therefore, $ind(T) = -1$.

Proposition 10.10. *For two Fredholm operators, $ind(T \circ S) = ind(T) + ind(S)$.*

Proof. See [1, Theorem 4.43]. □

Proposition 10.11. *When $X = Y$, I is a Fredholm operator and hence by perturbing by a compact operator K, we know $I + K$ is Fredholm of index 0.*

Proof. See [7, Appendix D Theorem 5], [1, Lemma 4.45]. □

10.2 Application of Fredholm theory - Solvability of second order elliptic operators

Theorem 10.12. *Suppose P is a second order elliptic operators in divergence form, then $P, P^* : H_0^1 \to H^{-1}$ are Fredholm. In particular, $N(P^*) = span\{v_1, \cdots, v_k\}$, $N(P) = span\{u_1, \cdots, u_l\}$ for some finite integer l, k and the solutions exists if $f \perp \{v_1, \cdots v_k\}$ ($f \in R(P) = N(P^*)^\perp$), while the solutions are unique modulo $u_1, \cdots, u_l$.*

Proof. We start with our operator $P : H_0^1 \to H^{-1}$. For the bilinear form

$$B(u, u) = \int a^{ij}\partial_i u \partial_j u + b^i\partial_i u + cu^2\, dx,$$

we want a coercive property $B(u,u) \geq C\|u\|^2_{H^1_0}$. To ensure that it holds, we choose $\lambda \gg 1$ and change P to $P+\lambda$, then

$$B_\lambda(u,u) \geq c\|\nabla u\|^2_{L^2} + \lambda\|u\|^2_{L^2} - C\left(\|u\|_{L^2}\|\partial u\|_{L^2} + \|u\|^2_{L^2}\right).$$

If $\lambda \gg 1$, then $B(u,u) \geq \|u\|^2_{H^1_0}$. Hence, $P+\lambda$ is invertible thanks to the Lax-Milgram theorem, that is, the solution exists and is unique.

Moreover, $u \to \lambda u$, $H^1_0 \mapsto H^{-1}$ is a compact perturbation, so P is a Fredholm operator. □

10.3 Application of Fredholm theory - Eigenvalues and eigenfunctions

We say λ is an eigenvalue if $\ker(P-\lambda) \neq \{0\}$, that is, $Pu = \lambda u$ has nontrivial solutions.

Fredholm theory tells us any $\lambda \in \mathbb{C}$ has finite multiplicity. And λ is an eigenvalue for P implies $\bar{\lambda}$ is an eigenvalue for P^* since $0 = ind(P) = ind(P-\lambda) = \dim N(P-\lambda) - \dim N(P^* - \bar{\lambda})$.

Where are these eigenvalues? Note that we can change P to $P+\mu$ to examine the eigenvalues, so that we can take μ sufficiently large so we have solvability. For $Pu = \lambda u$, we have $u = \lambda P^{-1}u$. By Rellich-Kondrachov, $P^{-1} = K : L^2 \to L^2$ is compact since both $H^1_0 \subset L^2$ and $L^2 \to H^{-1}$ are compact. Then $Ku = \lambda^{-1}u$ implies λ^{-1} is an eigenvalue of a compact operator.

A compact operator has finitely many eigenvalues or countably many with an accumulating point at 0.

Theorem 10.13. *P has countably many eigenvalues λ_n and* $\lim_{n\to\infty}|\lambda_n| = \infty$.

For $P = -\partial_i a^{ij}\partial_j + c$, we know P is self-adjoint and hence eigenvalues are real with orthogonal eigenfunctions corresponding to different eigenvalues. By normalizing it, it gives an orthonormal basis.

For the coercive case,

$$\lambda\|u\|^2_{L^2} = \langle Pu, u\rangle = B(u,u) \geq C\|u\|^2_{H^1_0}$$

So the eigenvalues can only go to the right. Even if our P is not coercive, we can still shift by μ to make it coercive, so that the eigenvalues are also accumulating at $+\infty$.

The picture in the non-symmetric case, there is still a barrier for eigenvalues on the left, but the eigenvalues can be complex numbers. However, the first eigenvalue is still real, which defines the barrier. We will study this in detail next time.

Example 10.14. In the case of $\mathbb{R}^n$, one can also apply Fredholm theory. For $-\Delta + V$, suppose V has sufficient decay at infinity, i.e. $|V| \lesssim R^{-\alpha}$. Due to the decay property of V, $V : H^1 \to L^2$ is a compact operator even if we don't have compactness theorems in the setting of $\mathbb{R}^n$. Indeed, this follows from the computation

$$\|Vu_n - Vu_m\|_{L^2} \lesssim \|\chi(u_n - u_m)\|_{L^2(B_{2R})} + R^{-\alpha}M,$$

where $\|u_n\|_{H^1} \leq M$ and χ is a bump function with $\chi \equiv 1$ in B_R such that $\chi(u_n - u_m) \in H^1_0(B_{2R})$. Therefore, for R sufficiently large, the second term is less than ε, the first term tends to zero as $n, m \to \infty$ thanks to the Rellich-Kondrachov theorem in the bounded domain B_{2R}. On the other hand, $-\Delta - \lambda$ is invertible when $\lambda < 0$ by applying Fourier transform. Thus, $-\Delta - \lambda + V$ is a Fredholm operator.

Eigenvalues and eigenfunctions (continued)

Date: February 21, 2023

For

$$\begin{cases} Pu = f, \quad P = -\partial_i a^{ij}\partial_j + b^j\partial_j + c, \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

we study $Pu = \lambda u$. By Fredholm theory, we find

$$(\lambda_k, u_k)$$

with $\lim_{k\to\infty}|\lambda_k| = \infty$. For λ_k complex, we also have u_k complex, then we can also change b^j, c to be complex but still need to keep the principal part real.

Proposition 11.1. *If coefficients are smooth, then eigenfunctions are also smooth.*

Proof. For $u_k \in L^2$ solves $Pu_k = \lambda_k u_k \in L^2$, by elliptic regularity (Theorem 9.3), we know $u_k \in H^2$. By induction, $u_k \in H^{2l}$ for any l. □

11.1 Visualization of eigenvalues in the self-adjoint and non-self-adjoint case

If P is self-adjoint, then the eigenvalues are real. We consider $P = -\partial_i a^{ij}\partial_j + c$ if all coefficients are real. If we consider the complex setting, a natural assumption is

$$P = -\partial_j^B a^{jk}\partial_k^B + c, \qquad \partial_j^B = \partial_j + iB_j,$$

with B_j and c real. Then ∂_j^B is skew-adjoint and hence P is still self-adjoint. An example of ∂_j^B is the electromagnetic potential. Going forward, we use the complex inner product

$$\int_\Omega Pu \cdot \bar{v} = \int_\Omega u \cdot \overline{Pv}.$$

If P self-adjoint, then $Pu = \lambda u$ implies

$$\int Pu \cdot \bar{u} = \int \lambda u \cdot \bar{u} = \lambda \|u\|_{L^2}^2,$$

where the left hand side is equal to its adjoint, so its real, which implies the realness of λ.

When P is not self-adjoint, we can write

$$P = P_{self} + P_{skew},$$

where $P_{self} = \frac{P+P^*}{2}$ is a second-order elliptic operator while $P_{skew} = \frac{P-P^*}{2}$ is a first order operator. For any eigenfunction u with eigenvalue λ, we compute

$$Pu \cdot \bar{u} = P_{self}u \cdot \bar{u} + P_{skew}u \cdot \bar{u} = \mathrm{Re}\lambda \|u\|_{L^2}^2 + i\mathrm{Im}\lambda \|u\|_{L^2}^2,$$

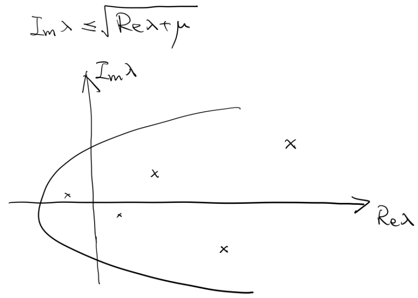
where $\mathrm{Re}\lambda \simeq P_{self}u \cdot \bar{u} \lesssim \|u\|_{H^1}^2$ and

$$\mathrm{Im}\lambda \simeq P_{skew}u \cdot \bar{u} \le \|u\|_{L^2}^2 + \|u\|_{L^2}\|\nabla u\|_{L^2} \le \|u\|_{H^1}\|u\|_{L^2}.$$

Moreover, if P_{self} is coercive, then $\mathrm{Re}\lambda \simeq P_{self}u \cdot \bar{u} \simeq \|u\|_{H^1}^2$ and this can be achieved by changing the operator P to $P + \mu$ for some sufficiently large μ as what we did in the proof of Theorem 10.12. By normalizing $\|u\|_{L^2} = 1$, we get

$$\mathrm{Re}\lambda \simeq \|u\|_{H^1}^2 \qquad |\mathrm{Im}\lambda| \le \|u\|_{H^1}.$$

Therefore, $|\mathrm{Im}\lambda| \leq \sqrt{\mathrm{Re}\lambda + \mu}$ for some sufficiently large μ which makes the approximation $P_{self} u \cdot \bar{u} \simeq \|u\|_{H^1}^2$ work. So the spectrum picture in the non-self-adjoint case can be sketched as follows.



11.2 Variational characterization of eigenvalues and eigenfunctions

11.2.1 Orthonormal eigenfunctions of a symmetric second order elliptic operator form an orthonormal basis

Proposition 11.2. *Suppose P is symmetric, then the L^2 normalized eigenfunctions form an orthonormal basis for L^2.*

Proof. For $P : H_0^1 \to H^{-1}$, we assume it is coercive (if not, we consider $P + \mu$). Then this implies the existence of $P^{-1} : H^{-1} \to H_0^1$, which follows from the Lax-Milgram theorem. Furthermore, we consider L^2 eigenfunctions (This is natural thanks to Proposition 11.1) and this restricts the consideration to the compact operator $K := P^{-1} : L^2 \to L^2$.

Suppose $\{\lambda_j\} \to 0, u_j$ are eigenvalues and orthonormal eigenfunctions for K. Set $V := span_{L^2}\{u_j\}$.

Since K is symmetric and compact, it follows from the spectrum theorem for compact operators ([23, Theorem 5.6]) that $\{u_j\}$ form an orthonormal basis if $R(K)$ is dense. Then it suffices to show $N(K) = N(K^*) = \overline{R(K)}^{\perp}$ is empty. This is trivial since K is invertible. $\square$

11.2.2 Variational characterization of the principle value

Now we still stick to the symmetric case with real coefficients so that the preceding proposition can be applied to obtain an orthonormal basis $\{u_j\}$ for L^2 which consists of eigenfunctions. For $u \in L^2$, $u = \sum c_j u_j$, then $\|u\|_{L^2}^2 = \sum c_j^2$, where $c_j = u \cdot u_j$. Then

$$Pu \cdot u = \sum c_j u_j \cdot \sum \lambda_j c_j u_j = \sum \lambda_j c_j^2$$

and $\|u\|_{H^1}^2 \simeq \sum (\lambda_j + \mu) c_j^2$ and $\|u\|_{H^2}^2 \simeq \sum (\lambda_j + \mu)^2 c_j^2$ and so on.

Proposition 11.3. *For P symmetric with real coefficients, the first eigenvalue satisfies*

$$\lambda_0 = \inf_{u \in H_0^1} \frac{B(u,u)}{\|u\|_{L^2}^2} = \inf_{u \in H_0^1, \|u\|_{L^2} = 1} B(u,u),$$

which is called the variational interpretation.

Proof. We keep using the notations above and suppose P is coercive by adding a μ if needed. Note that

$$B(u,u) = Pu \cdot u = \sum \lambda_j c_j^2 \geq \lambda_0 \sum c_j^2 = \lambda_0 \|u\|_{L^2}^2$$

with equality if and only if $Pu = \lambda_0 u$.

Now it suffices to show that if $u \in H_0^1$ with $\|u\|_{L^2} = 1$, then $B(u,u) = \lambda_0$ implies $Pu = \lambda_0 u$. (It is actually an equivalence relation but the other implication is trivial.) Recall that $u = \sum c_j u_j$ with $c_j = u \cdot u_j$. Since $\|u\|_{L^2} = 1$, we know $\sum_j c_j^2 = 1$. Hence,

$$\sum_j c_j^2 \lambda_0 = \lambda_0 = B(u,u) = \sum_j c_j^2 \langle Pu_j, u_j \rangle = \sum_j c_j^2 \lambda_j.$$

Therefore, $c_j = 0$ if $\lambda_j > \lambda_0$. Since λ_0 has finite multiplicity, $u = \sum_j c_j u_j$ is a finite sum and it satisfies $Pu = \lambda_0 u$, which completes the proof. □

Remark 11.4. Moreover, if $c_0 = 0$, then we can find λ_1 by using

$$\lambda_1 = \inf_{u \in H_0^1, u \perp u_0} \frac{B(u,u)}{\|u\|_{L^2}^2}.$$

This is kind of related to Lagrange multiplier.

By a side product of homework, $\partial_j |u| = \operatorname{sgn}(u) \partial_j u$ almost everywhere for $u \in H^1$. Therefore, $B(|u|, |u|) = B(u,u)$. If u is an eigenfunction, then $|u|$ is an eigenfunction and hence there exists a non-negative eigenfunction.

If there is another eigenfunction corresponding to λ_0, one can make a linear combination to let it have a zero, but this is impossible.

Proposition 11.5. *With the same assumption as in the preceding proposition, λ_0 is a simple eigenvalue and $u_0 > 0$.*

Proof. We have already derived that u_0 is non-negative. Thanks to the Harnack's principle which will be introduced in the remaining lectures, we know u_0 is strictly positive unless $u_0 \equiv 0$. (One needs to use the full generality of Harnack's principle when the operator has zeroth order term. See [11, Chapter 8].) If there exists another eigenfunction $\tilde{u}_0$ corresponding with λ_0 and linearly independent with u_0, then we can arrange that $|\tilde{u}_0 - cu_0|$ is not smooth for some c. However, notice that $\tilde{u}_0 - cu_0$ is still in H_0^1 and an eigenfunction corresponding to λ_0. This contradicts with the fact that eigenfunctions are smooth thanks to Proposition 11.1. □

Remark 11.6. Note that all the preceding propositions in this subsection combines to form an alternative proof of [7, Section 6.5.1, Theorem 2].

Theorem 11.7. *If P is not formally self-adjoint, then there exists a first eigenvalue $\lambda_0 \in \mathbb{R}$ and simple with $u_0 > 0$ and for any other λ_j, we have $\mathrm{Re}\lambda_j > \lambda_0$.*

One can find a proof in [7, Section 6.5.2], which shows the variational principle in this setting by using maximum principle. The idea is similar to the one presented above.

Examples of eigenvalue problem

Date: February 23, 2023

Today we do a wrap-up for the eigenvalue problems by providing some examples.

12.1 Basic examples with Dirichlet or periodic boundary condition

Example 12.1. In 1 dimension, we consider $P = -\partial_x^2$ in $[0, L]$ with Dirichlet boundary condition. The eigenfunctions and eigenvalues are

$$u_k = \sin\left(\frac{\pi k}{L}x\right), \quad \lambda_k = \left(\frac{\pi k}{L}\right)^2$$

for $k \geq 1$. If we choose Neumann boundary condition instead, then

$$u_k = \cos\left(\frac{\pi k}{L}x\right), \quad \lambda_k = \left(\frac{\pi k}{L}\right)^2$$

for $k \geq 0$. This means that we have an obstruction to solve the Neumann problems, which are the constant functions, that is, we can only solve uniquely up to constants for Neumann problems. Moreover, to ensure the existence of solutions, the source terms also need to be orthogonal to constant functions thanks to Fredholm theory (Theorem 10.12).n

Example 12.2. If we choose a periodic boundary condition $u(0) = u(L)$, $\partial_x u(0) = \partial_x u(L)$, then $u_0 = 1, \lambda_0 = 0$ is the first eigenvalue. One can view $[0, 1]$ as $\mathbb{S}^1$ when the boundary condition is periodic. We have

$$u_k^{\pm} = e^{\pm i \frac{2\pi k}{L} x}, \quad \lambda_k = \left(\frac{2\pi k}{L}\right)^2$$

for $k \geq 1$. Note that in this example, from the second eigenvalue, we start to have multiplicity.

Example 12.3. For the operator $P = -\partial_x a \partial_x + c$ in I, with Dirichlet boundary condition, it has a sequence of simple eigenvalues $\lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots$, which is studied by using the Sturm-Liouville theory. We omit the proof though it is not hard. It tells us u_k can only change signs exactly k times, which is called Sturm oscillation theory.

Example 12.4. For $P = -\Delta$ with Dirichlet boundary condition in $[0, \pi] \times [0, \pi]$, we have

$$u_{n,m} = \sin nx \sin my, \quad n, m \geq 1, \qquad \lambda_{n,m} = n^2 + m^2.$$

Note that we only need to consider the eigenfunctions in the form of separation of variables since the operator $-\Delta$ can be written into the sum of two operators commuting with each other, i.e. $[\partial_x^2, \partial_y^2] = 0$. This implies that they share common basis with $-\Delta$ at least in a heuristic level. (See [24, Problem 4.5] for a general statement of this fact.)

Example 12.5. For $P = -\Delta$ in $[0, 2\pi] \times [0, 2\pi]$ with periodic boundary condition in x, y, then we have

$$u_{n,m} = e^{inx} e^{imy}, \quad n, m \in \mathbb{Z}, \qquad \lambda_{n,m} = n^2 + m^2,$$

which is a problem on torus.

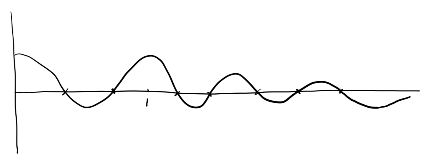
12.2 Laplacian on n-sphere

12.2.1 Laplacian with boundary conditions, Bessel functions

Example 12.6. For $D = \{|x| \leq 1\} \subset \mathbb{R}^2$, we consider $P = -\Delta$ with Dirichlet boundary condition. By writing it in (r, θ), we have $P = -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\theta^2$, where ∂_θ^2 is the Laplacian on the circle. Note that $[-\Delta, -\Delta_\theta] = 0$, which implies they share common basis in view of linear algebra of matrices. So it's natural to consider eigenfunctions in the form of separation of variables. Since $\partial_r, \partial_\theta$ commute, we know the eigenfunctions are a product of functions in θ and functions in r, namely $u(r,\theta) = v(r)w(\theta)$. From before, $w_k = e^{ik\theta}$. By plugging this back into the equation, we know

$$\left(-\partial_r^2 - \frac{1}{r}\partial_r + \frac{k^2}{r^2}\right) v = \lambda v,$$

which has variable coefficients and we need to solve it for each k. Unfortunately, there are no elementary solutions to this so that we need to introduce some special functions to solve this and what we obtain are called Bessel functions. Though it is impossible to write down the exact formula, we can obtain its asymptotic behavior. Moreover, by scaling, we can solve the equation for all λ given solutions when $\lambda = 1$, where λ is called the scaling parameter. Suppose we fix k and find a solution for $\lambda = 1$, which behaves like the following graph.



By noticing that the Dirichlet boundary condition for the eigenvalue problem requires $v(1) = 0$, so the choice of $\lambda_k = j_k$ are the specific scaling parameter such that the scaling moves the k-th zero to 1.

In the same spirit, we can solve the Neumann problem by looking for the specific scaling parameter such that the scaling makes $v'(1) = 0$.

Example 12.7. For $D = \{|x| \leq 1\} \subset \mathbb{R}^n$, we consider

$$P = -\Delta = -\partial_r^2 - \frac{n-1}{r}\partial_r - \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}}$$

with Dirichlet boundary condition, where the $\Delta_{\mathbb{S}^{n-1}}$ is the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$ as a Riemannian manifold. In the previous example, we knew the spectrum of the Laplacian on the circle so that this reduction helps us to find the spectrum of $-\Delta$. However, we do not know the eigenvalues of $\Delta_{\mathbb{S}^{n-1}}$ yet. If we do a separation of variables $u(r,\omega) = v(r)w(\omega)$ with $r \in [0,1]$, $\omega \in \mathbb{S}^{n-1}$, then we have

$$-\Delta_{\mathbb{S}^{n-1}} w = \mu w, \quad \left(-\partial_r^2 - \frac{n-1}{r}\partial_r + \frac{\mu}{r^2}\right) v = \lambda v.$$

We still get some Bessel functions if we knew w is an eigenfunction of $-\Delta_{\mathbb{S}^{n-1}}$ corresponding to μ. Though we do not even write out the exact formula for $-\Delta_{\mathbb{S}^{n-1}}$, we can obtain the spectrum by a trick introduced in the following example.

12.2.2 Spectrum of Laplacian on the sphere

Example 12.8. For the Laplacian $P = -\Delta_{\mathbb{S}^{n-1}}$ on $\mathbb{S}^{n-1}$, if w is an eigenfunction, that is, $-\Delta w = \mu w$, with w a function living on the sphere. The idea is to extend w to $\mathbb{R}^n$ by homogeneity. For $x = r\omega \in \mathbb{R}^n$, we can extend w to $\mathbb{R}^n$ by setting

$$u(x) = r^\sigma w(\omega).$$

We want to choose σ so that u is harmonic. We compute

$$\Delta u = \sigma(\sigma-1)r^{\sigma-2}w(\omega) + (n-1)\sigma r^{\sigma-2}w(\omega) + (-\mu)r^{\sigma-2}w(\omega),$$

which conveniently tells us σ should satisfy $\mu = \sigma(\sigma+n-2)$, which is a quadratic equation for σ. Since the growth of u at infinity is at most as a polynomial and is smooth away from 0, we know $u \in \mathscr{S}'$. Or to be more precise, one can apply [13, Theorem 7.1.18] directly to know $u \in \mathscr{S}'$. By applying Fourier transform to $-\Delta u = 0$, we know u can only be a harmonic polynomial, which implies that σ is a natural number. (We can also argue by the fact that harmonic functions are smooth.)

This in turn gives the eigenvalues of the spherical Laplacian. Given a harmonic polynomial u of degree $\sigma \in \mathbb{N}$, $u|_{\mathbb{S}^{n-1}}$ is an eigenfunction of $-\Delta_{\mathbb{S}^{n-1}}$ corresponding to $\mu = \sigma(\sigma+n-2)$. Therefore, we proved Theorem 12.9.

Theorem 12.9. *The spectrum of $-\Delta_{\mathbb{S}^{n-1}}$ is given by*

$$\sigma(-\Delta_{\mathbb{S}^{n-1}}) = \{\sigma(\sigma+n-2) : \sigma \in \mathbb{N}\}.$$

Remark 12.10. Though the spectrum is characterized by only one parameter for any dimension, it has very high multiplicities, which corresponds to how many independent harmonic polynomials you can find of degree σ and is roughly like $O(\sigma^{n-1})$.

Remark 12.11. See [12] for details of spherical harmonics and a decomposition of $L^2(\mathbb{S}^{n-1})$. In specific, see [12, Definition, Page 67] for the reason why we can extend this by homogeneity.

Remark 12.12. Note that the discreteness of the spectrum follows from the compactness of the sphere. We can prove Rellich-Kondrachov compactness theorem for the sphere, which is identical to the one on a connected and bounded domain. This can be applied to derive the compactness of inverse operators like what we did in the proof of Proposition 11.2 and hence implies the discreteness of the spectrum.

12.2.3 Examples with non-compact domains

Now we consider an example with non-compact domains.

Example 12.13. If $-\Delta u = \lambda u$ in $\mathbb{R}^n$, then by Fourier transform, $(\xi^2 - \lambda)\widehat{u} = 0$. Thus,

$$\operatorname{supp}\widehat{u} \subset \{|\xi| = \sqrt{\lambda}\},$$

which implies u cannot be in L^2 since the sphere is of measure zero. Therefore, $-\Delta$ only admits generalized eigenvalues $\lambda_\xi = \xi^2$ with generalized eigenfunctions $u_\xi = e^{ix\xi}$.

Remark 12.14. Suppose

$$u = \mathcal{F}^{-1}\left(g(\theta)\delta_{\mathbb{S}^1}\right).$$

For instance, in 2 dimensions, we write

$$u(x) = \int_{\omega \in S^1} g(\omega) e^{ix\cdot\omega} = \int g(\cos\theta, \sin\theta) e^{i(x_1\cos\theta + x_2\sin\theta)}\, d\theta,$$

where $\omega = (\cos\theta, \sin\theta)$, which can be analyzed by the idea of stationary phase method by viewing $\lambda \sim |x|$ and hence has $\dfrac{1}{\sqrt{|x|}}$ decay. (In fact, due to the rotational symmetry, one can assume $x_1 = |x|$ and $x_2 = 0$.) Then it is natural to think of stationary phase.

In general dimensions, one can show that u is smooth with $|x|^{-(n-1)/2}$ decay by using a general stationary phase method. Recall that when we implement the stationary phase method, we need to consider the number of nonzero eigenvalues of the Hessian, which corresponds to the non-vanishing curvature of the sphere. (If one represent the ball locally as a graph of a function F, then the the number of nonzero eigenvalues of the Hessian, which corresponds to the non-vanishing curvature of the sphere.) For $\mathbb{S}^{n-1}$, we know it has exactly $n-1$ non-vanishing curvature, which gives $|x|^{-(n-1)/2}$ decay. The point is, these generalized eigenfunctions are almost L^2 with a lack of $1/2$ decay.

The conclusion for this example is $\sigma(-\Delta) = \mathbb{R}^+$, which is a continuous spectrum.

Remark 12.15. For some $-\Delta+V$ with V periodic, you may see band structure in its spectrum, that is, combination of continuous and discrete spectrum.

12.3 Hermite operator (Harmonic Oscillator)

For $-\Delta$ on $\mathbb{R}^n$, the reason why we do not have compact theorems is due to the translations. To kill the possibility of translation, we add a potential to it.

In $\mathbb{R}^n$, we consider the *Hermite operator* $-\Delta + |x|^2 := H$, which corresponds to

$$B(u,u) = \int Pu \cdot u = \int |\nabla u|^2 + |xu|^2\, dx := \|u\|^2_{H^1_H}.$$

12.3.1 Compactness embedding $H^1_H \subset\subset L^2$

Heuristically, given a function $u \in H^1_H$, if we consider the enemy for Rellich-Kondrachov produced by translation as in Section 2.5, then we would notice that when the translational parameter n is large enough, the x in the term $|xu|^2$ kicks in, which makes the norm sufficiently large. Therefore, we would expect that we have compact embeddings.

In fact, the same kind of proof by contradiction for Poincaré's inequality in [7] applies.

Proposition 12.16. *We have the compact embedding $H^1_H \subset L^2$.*

Proof. **Step 1 : H^1_H continuously embeds into L^2**

First, we show $H^1_H \subset L^2$ is a continuous embedding. It suffices to show

$$\|u\|_{L^2} \le C\|u\|_{H^1_H}.$$

Suppose not by contradiction, then there exists $\{u_n\}$ such that

$$\|u_n\|_{L^2} \ge n\|u_n\|_{H^1_H}.$$

Without loss of generality, we assume $\|u_n\|_{L^2} = 1$. Since $\|\nabla u_n\|_{L^2} \le 1/n$, we know $\|u_n\|_{H^1} \le 2$. Therefore, thanks to the Rellich-Kondrachov compactness theorem, $u_n \to u$ in $L^2(B_R)$ by passing to a subsequence. Moreover, since $R\|u_n\|_{L^2(B_R^c)} \le \|x\cdot u_n\|_{L^2} \le 1/n$, we know $u_n \to 0$ in $L^2(B_R^c)$. Here we abuse notation a little bit to denote u for the limit of u_n in $L^2(\mathbb{R}^n)$. Then we know that supp$u \subset \{0\}$ since we can freely choose $R > 0$.

However, since $\|\nabla u_n\|_{L^2} \le 1/n \to 0$, $\nabla u_n \to 0$ in L^2 and hence in $\mathcal{D}'$, which implies $u = 0$, which contradicts the fact that $\|u_n\|_{L^2} = 1$ and $u_n \to u$ in L^2. Thus, we complete the proof of continuous embedding.

Step 2 : H_H^1 continuously embeds into H^1

Furthermore, since the estimate $\|\nabla u\|_{L^2} \le \|u\|_{H_H^1}$ is trivial, we know that $H_H^1 \subset H^1$.

Step 3 : H_H^1 compactly embeds into L^2

We just need to modify the proof of Rellich-Kondrachov theorem a little bit. Suppose $\|u_n\|_{H_H^1} \le C$ holds uniformly. In the proof of Theorem 3.1, we examine the last step. For each $\delta > 0$, we can choose R sufficiently large such that $\|u_n\|_{L^2(B_R^c)} \le \delta$ which can be achieved since

$$R\|u_n\|_{L^2(B_R^c)} \le \|x\cdot u_n\|_{L^2(B_R^c)} \le C.$$

Then we apply the arguments exactly like what we did in B_R to obtain a ε such that $\|u_n^\varepsilon - u_n\|_{L^2(B_R)} \le \delta$ for all n. Moreover, we can select a subsequence $\{n_j\}$ and N sufficiently large such that for $j, k > N$, $\|u_{n_j}^\varepsilon - u_{n_k}^\varepsilon\|_{L^2(B_R)} \le \delta$. Therefore, for $j, k > N$,

$$\begin{aligned}&\|u_{n_j} - u_{n_k}\|_{L^2(\mathbb{R}^n)} \le \|u_{n_j} - u_{n_k}\|_{L^2(B_R)} + \|u_{n_j} - u_{n_k}\|_{L^2(B_R^c)}\\ \le&\|u_{n_j}^\varepsilon - u_{n_k}^\varepsilon\|_{L^2(B_R)} + \|u_{n_j}^\varepsilon - u_{n_j}\|_{L^2(B_R)} + \|u_{n_k}^\varepsilon - u_{n_k}\|_{L^2(B_R)} + \|u_{n_j}\|_{L^2(B_R^c)} + \|u_{n_k}\|_{L^2(B_R^c)} \le 5\delta,\end{aligned}$$

which completes the proof by a following diagonal argument on choosing subsequences. □

Remark 12.17. From the preceding proposition, H_H^1 is obviously a Hilbert space. Suppose u_n is Cauchy in H_H^1, then $u_n \to u$ in H^1 by the continuous embedding. Moreover, xu_n is Cauchy in L^2 so it converges to some $v \in L^2$. However, $u_n \to u$ in $\mathcal{D}'$ and hence $xu_n \to xu$ in $\mathcal{D}'$, so $v = xu \in L^2$, which completes the proof.

12.3.2 Spectrum of the operator $H = -\Delta + |x|^2$

Now we can apply the Lax-Milgram theorem or Riesz representation theorem to $B(u, v)$ to obtain the following result : for any $f \in L^2 = (L^2)^* \subset (H_H^1)^*$, one can find a weak solution $u \in H_H^1$ in the sense of $B(u, u) = \langle f, u\rangle$ and hence $\|u\|_{H_H^1} \le \|f\|_{L^2}$, which gives the boundedness of the inverse $L^2 \to H_H^1$. Moreover, by the coercivity, there are no negative eigenvalues for P.

On the other hand, the preceding proposition has an easy corollary that the spectrum of H is discrete thanks to the compactness of the inverse

$$L^2 \to H_H^1 \subset L^2.$$

Now we compute the spectrum of this operator. Since we have the decomposition

$$P = -\partial_1^2 + x_1^2 - \partial_2^2 + x_2^2 - \cdots,$$

it suffices to consider this in 1 dimension for

$$P = -\partial_x^2 + x^2 = -(\partial_x - x)(\partial_x + x) + 1.$$

We write $P-1 = -(\partial_x - x)(\partial_x + x)$. If $(\partial_x + x)u = 0$, then we can explicitly solve $u_0 = e^{-x^2/2}$, since it is positive, we know this corresponds to the first eigenvalue. (See [7, Section 6.5.2] for a similar result. Though we do not have Dirichlet boundary value, the $x \cdot u \in L^2$ still gives some decay at infinity.) (Though we are not in the Dirichlet boundary condition, we also require some decay at infinity.) We write

$$(\partial_x + x)P = P(\partial_x + x) + 2(\partial_x + x),$$

which means that $(\partial_x + x)u$ corresponds to $(\lambda - 2)$ and $(\partial_x - x)u$ corresponds to $(\lambda + 2)$ if u is an eigenfunction corresponding to λ. Therefore, eigenvalues are $1 + 2\mathbb{N}^*$ corresponding to $u_0 = e^{-x^2/2}$ and

$$u_k = (\partial_x - x)^k e^{-x^2/2} = p_k(x) e^{-x^2/2},$$

where p_k's are called the Hermite polynomials.

Remark 12.18. In general, suppose an operator P is of the form $P = -\Delta + V(x)$. The discrete spectrum depends on all the properties of V while the essential spectrum or continuous spectrum, only depends on properties of V at infinity.

Green functions and an intro to unique continuation

Date: February 28, 2023

13.1 Green functions : an analogue of fundamental solutions

We consider

$$\begin{cases} -\Delta u = f, \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Note that $u(x) = \int K(x-y)f(y)\,dy$, where K only depends on $x-y$ thanks to the translation invariance in $\mathbb{R}^n$. However, when the setting is a bounded domain, we would only expect to have a solution to $f = \delta_y$ for any fixed $y \in \Omega$ of the form $K = K_y(x) = G(x,y)$. If we had something like this, then we can find

$$u(x) = \int G(x,y)f(y)\,dy.$$

Our first guess would be $G_0(x,y) = K(x-y)$, then $\Delta G_0 = \delta_x$. However, $G_0|_{\partial\Omega} \neq 0$, so we need to introduce an error

$$G(x,y) = G_0(x,y) + R(x,y)$$

to force it to satisfy the boundary condition. For y fixed,

$$\begin{cases} -\Delta_x R(x,y) = 0, \\ R(x,y) = -G_0(x,y), \quad x \in \partial\Omega. \end{cases}$$

Solving this, we know that $R(\cdot,y)$ is harmonic and hence smooth as a function of x, which implies $G_0(\cdot,y)$ is smooth at the boundary thanks to the boundary condition $R(x,y) = -G_0(x,y)$. Therefore, the G given here allows us to solve the boundary value problem in a bounded domain.

Definition 13.1. We say G is the Green function for our boundary value problem.

Proposition 13.2. *The Green function satisfies the symmetric condition $G(x,y) = G(y,x)$.*

Remark 13.3. This symmetry holds for all the self-adjoint operators with Dirichlet boundary condition.

Proof. We denote $-\Delta_D$ to emphasize the boundary condition is Dirichlet. Thanks to the Dirichlet boundary condition, we compute

$$\int_\Omega -\Delta_D u \cdot v = \int_\Omega u \cdot (-\Delta_D v)$$

if we set

$$u(x) = G(x,y_1), \quad v(x) = G(x,y_2), \quad y_1, y_2 \in \Omega.$$

Note that this implies $\delta_{y_1}(v) = \delta_{y_2}(u)$, which completes the proof. $\square$

The symmetry property implies

$$G(x,y) = K(x-y) + R(x,y),$$

where R is harmonic both in x and y and also symmetric. There is one subtlety, as $y \in \Omega$ approaches the boundary, the smoothness is not uniform because the boundary condition becomes more and more singular as one pushes y to the boundary. It is useful to work out some simple examples.

Example 13.4. Set $\Omega = \mathbb{H}$ to be the half plane. Then $G(x,y) = K(x-y) - R(x,y)$. To find $R(x,y)$, we reflect y to y^* about $\partial\mathbb{H}$. By noticing $|x-y^*| = |x-y|$, we can choose $R(x,y) = K(x-y^*)$, which is smooth in $\mathbb{H}$ since the singularity is out of our domain.

Example 13.5. Set $\Omega = B$ to be the unit ball. Now we look for conformal symmetries, by which we mean $-\Delta \mapsto -f(x)\cdot\Delta$, which works well on harmonic functions. For conformal symmetries, distances are multiplied by $f(x)$ but it is angle preserving. A good conformal symmetry for the ball is the inversion, that is, $|y^*|\cdot|y| = 1$.

We compare $|x-y|$ with $|x-y^*|$ for $x \in \partial B$ by writing

$$|x-y^*|^2 = \left|x - \frac{y}{|y|^2}\right|^2 = 1 + \frac{1}{|y|^2} - 2\frac{x\cdot y}{|y|^2} = \frac{1}{|y|^2}|x-y|^2.$$

Therefore,

$$G(x,y) = \begin{cases} K(x-y) - |y|^{-(n-2)}K(x-y^*), & n \geq 3, \\ K(x-y) - K(x-y^*) + \ln|y|, & n = 2 \end{cases} = K(x-y) - K(|y|(x-y^*)).$$

If the boundary condition is nonzero, say

$$\begin{cases} -\Delta u = 0, \text{ in } \Omega, \\ u = g \text{ on } \partial\Omega. \end{cases}$$

We extend u by 0 outside Ω and denote it by $\bar{u}$. For $\bar{u}$, when you differentiate once, you see the jump at the boundary and hence get a dirac mass at the boundary. When you differentiate the second time, you also see the jump of normal derivative, and therefore you get

$$-\Delta\bar{u} = u|_{\partial\Omega}\cdot\delta'_{\partial\Omega} + \frac{\partial u}{\partial\nu}|_{\partial\Omega}\cdot\delta_{\partial\Omega},$$

where one can realizing this heuristic idea by acting on $\phi \in \mathcal{D}$:

$$\langle\Delta(\bar{u}\cdot 1_\Omega), \phi\rangle = \int_\Omega u\Delta\phi\,dx.$$

13.2 Introduction to potential theory

If we knew both u and $\frac{\partial u}{\partial\nu}$ on $\partial\Omega$, then

$$\bar{u}(x) = \int_{\partial\Omega} u(y)\frac{\partial}{\partial\nu_y}K(x,y)\,dy - \int_{\partial\Omega}\frac{\partial u}{\partial\nu_y}K(x,y)\,dy.$$

To justify this, we just need to integrate by parts to compute

$$\int -\Delta u(y) \cdot K(x-y)\, dy.$$

Unfortunately, given Dirichlet boundary condition or Neumann boundary condition, we cannot know u and $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$ at the same time. If we try to compute the contribution

$$u_0(x) = -\int_{\partial\Omega} g(y) \cdot \frac{\partial}{\partial \nu} K(x,y)\, dy,$$

then one would notice that we cannot determine the boundary value. Instead, we can know the jump of u_0 at the boundary is $[u_0]|_{\partial\Omega} = g$ provided $g \in C(\partial\Omega)$. One can find this result stated as a corollary of [9, Theorem 3.22], which says that

$$\lim_{x\to\partial\Omega, x\in\Omega} \int_{\partial\Omega} g(y) \cdot \frac{\partial}{\partial \nu} K(x,y)\, dy = \frac{1}{2} g(x_0) + u_0(x_0),$$
$$\lim_{x\to\partial\Omega, x\in\Omega^c} \int_{\partial\Omega} g(y) \cdot \frac{\partial}{\partial \nu} K(x,y)\, dy = -\frac{1}{2} g(x_0) + u_0(x_0).$$

The operator $-\frac{\partial}{\partial\nu} K : g \mapsto u_0$ is called the *double layer potential* with moment g. The phenomenon that approaching from inside of Ω and outside of Ω have different limits is in the same spirit of homogeneous distributions of -1, $\frac{1}{x+i0}$ and $\frac{1}{x-i0}$, which we introduced last semester using approximation from upper and lower half plane.

Also, one can look at

$$h \mapsto \int h(y) K(x,y)\, dy,$$

which is called the *single layer potential* with moment h.

Since $\dim \partial\Omega = n-1$, $K(x,y) = |x-y|^{2-n}$, we know $\frac{\partial}{\partial\nu} K(x,y)$ is an operator of order 0 and $K(x,y)$ is an operator of order -1.

Single and double layer potentials, which are good Fredholm operators and leads to the solvability results of the boundary value problems. These operators can be studied by the Calderon-Zygmund operator theory.

13.3 Introduction to unique continuation, Cauchy-Kowalevski theorem

A question is :

Can the solution to $-\Delta u = 0$ vanish in an open set ?

The answer is no because u is analytic. This is a simple example of unique continuation. In fact, the proof only requires that u and all its derivatives vanish at a single point. The property is worth having a name.

Definition 13.6. If the solution u satisfies the following property :

If u vanishes of infinite order at x_0, then $u \equiv 0$.

then we say it satisfies the strong unique continuation.

Example 13.7. Given $\Gamma \subset \partial\Omega$, we consider

$$\begin{cases} -\Delta u = 0 \text{ in } \Omega, \\ u = 0 \text{ in } \Gamma \subsetneq \partial\Omega, \end{cases}$$

with boundary value prescribed only on Γ, the solution is not unique since we can extend the boundary condition to the whole boundary in various ways and solve them to obtain different solutions by existence.

Example 13.8. If we put an additional condition,

$$\begin{cases} -\Delta u = 0 \text{ in } \Omega, \\ u = 0 \text{ in } \Gamma \subset \partial\Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ in } \Gamma \subset \partial\Omega, \end{cases}$$

where $\Gamma \subset \partial\Omega$ is open. By making an extension by 0 to $\bar{u}$, we have $-\Delta\bar{u} = 0$ around Γ thanks to the following observation. By flattening the boundary, we notice

$$u = 0, \quad (\partial')^\alpha u = 0, \quad \partial_n u = 0.$$

Then thanks to the equation, we know all the derivatives at the boundary are zero, which means that $\bar{u}$ vanishes on Γ of infinite order. One can view this as another unique continuation property.

This example motivates the study of the strategy to find analytic solutions :

$$\begin{cases} -\Delta u = 0, \\ u = f \text{ on } \Gamma, \\ \frac{\partial u}{\partial \nu} = g \text{ on } \Gamma, \end{cases}$$

where $\Gamma = \partial\Omega$ is smooth.

Suppose f, g are analytic, we can solve the problem by computing all derivatives of u on Γ using the same idea as in the preceding example, which is the Cauchy–Kowalevski theorem. The same computation applies for the full Taylor series. If the Taylor series is convergent, then it is a local solution.

We try to solve

$$P(x, D) = \sum_{|\alpha| \le K} c_\alpha(x) D^\alpha,$$

what we care about Γ is the normal direction with normal vector N. We look at the principal symbol

$$P_0(x, \xi) = \sum_{|\alpha| = K} c_\alpha(x) \xi^\alpha.$$

Definition 13.9. The boundary Γ is non-characteristic for P if $P_0(x, N) \neq 0$, that is, the principal symbol does not vanish along the normal direction.

This non-characteristic property will take place of the condition "we can compute the full Taylor series" in the previous baby version of Cauchy–Kowalevski theorem. This helps to determine some derivatives by using the equation itself as what we did in the preceding example.

Theorem 13.10 (Cauchy–Kowalevski)**.** *If $\partial\Omega = \Gamma$ is non-characteristic for P, then we have local solvability.*

Next time, we want to move away from the analytic class.

Unique continuation property and Carleman estimates

Date: March 2, 2023

14.1 Unique continuation without analyticity condition, Carleman estimates

We state our main theorem today in a general manner, but we only prove for $-\Delta$.

Theorem 14.1. *Let $P = -\partial_i a^{ij}\partial_j + b^j\partial_j + c$, where a^{ij} is Lipschitz and $b^j, c \in L^\infty$. If $Pu = 0$, and $u = 0$ in an open set, then $u \equiv 0$.*

As a remark, note that the regularity assumed here is stronger than the one in the existence theorem.

For simplicity, we only prove this when $a^{ij} = I$ to present the main idea. Without loss of generality, we assume $u = 0$ in $B(0,1)$. By making an inversion $x \mapsto x^* = \frac{x}{|x|^2}$, we know that P has a similar form in the sense that

$$-\Delta \mapsto -c(x)\Delta,$$

where the constant can be divided from both sides. Then what we need to do is to push the boundary of the unit ball inward.

We consider a small ball centered at a boundary point x_0 with a cutoff χ selecting this ball. Then

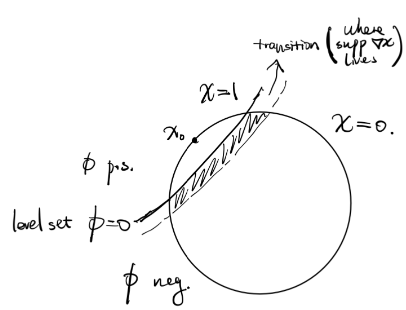
$$-\Delta v = -\Delta(\chi u) = \chi f + 2\nabla\chi\cdot\nabla u + \Delta\chi\cdot u,$$

where we just think of the equation as a perturbative way with $f = b^j\partial_j u + cu$. Therefore,

$$\|v\|_{H^2} = \|\chi u\|_{H^2} \lesssim \|\chi f + 2\nabla\chi\cdot\nabla u + \Delta\chi\cdot u\|_{L^2} \lesssim \|\chi f\|_{L^2} + \|u\|_{H^1(\mathrm{supp}\nabla\chi)}$$

thanks to the elliptic regularity. Since $f = b^j\partial_j u + cu$, we can absorb $\|\chi f\|_{L^2}$ to the left hand side by selecting the ball sufficiently small.

However, the last term is not small and difficult to manage. The very nice idea addresses this problem is that we do not weight thing properly in the preceding estimate. We want to add some weights which is large where we want to show u is zero (near x_0) but is small in $\mathrm{supp}\nabla\chi$. The idea is to add a weight which is large near x_0 and small in the shadow region.



To realize this, we need to choose one parameter family of weights. The idea is due to Carleman in 1930s, then Aronszajn generalized to higher dimensions in 1950s. See [15] for a brief history of the results on Carleman estimates. We choose an exponential weight $e^{\tau\phi}$, where τ is a large parameter. Moreover, we need the constant uniform in τ. First, we show Carleman estimates and then use this to prove unique continuation.

Theorem 14.2 (Carleman estimates)**.** *Suppose* $-\Delta v = f$ *and*

$$\tau^{\frac{3}{2}}\|e^{\tau\phi}v\|_{L^2} + \tau^{\frac{1}{2}}\|e^{\tau\phi}\nabla v\|_{L^2} \le C\|e^{\tau\phi}f\|_{L^2},$$

where the constant C *is uniform in* τ*.*

Remark 14.3. If you put H^2 on the left hand side, then the constant will blow up as $\tau \to \infty$. Heuristically, one would expect $\tau^{-\frac{1}{2}}$ as the coefficients for $\|e^{\tau\phi}\nabla^2 v\|_{L^2}$.

Proof of unique continuation property assuming Carleman estimates. Assuming this first, we show how to invoke Carleman estimates to prove unique continuation. For $v = \chi u$, we write

$$-\Delta u = B\nabla u + cu, \; -\Delta v = B\nabla v + cv + 2\nabla u \cdot \nabla\chi + u\Delta\chi - B(\nabla\chi)u,$$

where the last three terms are supported in the shadow region. By applying Carleman estimates with $v = \chi u$, we obtain

$$\tau^{\frac{3}{2}}\|e^{\tau\phi}v\|_{L^2} + \tau^{\frac{1}{2}}\|e^{\tau\phi}\nabla v\|_{L^2} \lesssim \|e^{\tau\phi}\nabla v\|_{L^2} + \|e^{\tau\phi}v\|_{L^2} + \|e^{\tau\phi}u\|_{H^1(\mathrm{supp}\nabla\chi)},$$

where we use $B, c \in L^\infty$. Moreover, we use the fact $\phi \le 0$ in $\mathrm{supp}\nabla\chi$ to see

$$\tau^{\frac{3}{2}}\|e^{\tau\phi}v\|_{L^2} + \tau^{\frac{1}{2}}\|e^{\tau\phi}\nabla v\|_{L^2} \lesssim \|u\|_{H^1(\mathrm{supp}\nabla\chi)} \le C\|u\|_{H^1}.$$

Now, let $\tau \to \infty$, we know $v = 0, \nabla v = 0$ in $\{\phi > 0\}$. Otherwise, the left hand side would tend to infinity, which violates the boundedness from above by $C\|u\|_{H^1}$. Therefore, by repeating this near each point on the boundary, we shrink the ball a little bit, which proves Theorem 14.1. ☐

14.2 Proof of Carleman estimate (Theorem 14.2)

Now we want to prove Carleman estimates, which is sort of one level up from elliptic regularity. We want to choose good weights ϕ to realize the picture above and obtain the estimates in Theorem 14.2. Note that not all weights would make the Carleman estimates to be true after putting it into the inequality. We need to determine what functions ϕ are good weights in Carleman estimates. Note that we want the estimates to be uniform with respect to the exponential weight, so when we prove the estimate, we want to take the weight out of the picture. Hence, we do a substitution $w = e^{\tau\phi}v$. If $-\Delta v = f$, then we start to derive the equation for w by writing

$$-e^{\tau\phi}\Delta v = e^{\tau\phi}f.$$

We compute

$$e^{\tau\phi}\partial_j v = \partial_j\left(e^{\tau\phi}v\right) - \tau\partial_j\phi(e^{\tau\phi}v) = (\partial_j - \tau\phi_j)(e^{\tau\phi}v)$$

and

$$-\sum_j(\partial_j - \tau\phi_j)^2 w = e^{\tau\phi}f := g.$$

For this, we need to prove the estimate

$$\tau^{\frac{3}{2}}\|w\|_{L^2} + \tau^{\frac{1}{2}}\|\nabla w\|_{L^2} \lesssim \|g\|_{L^2}.$$

Heuristically, if $\phi_j \sim 1$, then we want to think of τ having the same strength as a derivative when we consider the symbol, that is, ξ is comparable to τ in the following. The reason why we want this is the constant in the Carleman estimates should be independent of τ. Then the estimate we want above looks like an elliptic estimate. Naturally, we want to determine whether the operator

$$P_\tau := -\sum_j (\partial_j - \tau\phi_j)^2$$

is elliptic. The symbol is given by

$$P_\tau(x,\xi) = -\sum_j (i\xi_j - \tau\phi_j)^2 = |\xi|^2 - \tau^2|\nabla\phi|^2 + 2i\tau\phi_j\xi_j,$$

where the last term is imaginary. If $P_\tau(x,\xi) = 0$, then $\xi \perp \nabla\phi$ and $|\xi|^2 = \tau^2|\nabla\phi|^2$. The first condition means ξ lives on a plane which is perpendicular to $\nabla\phi$ and the second condition means that ξ lives on a sphere. However, the intersection of a sphere with a plane passing the origin is nontrivial. Thus, P_τ is not elliptic.

The second nice property to examine is the symmetry of the operator. Unfortunately, P_τ is not symmetric since the symbol is not real. We split the operator into two parts,

$$P_\tau = P_\tau^s + P_\tau^a,$$

where one is the symmetric part $P_\tau^s = -\Delta - \tau^2|\nabla\phi|^2$, the other is the anti-symmetric part $P_\tau^a = \tau(\partial_j\phi_j + \phi_j\partial_j)$. By quadratic formula and the symmetry , we compute

$$\|g\|_{L^2}^2 = \|P_\tau^s w + P_\tau^a w\|_{L^2}^2 = \|P_\tau^s w\|_{L^2}^2 + \|P_\tau^a w\|_{L^2}^2 + 2\mathrm{Re}\langle P_\tau^a w, P_\tau^s w\rangle = \|P_\tau^s w\|_{L^2}^2 + \|P_\tau^a w\|_{L^2}^2 + \mathrm{Re}\langle [P_\tau^s, P_\tau^a]w, w\rangle$$

Note that $[P_\tau^s, P_\tau^a]$ is second order, so we examine whether this is elliptic or not by computing $[-\Delta - \tau^2|\nabla\phi|^2, \tau(\partial_j\phi_j + \phi_j\partial_j)]$. We write

$$[\Delta, \partial_j\phi_j + \phi_j\partial_j]u = \partial_k^2(\partial_j(\phi_j u)) - \partial_j(\phi_j\partial_k^2 u) + \partial_k^2(\phi_j\partial_j u) - \phi_j\partial_j\partial_k^2 u = 4\phi_{jk}\partial_k\partial_j u + 6\phi_{jjk}\partial_k u + \phi_{jjkk}u$$

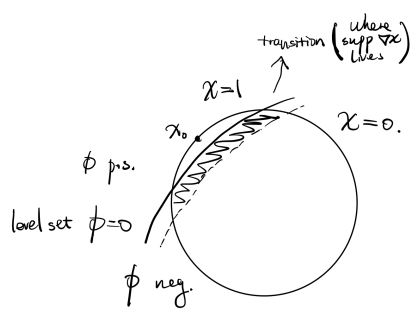
and hence

$$[-\Delta - \tau^2|\nabla\phi|^2, \tau(\partial_j\phi_j + \phi_j\partial_j)] = -\tau[\Delta + \tau^2|\nabla\phi|^2, (\partial_j\phi_j + \phi_j\partial_j)] = \tau\left(-4\partial_k\phi_{jk}\partial_j + 4\tau^2\phi_j\phi_{kj}\phi_k\right) + l.o.t.$$

Therefore, the principal symbol is

$$4\tau\phi_{jk}\xi_k\xi_j + 4\tau^3\phi_j\phi_{jk}\phi_k,$$

which is elliptic if ϕ is strictly convex ($D^2\phi > 0$) with τ, ξ viewed as the same strength keeping in mind. Therefore, we need to flip the concavity to convexity for the curve of level set as shown in the picture below.



Therefore, the commutator C satisfies

$$\langle Cw, w\rangle \simeq \tau\|\nabla w\|_{L^2}^2 + \tau^3\|w\|_{L^2}^2.$$

Definition 14.4. We say ϕ is strong pseudo-convex with respect to P if

$$[P_\tau^s, P_\tau^a] > 0$$

on $P_\tau = 0$ $(\mathrm{Re}P_\tau(x,\xi) = \mathrm{Im}P_\tau(x,\xi) = 0)$.

Remark 14.5. It seems like this definition still needs to be checked by computing the commutator. However, from the perspective of microlocal analysis, one can also paraphrase the condition $[P_\tau^s, P_\tau^a] > 0$ by doing an algebraic computation of the Poisson bracket $\{P_\tau^s(x,\xi), P_\tau^a(x,\xi)\}$.

See the unfinished book [22] for a full account of unique continuation.

Strong unique continuation principle

Date: March 7, 2023

We study $-\Delta u = Vu$ with $V \in L^\infty$.

Theorem 15.1 (Strong unique continuation property). *If u vanishes at x_0 of infinite order, then $u \equiv 0$.*

Note that we only have the regularity $V \in L^\infty$ here, so we need to make sense of "vanishing of infinite order" first. If u is analytic and vanishes of infinite order, then $\|u\|_{L^2(B(x_0,r))} \leq C_N r^N$ for any N as $r \to 0$. This motivates the following definition.

Definition 15.2. We say u vanishes of infinite order at x_0 if for any N,

$$\|u\|_{L^2(B(x_0,r))} \leq C_N r^N$$

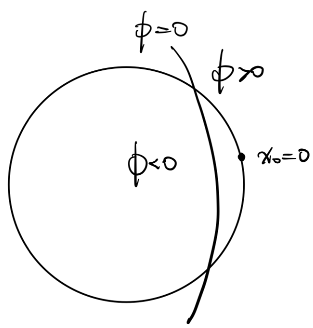
as $r \to 0$.

Our objective is to find a good Carleman estimate for Δ. By translation, we assume without loss of generality that $u(0) = 0$. Then we need to push the zero set at 0 outward. If we try to write down a Carleman estimate of the form :

$$\|e^{\tau\phi} u\| \leq \|e^{\tau\phi}\Delta u\|,$$

then we need to choose ϕ to blow up at 0 and decrease as it moves out. So this looks like the picture last time.

We also know that we need convexity of ϕ.



The idea is to choose $\phi = -\ln r = -\ln|x|$. However, thanks to scaling property, we put $|x|^{-\tau-2}$ in front of u as the weight instead of $|x|^{-\tau}$.

Proposition 15.3 (Carleman estimates). *Suppose u satisfies $-\Delta u = Vu$ with $V \in L^\infty$ and u vanishes of infinite order at 0, the following Carleman estimate*

$$\||x|^{-\tau-2}u\| \leq \||x|^{-\tau}\Delta u\|_{L^2}$$

holds uniformly for τ away from half integers and integers.

Remark 15.4. One can also add a gradient term to the left hand side as the Carleman estimates discussed last time.

Remark 15.5. We can see why we need the condition that u vanishes of infinite order at 0 for Proposition 15.3 to hold from the example of harmonic functions. If we choose a harmonic function which is 1 at 0 and cut it off away from some large ball, then the left hand side would be small enough with the left hand side blows up for τ big enough.

15.1 Carleman estimates implies strong unique continuation property

Write $v = \chi u$ with $\chi = 1$ in $B(0, R)$ and $\chi = 0$ outside $B(0, 2R)$. For v,

$$-\Delta v = Vv + f,$$

where f is the truncation error, which lives in the transition region. Then we compute

$$\||x|^{-\tau-2}v\|_{L^2} \lesssim \||x|^{-\tau}\Delta v\|_{L^2} \lesssim \||x|^{-\tau}Vv\|_{L^2} + \||x|^{-\tau}f\|_{L^2},$$

where the first term on the right can be absorbed into the left when R small since $V \in L^\infty$ implies $|V| \ll |x|^{-2}$ near 0, i.e. $|V| \leq \delta|x|^{-2}$ for δ small enough. In fact, if we weaken the condition on V so that we only have $|V| \leq C|x|^{-2}$ for some large constant C, then we cannot derive the same conclusion. This phenomenon here is also related to spherical harmonics and Laplacian on the circle. Moreover, $\||x|^{-\tau}f\|_{L^2} \lesssim R^{-\tau}$ by the support property.

Therefore,

$$\||x|^{-\tau-2}u\|_{L^2(B_R)} \leq \||x|^{-\tau-2}v\|_{L^2} \lesssim R^{-\tau},$$

which implies

$$\left\||x|^{-2}(R/|x|)^\tau u\right\|_{L^2(B_R)} \lesssim 1.$$

When $|x| < R$ and $\tau \to \infty$, the left hand side blows up unless $u \equiv 0$ in $|x| \leq R$, which completes the proof.

In fact, the proof above is not rigorous and we do not take advantage of the vanishing of infinite order. When absorbing the term $\||x|^{-\tau}Vv\|_{L^2}$ to the left hand side, we need to make sure that it is indeed finite, which requires the property of vanishing to infinite order. We prove that $\||x|^{-\tau}u\|_{L^2(B_R)} < \infty$. We use dyadic decomposition and consider

$$\||x|^{-\tau}u\|_{L^2(2^{j-1}<|x|<2^j)} \leq \|u\|_{L^2(2^{j-1}<|x|<2^j)}2^{-j\tau} \leq C_N 2^{j(N-\tau)}.$$

By choosing $N > \tau + 1$, we know

$$\||x|^{-\tau}u\|_{L^2(|x|<2^J)} \leq \sum_{j=-\infty}^{J} C_N 2^{j(N-\tau)} \lesssim 2^{J(N-\tau)},$$

which is finite.

15.2 Proof of the Carleman estimate (Proposition 15.3)

We write $P = -\Delta$, $P_\tau = -e^{\tau\phi}Pe^{-\tau\phi}$ and $P_\tau = P_\tau^s + P_\tau^a$. What we want is $[P_\tau^s, P_\tau^a] > 0$. If we choose $\phi = -\ln|x|$ and mimic the computations last time, we would see $[P_\tau^s, P_\tau^a] = 0$, which means that the weight $|x|^{-\tau}$ is degenerate pseudo-convex.

This time, we set $P_\tau = |x|^{-\tau}\Delta|x|^{\tau+2}$ with $w = |x|^{-\tau-2}u$, then we prove

$$\|w\|_{L^2} \leq \|P_\tau w\|_{L^2}.$$

By switching to the polar coordinates $x = r\Theta$, we just discard the Jacobian since they are the same at both sides, we have

$$\|w\|_{L^2(\mathbb{R}^+\times\mathbb{S}^{n-1})} \leq \|P_\tau w\|_{L^2(\mathbb{R}^+\times\mathbb{S}^{n-1})}.$$

Recall $\Delta = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_S$, so

$$\begin{aligned} P_\tau =& r^{-\tau}\left(\partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_S\right) r^{\tau+2} = r^{-\tau}\left(\partial_r^2 r^2 + \frac{n-1}{r}\partial_r r^2 + \Delta_S\right) r^\tau \\ =& r^{-\tau}\left(r\partial_r\cdot r\partial_r + 3r\partial_r + 2 + (n-1)r\partial_r + 2(n-1) + \Delta_S\right) r^\tau. \end{aligned}$$

To make $\partial_s = r\partial_r$, we need to have $dr = e^s ds$, that is, setting $r = e^s$. Note that the substitution $r = e^s$ will turn the scaling invariance to a translation invariance and hence the operator would be turned into a constant coefficients operator. In fact, we can predict from this invariance heuristic before we did the computation. Then we have

$$P_\tau = e^{-\tau s}(\partial_s^2 + (n+2)\partial_s + 2n + \Delta_S)e^{\tau s}.$$

Since $e^{-\tau s}\partial_s e^{\tau s} = \partial_s + \tau$, we have

$$P_\tau = (\partial_s + \tau)^2 + (n+2)(\partial_s + \tau) + 2n + \Delta_S.$$

To prove $\|w\|_{L^2} \leq \|P_\tau w\|_{L^2}$, we have $P_\tau(\xi,\mu) = -\xi^2 + \tau^2 + (n+2)\tau + 2n - \mu + i(2\tau + n + 2)\xi$, where ξ corresponds to s and μ is the eigenvalue of Δ_S. We want

$$|P_\tau(\xi,\mu)| \geq C$$

uniformly in ξ, μ. Suppose by contradiction that $P_\tau(\xi,\tau) = 0$, then

$$(2\tau + n + 2)\xi = 0 \text{ and } -\xi^2 + \tau^2 + (n+2)\tau + 2n - \mu.$$

If τ cannot be half integers or integers, then $\tau \neq -\frac{n+2}{2}$ and hence $\xi = 0$. Furthermore, $\tau^2 + (n+2)\tau + 2n - \mu = 0$. From the previous lecture, we know $\mu = \sigma(\sigma + n - 2)$ for $\sigma \in \mathbb{Z}$, then

$$\tau^2 + (n+2)\tau + 2n = \sigma(\sigma + n - 2),$$

which implies $\tau = \sigma - 2$. Therefore, if τ cannot be half integers or integers, then $|P_\tau(\xi,\mu)| > 0$. The final step is to show that when

$$|\tau - \sigma'| \geq C, \quad \forall \sigma' \in \mathbb{Z} \text{ and } \tau \neq -\frac{n+2}{2},$$

we know

$$|P_\tau(\xi,\mu)| \geq \tilde{C}$$

thanks to the coercivity at infinity in τ.

Remark 15.6. In the sketch of proof above, we only consider the change of operator itself. If we also take the change of Jacobian into account, then one would notice that the only difference is that we may end up with another term e^{cs} for some constant c in the estimates. However, this would not affect the conclusion.

Before we end the lecture, we make a comment on why we need to avoid τ half-integers. We consider homogeneous harmonic polynomials $u = p_k(x)$ of order k as our starting point to construct counterexamples. Since p_k is not of compact support, we take $u_{\varepsilon,R} = \chi_{\varepsilon,R} p_k(x)$, where $\chi_{\varepsilon,R} = \chi^1_\varepsilon(x) \cdot \chi^2_R(x)$ and

$$\chi^1_\varepsilon(x) = \chi^1(x/\varepsilon), \quad \chi^2_R(x) = \chi^2(x/R), \quad \chi^1(y) = 1 - \chi^2(y)$$

and $\chi^2 \in \mathcal{D}$ is 1 near 0 and 0 outside a large region, which is between 0 and 1 in the transition region.

If u is a homogeneous harmonic polynomial, the left hand side is approximately

$$\int ||x|^{-\tau-2} u_{\varepsilon,R}|^2 \, dx \simeq \int_\varepsilon^R r^{-2(\tau+2)} \cdot r^{2k} r^{n-1} \, dr$$

while the right hand side is

$$\begin{aligned}
\int ||x|^{-\tau} \Delta u_{\varepsilon,R}|^2 \, dx &\simeq \int_\varepsilon^{2\varepsilon} r^{-2\tau} \cdot (r^{2(k-1)} |(\chi^1_\varepsilon)'|^2 + r^{2k} |(\chi^1_\varepsilon)''|^2) r^{n-1} \, dr \\
&\quad + \int_R^{2R} r^{-2\tau} \cdot (r^{2(k-1)} |(\chi^2_R)'|^2 + r^{2k} |(\chi^2_R)''|^2) r^{n-1} \, dr \\
&\simeq \int_\varepsilon^{2\varepsilon} r^{-2\tau} \cdot r^{2(k-2)} r^{n-1} \, dr + \int_R^{2R} r^{-2\tau} \cdot r^{2(k-2)} r^{n-1} \, dr,
\end{aligned}$$

where the LHS and RHS have the same power of r in the integrand. Therefore, if $-2(\tau + 2) + 2k + (n-1) = -1$, which is $\tau = -2 + k + \frac{n}{2}$, then the left hand side is like $\log(R/\varepsilon)$ and the right hand side only has contribution from the intermediate region, which is bounded (by $C \ln 2$), which violates the Carleman estimates in Proposition 15.3. Note that for any other τ except $\tau = -2 + k + \frac{n}{2}$, this is not a counterexample anymore and this is an explicit example which tells us why we need to avoid half integers.

In fact, we do not need τ to be positive in the computation above. If we choose $\tau \to -\infty$ away from half integers and integers, then the weight blows up at infinity instead of at 0, which leads us to the unique continuation from infinity.

Theorem 15.7 (Unique continuation from ∞)**.** *Suppose $-\Delta u = Vu$ with $|V| \ll |x|^{-2}$ near ∞. If u vanishes of infinite order at ∞, then $u \equiv 0$.*

Since an inversion can turn the Laplacian to a multiple of Laplacian, this can be proved by making an inversion and turn the case at infinity to the case at zero.

Hopf's lemma, Harnack principle and regularity for Perron's method

Date: March 9, 2023

Note that the solutions provided by Perron's method are just bounded, but one can require their regularity. The method applies to both linear and nonlinear elliptic equations.

16.1 Hopf's lemma for $-\Delta u = 0$

First, recall the maximum principle we introduced before for certain class of second order elliptic operators.

Theorem 16.1 (Weak maximum principle)**.** *Suppose $Pu \leq 0$, then $\max_{\Omega} u = \max_{\partial\Omega} u$.*

Theorem 16.2 (Strong maximum principle)**.** *Suppose the maximum of u to $Pu = 0$ is attained inside, then $u \equiv C$.*

We consider $-\Delta u = 0$ in Ω. Suppose $x_0 \in \partial\Omega$ is a maximum point of u and that u does not vanish in the whole region Ω. Since x_0 is a maximum point, then $T \cdot \nabla u(x_0) = 0$ for T tangent to $\partial\Omega$ at x_0. (This requires $\partial\Omega \in C^1$.) For N outer normal, $N \cdot \nabla u(x_0) \geq 0$. By Hopf's lemma, $\frac{\partial u}{\partial \nu}(x_0) > 0$, that is, the inequality is strict. (To apply this, a sufficient condition is C^2. Instead, we only need one sided C^2, which is the so-called ball condition, which is depicted below.)

Proposition 16.3 (Hopf's lemma)**.** *Suppose u solves $-\Delta u = 0$, then there is some $x_0 \in \partial\Omega$ such that $u(x_0) = M = \max u$ and $u(x) < M$ in Ω. Suppose $B(x_1, r) \subset \Omega$ is a ball centered at x_1 tangent with $\partial\Omega$ at x_0. Then $\frac{\partial u}{\partial \nu}(x_0) > 0$.*

Proof. Since $B(x_1, r/2) \subset B(x_1, r) \subset \Omega$, $u(x) \leq M - \delta$ for some $\delta > 0$ for all $x \in B(x_1, r/2)$. In the annulus, $-\Delta u = 0$. We lift u inside the annulus

$$\tilde{u} = u + \varepsilon(K(x - x_1) - K(r)),$$

where K is the fundamental solution. Note that $\tilde{u}$ is still harmonic. On the outer boundary $\partial B(x_1, r)$, $\tilde{u} = u \leq M$. On the inner boundary $\partial B(x_1, r/2)$, $\tilde{u} \leq M - \delta + \varepsilon(K(r/2) - K(r)) \leq M$ for ε small enough. Now we apply the maximum principle in the region $B(x_1, r) \setminus B(x_1, r/2) := A$, then we know $\max_A \tilde{u} \leq \max_{\partial A} \tilde{u} = M$. Therefore,

$$\frac{\partial \tilde{u}}{\partial \nu}(x_0) \geq 0$$

and hence

$$\frac{\partial u}{\partial \nu}(x_0) \geq -\varepsilon \frac{\partial K}{\partial \nu}(x_0) > 0.$$

$\square$

Remark 16.4. For general second elliptic operator P with $c \equiv 0$, one can also prove Hopf's lemma. Moreover, Hopf's lemma leads to strong maximum principle. See [7, Section 6.4.2].

16.2 Harnack's inequality

Harnack's inequality is a type of estimates similar to the maximum principle. Given $P = -a^{ij}\partial_i\partial_j + b^i\partial_i + c$ with $a^{ij} \in L^\infty$, bounded and uniform elliptic. Suppose $Pu = 0$ in Ω, then the maximum principle holds. In addition, we assume $u > 0$ in Ω and $u \in C^2(\Omega)$. (u may blow up near the boundary.) Suppose $K \subset \Omega$, then Harnack's inequality allows us to compare the minimum and maximum of u in K :

$$\min_K u \le \max_K u \le C \min_K u.$$

where C only depends on K, Ω, P.

Theorem 16.5 (Harnack's inequality)**.** *Assume $a^{ij} \in L^\infty$ uniformly elliptic, $b^j \in L^\infty$, then there exists $C = C(K, \Omega, a, b)$ such that $\max_K u \le C \min_K u$ for all $u \ge 0$ in Ω.*

Remark 16.6. Note that if u vanishes at one point, then u vanish in Ω in this case. It looks like the strong unique continuation, but in this case, we require $u \ge 0$.

Remark 16.7. If a^{ij}, b^j, c are bounded measurable, see [11, Chapter 8] for a detailed proof. If a^{ij} is barely C^2 and $b^j, c \equiv 0$, see [7, Section 6.4] for a proof. We only give a proof for $P = -\Delta$ using Green's functions.

Proof. Recall that the Green's function $G(x, y)$ in Ω which satisfies $PG(x, y) = \delta_y$, $G(x, y) = 0$ if $y \in \partial\Omega$, helps us to solve $Pu = f, u|_{\partial\Omega} = 0$ by $u(x) = \int G(x, y)f(y)\,dy$. By Green's theorem,

$$\int_\Omega Pu \cdot v = \int_\Omega u \cdot Pv - \int_{\partial\Omega} u\frac{\partial v}{\partial\nu} - v\frac{\partial u}{\partial\nu}\,d\sigma.$$

Choosing $v = G(x, y)$, then $u(y) = \int G(y, x)f(x)\,dx - \int_{\partial\Omega} u(y)\frac{\partial G}{\partial\nu_y}(x, y)\,d\sigma$. Therefore, this tells us the Green's function can also give a representation of the solution when $f = 0$, g nontrivial

$$\begin{cases} Pu = 0 \text{ in } \Omega, \\ u = g \text{ on } \partial\Omega, \end{cases}$$

by $u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial\nu_y}(x, y)\,d\sigma(y)$.

For $x \in \Omega$ fixed, $G(x, y)$ (solves $-\Delta_y G(x, y) = \delta_x(y)$) satisfies $-\Delta_y G(x, y) = 0$ when $y \in \partial\Omega$. Thanks to the maximum principle, $\frac{\partial G}{\partial\nu_y}(x, y) \le 0$ for $y \in \partial\Omega$ and by Hopf's lemma, we know $\frac{\partial G}{\partial\nu_y}(x, y_0) < 0$ for some $y_0 \in \partial\Omega$. Therefore, $u(x) > 0$ if g is positive on the boundary. Note that the inequality $\frac{\partial G}{\partial\nu_y}(x, y) < 0$ is only uniform (in x) when x is away from the boundary. (This is why we want to consider in a compact set K.)

Since K is compact, then it suffices to prove the Harnack's inequality for a ball. For $x \in B$, apply Green's formula in $2B$, then

$$u(x) = \int_{\partial(2B)} g(y) \cdot \left(-\frac{\partial}{\partial\nu}G(x, y)\right)\,d\sigma.$$

As long as $x \in B$, $-\frac{\partial}{\partial\nu}G(x, y)$ is strictly positive by Hopf's lemma and hence

$$C_1 \le -\frac{\partial}{\partial\nu}G(x, y) \le C_2$$

uniformly for all $x \in B$. Therefore,

$$C_1 \int_{\partial(2B)} g(y)\, d\sigma \leq u(x) \leq C_2 \int_{\partial(2B)} g(y)\, d\sigma.$$

In particular,

$$\max_B u \leq \frac{C_2}{C_1} \min_B u.$$

□

Remark 16.8. What matters here is the size of the Green's function. But one would not see this phenomenon if one reads the proof in [7].

Remark 16.9. Note that the constant C dependes on the scale $\operatorname{diam}\Omega$. One can use a scaling argument to see how the constant changes and it is better in small scales.

16.3 Application of Harnack's inequality - Improving regularity from L^∞ to C^ε

The important case for Harnack's inequality is the case when the coefficients are bounded and measurable. This is the key ingredient for upgrading the regularity from continuous solutions to the ones with higher regularity to nonlinear elliptic equations.

The hardest step is the first step to improve $u \in L^\infty$ to $u \in C^\varepsilon$. Then the steps are improving to $u \in Lip^1$ and then $u \in C^{1,\varepsilon}$. Note that we cannot obtain $u \in C^2$ in general. The Harnack's inequality can help us to acheive the first step.

Proposition 16.10. *Suppose Harnack's inequality holds for P, then $u \in L^\infty$ implies $u \in C^\varepsilon$ for some small ε.*

Remark 16.11. We do not require u to be positive in the assumptions.

Proof. We want to show

$$|u(x) - u(y)| \leq Cr^\varepsilon, \quad |x - y| \leq r.$$

Since $u \in L^\infty$, this holds for $r = 1$ with some C. We argue by induction on r. Set

$$A(R) := \sup_{|x-y|\leq R} |u(x) - u(y)|.$$

Step 1 : Proving $A(R) \leq LA(2R)$ with a universal constant $L < 1$ will suffice
Indeed, this implies

$$A(1/2^n) \leq L^n A(1) = (\frac{1}{2^n})^\varepsilon A(1),$$

which completes the proof for $r = \frac{1}{2^n}$ for any n and is enough.

Step 2 : Induction on r

For $B_r \subset B_{2r}$, we know u varies by at most $A(2r)$ with $M \leq u \leq M + A(2r)$ in $B(2r)$. By applying Harnack's inequality to $v = u - M \geq 0$ on $\Omega = B(2r)$ with $K = \overline{B(r)}$, we get

$$\max_{B_r} v \leq C(r) \min_{B_r} v.$$

Moreover, $0 \leq \min_{B_r} v \leq \max_{B_r} v \leq A(2r)$. Note that we can choose $C(r)$ independent of r for all $|r| \leq 1$ since

$$\min_{B_r} v \geq \min_{B_1} v \geq C(1) \max_{B_1} v \geq C(1) \max_{B_r} v.$$

We denote $C = C(1)$. Then there are two cases to consider :

(1) $\max_{B_r} v \leq \frac{1}{2}A(2r)$: Then it is obvious that

$$\max_{B_r} v - \min_{B_r} v \leq \frac{1}{2}A(2r),$$

which implies $A(r) \leq \frac{1}{2}A(2r)$.

(2) $\max_{B_r} v \geq \frac{1}{2}A(2r)$: Then $\min_{B_r} v \geq \frac{1}{2C}A(2r)$ and hence

$$\max_{B_r} v - \min_{B_r} v \leq A(2r) - \frac{1}{2C}A(2r) = (1 - \frac{1}{2C})A(2r).$$

Then we just choose $L = \max\{\frac{1}{2}, 1 - \frac{1}{2C}\}$. □

Remark 16.12. In the proof of Proposition 11.5, we also mentioned that we can use Harnack's principle to derive the strict positivity of the eigenfunction corresponding to the first eigenvalue.

Nonlinear elliptic PDEs

Date: March 14, 2023

We begin by a simple classification of elliptic PDEs :

(1) Linear equations : $-\Delta u = f$, $-\partial_i a^{ij}\partial_j u = f$.
(2) Semilinear equations : $-\Delta u = f(u) + g$.
(3) Quasilinear equations : $-\partial_i a^{ij}(u)\partial_j u = f$.
(4) Fully nonlinear equations : $F(D^2 u) = 0$.

A typical class of semilinear equations is of the form

$$-\Delta u \pm u|u|^{p-1} = f.$$

We would focus on this equation in a moment.

17.1 Fully nonlinear equations

We say $F(D^2 u) = 0$ is elliptic if $F(Y) \geq F(X)$ for all X, Y symmetric and $Y \geq X$. To compare with the definition in the linear case, we record the definition of uniform ellipticity in [5].

Definition 17.1. F is uniformly elliptic if there are two positive constants $\lambda \leq \Lambda$ such that for any symmetric matrix M and $x \in \Omega$,

$$\lambda\|N\| \leq F(M+N, x) - F(M, x) \leq \Lambda\|N\|, \quad \forall N \geq 0,$$

where $\|N\|$ denotes the $L^2 - L^2$ norm of N, which is the largest eigenvalue whenever $N \geq 0$.

A classical example is the Monge-Ampere equation. Set $F(X) = \det X$, then F is elliptic when we restrict to all symmetric nonnegative matrices. Equivalently,

$$\det D^2 u = f(x)$$

is only elliptic for functions u that are strictly convex in Ω. Therefore, for such a solution u to exist, we must have f positive.

17.2 Semilinear equations with the favorable sign

For the favorable semilinear equation

$$-\Delta u + u|u|^{p-1} = f,$$

we can think of this variationally by viewing

$$\mathcal{L}(u) = \int \frac{1}{2}|\nabla u|^2 + \frac{1}{p+1}|u|^{p+1} - fu\, dx.$$

For a convex variational problem

$$\mathcal{L} : \dot{H}^1(\mathbb{R}^n) \to \mathbb{R} \text{ or } \mathcal{L} : H_0^1(\Omega) \to \mathbb{R},$$

in order to look for a minimizer of $\mathcal{L} : \dot{H}^1 \to \mathbb{R}$, we first notice that $\mathcal{L}$ is smooth if $\displaystyle\int |u|^{p+1}\, dx < \infty$. In $\mathbb{R}^n$, the inequality

$$\|u\|_{L^{p+1}} \leq \|\nabla u\|_{L^2}$$

only holds for $p = p_c = 1 + \frac{4}{n-2}$. On the other hand, we need $p \le p_c$ in Ω.

If boundedness does not hold but $\mathcal{L}$ is still convex, then we can consider

$$\mathcal{L} : \mathcal{D}(\mathcal{L}) \to \mathbb{R}$$

restricted on the domain of $\mathcal{L} \subset H^1$ to prove the existence of a minimizer in the domain.

For this favorable sign, we have maximum principle for positive solutions, which can be proved as usual. Then one can adapt Perron's method to this equation.

17.3 Semilinear equations with the unfavorable sign

For the unfavorable sign, we have

$$\mathcal{L}(u) = \int \frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1} - fu\,dx.$$

This is not convex and not coercive. Indeed, for $\phi \in \mathcal{D}$,

$$\mathcal{L}(\lambda\phi) \to -\infty, \text{ as } \lambda \to \infty$$

provided $p + 1 > 2$, then it cannot satisfy

$$\lim_{\|u\|\to\infty} \mathcal{L}(u) = \infty.$$

The idea to solve this is to apply the perturbative methods. Obviously, for $f = 0$, $u = 0$ is a solution. Then we want to find a small solution u provided f small. To make it simpler, we consider the critical exponent $p = p_c$ case with the domain $\dot{H}^1(\mathbb{R}^n)$. For

$$-\Delta u = u|u|^{p_c-1} + f,$$

with $f \in \dot{H}^{-1}$ small, we phrase the problem as a fixed point problem, that is,

$$u = (-\Delta)^{-1}\left(u|u|^{p_c-1} + f\right)$$

and denote the right hand side by $N(u) + u_f$, where $u_f = (-\Delta)^{-1}f \in \dot{H}^1$. Since $u \in \dot{H}^1 \subset L^{2^*}$,

$$u \cdot |u|^{p_c-1} \in L^{\frac{2^*}{p_c}} \subset \dot{H}^{-1}, \tag{17.1}$$

where the last inclusion follows from

$$\frac{2^*}{p_c} = \frac{2n/(n-2)}{(n+2)/(n-2)} = \frac{2n}{n+2} = (2^*)'.$$

From this, one can see the importance of the choice of $p = p_c$. Therefore, $N(u) \in \dot{H}^1$ and

$$\tilde{N} : \dot{H}^1 \to \dot{H}^1, \quad u \mapsto N(u) + u_f, \tag{17.2}$$

which is a fixed point problem.

Remark 17.2. For $p \ne p_c$, we need to replace $\dot{H}^1$ by other homogeneous $\dot{H}^s$, where s can be determined by the scaling property so that an analogous inclusion (17.1) holds.

Remark 17.3. We need to work in the dimensions where the homogeneous Sobolev spaces are not a quotient space so that we have the Sobolev embeddings as desired.

In functional analysis, we have Banach contraction principle.

Definition 17.4. Given a complete metric space X and a map $N : X \to X$, we say N is a contraction if $d(N(x), N(y)) \le Ld(x, y)$ with $L < 1$.

Theorem 17.5 (Banach contraction principle)**.** *If $N : X \to X$ is a contraction, then it has a unique fixed point.*

Proof. First, we prove uniqueness by contradiction. Suppose $u = N(u)$, $v = N(v)$, then

$$d(u, v) = d(N(u), N(v)) \le Ld(u, v),$$

which implies $d(u, v) = 0$.

Then we start from u_0 and define a sequence $u_{n+1} = N(u_n)$ to show that u_n converges. We compute

$$d(u_{n+1}, u_n) = d(N(u_n), N(u_{n-1})) \le Ld(u_n, u_{n-1}),$$

and by iterating, the distance between consecutive elements would goes to 0 exponentially as $d(u_{n+1}, u_n) \le L^n d(u_0, u_1)$, which implies $\{u_k\}$ is Cauchy. By completeness, $u_n \to u$ and we can pass the limit. $\square$

Note that $N(u)$ is superlinear, so it cannot be Lipschitz if we consider H^1. Instead, we choose $X = B_{H^1}(0, R)$, we need to choose R carefully. If R is too big, then we lose Lipschitz condition, while if R is too small, then the map (17.2) cannot take the ball X to itself.

Since

$$\begin{aligned} \|N(u)\|_{\dot{H}^1} &\le \|u|u|^{p-1}\|_{\dot{H}^{-1}} \le C\|u\|_{\dot{H}^1}^p, \\ \|N(u) + u_f\|_{\dot{H}^1} &\le CR^p + \|f\|_{\dot{H}^{-1}} \end{aligned} \tag{17.3}$$

provided that $\|u\|_{\dot{H}^1} \le R$. Therefore, if $\|f\|_{\dot{H}^{-1}} \ll 1$, then we can choose $R = \tilde{C}\|f\|_{\dot{H}^{-1}}$ so that $\tilde{N} : X \to X$.

Furthermore, in order to prove the Lipschitz property, we expect $|N(u) - N(v)|$ is bounded from above by the product of $|u - v|$ and a function of homogeneity $p - 1$. Indeed, by making an algebraic substitution $w = u/v$, one can show

$$\left| u|u|^{p-1} - v|v|^{p-1} \right| \le C|u - v| \cdot \left(|u|^{p-1} + |v|^{p-1} \right).$$

Moreover, this implies

$$\begin{aligned} \|N(u) - N(v)\|_{\dot{H}^1}^q \le & \|u|u|^{p-1} - v|v|^{p-1}\|_{\dot{H}^{-1}}^q \lesssim \left\| |u - v| \cdot \left(|u|^{p-1} + |v|^{p-1} \right) \right\|_{L^q}^q \\ \lesssim & \|u - v\|_{L^{2^*}} \left(\|u\|_{L^{2^*}}^{p-1} + \|v\|_{L^{2^*}}^{p-1} \right) \lesssim \|u - v\|_{\dot{H}^1} \left(\|u\|_{\dot{H}^1}^{p-1} + \|v\|_{\dot{H}^1}^{p-1} \right), \end{aligned}$$

where $q = 2^*/p$ is the one in (17.1). This implies that $\tilde{N}$ is a contraction if and only if $L = CR^{p-1} < 1$.

Theorem 17.6. *If $\|f\|_{\dot{H}^{-1}} \ll 1$, then there exists a unique solution $u \in \dot{H}^1$ such that $\|u\|_{\dot{H}^1} \ll 1$, where the uniqueness only holds in a small ball in $\dot{H}^1$.*

To construct a solution numerically, we start with a guess $u_0 = 0$ and define $u_{n+1} = N(u_n) + u_f$, which implies a convergence $u_n \to u$ in $\dot{H}^1$, which makes it possible to run iteration. In fact, if we choose u_0, u_1 in the initial step such that $u_1 > u_0$ pointwisely, then we write

$$u_{n+1} - u_n = N(u_n - u_{n-1}) = (-\Delta)^{-1}(u_n|u_n|^{p-1} - u_{n-1}|u_{n-1}|^{p-1}),$$

which implies $u_{n+1} - u_n \geq 0$ pointwisely if $u_n - u_{n-1} \geq 0$ pointwisely thanks to the maximum principle for the Laplacian operator $-\Delta$.

Remark 17.7. This method also works for the favorable sign case.

Remark 17.8. The iteration scheme even works for quasilinear problem. Instead, we cannot think of this as a contraction. The idea is to use a two layer scheme by considering in two different Sobolev spaces where we prove boundedness in the top Sobolev space and we have convergence in a weaker Sobolev space. We will see the same idea in proving local existence of hyperbolic systems.

17.4 Mountain pass theorem

Another useful tool to study the unfavorable sign case variationally is the mountain pass theorem. We briefly introduce the idea and refer to [7, Chapter 8.5] for details. Recall the setting

$$\mathcal{L}(u) = \int \frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\, dx, \quad f = 0$$

with $p > 1$. We look for critical points of $\mathcal{L}$. Note that $\mathcal{L}$ is not convex and not coercive. A simple observation is that $u = 0$ is a local minimum which follows from the fact that the first term beats the second one for small $u \in H_0^1(\Omega)$ thanks to $p + 1 > 2$. This implies a convex behavior near 0. However, we have u_0 such that $\mathcal{L}(u_0) \leq 0$. Heuristically, it would pass a saddle point before $\mathcal{L}$ becomes negative so that we would expect there exists a critical point. The mountain pass theorem basically proves the following heuristic fact : if we choose the trajectory from 0 to u_0 minimizing the maximum height, then it corresponds to a saddle point.

Theorem 17.9 (Mountain pass theorem)**.** *Suppose $\mathcal{L} \in C^1(H;\mathbb{R})$ and $\mathcal{L}' : H \to H$ is Lipschitz on bounded sets of H, which satisfies the Palais-Smale condition. Suppose also*

- $\mathcal{L}(0) = 0$,
- *there exists $r, a > 0$ such that $\mathcal{L}(u) \geq a$ if $\|u\| = r$,*
- *there exists an element $u_0 \in H$ with $\|u_0\| > r$ and $\mathcal{L}(u_0) \leq 0$.*

Define all the trajectories connecting 0 to u_0 by

$$\Gamma := \{g \in C([0,1]; H) : g(0) = 0, g(1) = u_0\}.$$

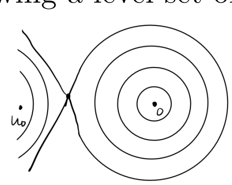
Then

$$c = \inf_{g\in\Gamma} \max_{0\leq t\leq 1} \mathcal{L}(g(t))$$

is a critical value of $\mathcal{L}$, that is, $K_c = \{v \in H : \mathcal{L}(v) = c, \mathcal{L}'(v) = 0\} \neq \varnothing$.

Remark 17.10. In an infinite dimensional H, $|\nabla u|^2$ wins most of the time (up to finite dimensional directions). On the other hand, $|u|^{p+1}$ only beats $|\nabla u|^2$ in very few directions.

We demonstrate the idea by drawing a level set of some functional $\mathcal{L}$.



Second order parabolic equations - Introduction and L^2 theory

Date: March 16, 2023

Recall the model problem for heat equation

$$\begin{cases} (\partial_t - \Delta)u &= f \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^n, \\ u(0) &= u_0. \end{cases}$$

is a special case for the following general problem

$$\begin{cases} (\partial_t - P)u &= f, \\ u(0) &= u_0. \end{cases}$$

In this general case, we focus on two different domains. If we consider $\mathbb{R}^+ \times \mathbb{R}^n$, this is similar to the heat equation model problem. If we consider in a bounded domain $\mathbb{R}^+ \times \Omega$ with $\Omega \subset \mathbb{R}^n$, we need to add some boundary condition to the equation.

18.1 Recap of the properties of heat equation

Last semester, we discuss the heat propagation and a probabilistic interpretation of the heat equation. The key properties of the model problem are as follows :

(1) (forward) fundamental solutions : $K(t,x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}} 1_{t\geq 0}$.

(2) solutions given by convolution :

$$u(t) = K(t) *_x u_0 + K *_{t,x} f.$$

(3) regularizing effect : If $f = 0, u_0 \in \mathscr{S}'$, then

$$u \in C^\infty((0,\infty) \times \mathbb{R}^n).$$

This can be proved as follows. First, we write $\widehat{u}(t,\xi) = e^{-t|\xi|^2}\widehat{u}_0(\xi)$ and hence

$$u(t,x) = \langle \widehat{u}_0(\xi), e^{-t|\xi|^2} e^{ix\cdot\xi} \rangle_{\mathscr{S}'_\xi, \mathscr{S}_\xi},$$

where the smoothness follows from the fact that

$$\nabla_{t,x}\left(e^{-t|\xi|^2} e^{ix\cdot\xi}\right) \in \mathscr{S}_\xi.$$

(4) infinite speed of propagation : Since $\operatorname{supp} K(t) = \mathbb{R}^d$, we know that $\operatorname{supp} u = \mathbb{R}^d$ even if u_0 is compactly supported.

(5) forward evolution : The heat equation is ill-posed as a backward evolution since we only have one fundamental solution and it is supported in $\{t \geq 0\}$. This is due to the fact $\frac{1}{i\tau + |\xi|^2} \in L^1_{loc}$.

(6) energy dissipation : Heuristically, by a direct computation,

$$\frac{d}{dt}\|u(t)\|^2_{L^2} = 2\int u \cdot u_t \, dx = 2\int u \cdot \Delta u \, dx = -2\int |\nabla u|^2 \, dx, \tag{18.1}$$

which implies

$$\int_0^\infty \int |\nabla u|^2 \, dx \, dt < \infty$$

since $\|u(t)\|_{L^2}^2$ is decreasing while it is always non-negative. The phenomenon that $\|u(t)\|_{L^2}^2$ is non-increasing is called the dissipation of energy. In particular, u becomes L^2H^1. The computation above can be made rigorous after we showed the existence theorem, Theorem 18.2, by applying an approximation argument to

$$\|u(t)\|_{L^2}^2 = \|u(0)\|_{L^2}^2 - 2\int_0^\infty \int |\nabla u|^2 \, dx \, dt.$$

To make this argument rigorous, we need to assume u has the regularity proved in the existence theory so that we can make approximation argument. This is just the energy estimate we are going to prove in the following.

In fact, we can show a stronger result that u becomes H^1 immediately after initial time. Moreover, we compute

$$\frac{d}{dt}\left(t\|\nabla u\|_{L^2}^2\right) = \|\nabla u\|_{L^2}^2 + 2t\int \nabla u \cdot \nabla u_t \, dx = \|\nabla u\|_{L^2}^2 - 2t\int |\Delta u|^2 \, dx \le \|\nabla u\|_{L^2}^2,$$

then this implies

$$t\|\nabla u\|_{L^2}^2 \lesssim \int_0^t \|\nabla u\|_{L^2}^2(s) \, ds \lesssim \|u_0\|_{L^2}^2,$$

where the last inequality follows from (18.1). Note that the choice of t in the first quantity comes from the parabolic scaling.

We also make a remark that, for $u_0 \notin \dot{H}^1$, we need to be careful if we want to use "$t \cdot \|\nabla u(0)\|_{L^2}^2 = 0$" at $t = 0$. To justify this, one needs to first compute for $u_0 \in \mathcal{D}$ and then do an approximation in the final estimate we get. See Remark 18.6 at the end of this section for a detailed proof.

Another approach to justify this is to do for each Littlewood-Paley piece since for each dyadic frequency region, it is finite. This is simple due to the fact that if (u, u_0) is a solution, then $(P_j u, P_j u_0)$ is also a solution. We just need to compute

$$t\|\nabla u(t)\|_{L^2}^2 = t\sum_j \|\nabla P_j u(t)\|_{L^2}^2 \lesssim \sum_j \int \|\nabla P_j u(s)\|_{L^2}^2 \, ds = \int \|\nabla u(s)\|_{L^2}^2.$$

Daniel Tataru mentioned that this fact also holds in the variable coefficient case. If we still want to apply Littlewood-Paley theory, then we can only expect a weaker estimate like $\|P_j u\|_{L^2} \le C_N(1 + t2^{2j})^{-N}$ instead of the strongest ones $\|P_j u\|_{L^2} \le Ce^{-Ct2^{2j}}$ due to the difference in the heat kernel. Also, we need a bootstrap argument to conclude the argument.

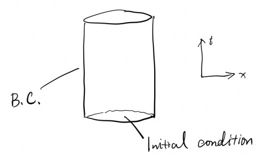
(7) parabolic scaling : If $u(x,t)$ solves the homogeneous heat equation $(\partial_t - \Delta)u = 0$, then $u(\lambda x, \lambda^2 t)$ also solves the equation. Due to the scaling, we focus on parabolic cylinders, $[t, t+R^2] \times B(x_0, R)$ instead of balls.

18.2 L^2 theory for parabolic equations - energy estimates

We consider

$$\begin{cases} (\partial_t + P)u = f & \text{in } \Omega \times [0,T], \\ u(t=0) = u_0 & \text{in } \Omega, \\ u(t,x) = 0 & \text{in } \partial\Omega \times [0,T], \end{cases}$$

where $P = -\partial_i a^{ij}\partial_j + b^j\partial_j + c$ with $a^{ij}, b^j, c \in L^\infty$ and (a^{ij}) elliptic and symmetric.



As we discussed for the elliptic equations, we start by showing an energy estimate so that it gives a uniqueness theorem at least. Furthermore, by a duality argument, we can show existence. We would expect the same idea applies to parabolic equations.

Suppose $u_0 \in L^2$ and we would like to track the L^2 size of the function, which is exactly as the previous computation. We compute

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 = 2\int u \cdot \partial_t u\, dx = -2\int u \cdot (Pu - f)\, dx = -2B(u,u) + \int u \cdot f\, dx.$$

As before,

$$B(u,u) = \int_\Omega a^{ij}\partial_i u \partial_j u + b^i \partial_i u \cdot u + cu^2\, dx,$$

where the first term is called the principal part and the last two are the lower order terms. The principal part is comparable to $\|\nabla u\|_{L^2}^2$, while we may not expect a sign in the lower order terms. Thus,

$$B(u,u) \geq C_1\|\nabla u\|_{L^2}^2 - C_2\|u\|_{L^2}^2.$$

Suppose $f = 0$, then

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 \leq -C_1\|\nabla u\|_{L^2}^2 + C_2\|u\|_{L^2}^2 \tag{18.2}$$

and hence

$$\frac{d}{dt}e^{-C_2 t}\|u(t)\|_{L^2}^2 \leq -C_1 e^{-C_2 t}\|\nabla u\|_{L^2}^2,$$

which implies

$$\|u(t)\|_{L^2}^2 \leq e^{C_2 t}\|u(0)\|_{L^2}^2 - C_1 e^{C_2 t}\int_0^t e^{-C_2 s}\|\nabla u(s)\|_{L^2}^2\, ds.$$

Since we focus on a compact t region, then we obtain our uniform energy bound

$$\|u\|_{L_t^\infty L_x^2}^2 + \|\nabla u\|_{L_t^2 L_x^2}^2 \lesssim \|u_0\|_{L^2}^2. \tag{18.3}$$

The first term on the left hand side means that if you start from $u_0 \in L^2$, then the solution keeps in the same function space. On the other hand, the second term measures the dissipation. In fact, (18.3) can be obtained by integrating (18.2) on both sides.

From (18.3), we expect the existence theory ensures $u \in C_t(L_x^2(\Omega)) \cap L_t^2 H_0^1(\Omega)$. Now we consider nontrivial f and we need to be able to estimate

$$I = \int_0^T \int_\Omega u \cdot f\, dx.$$

If we use the uniform bound for u, then

$$|I| \leq \int_0^T \|u(t)\|_{L^2}\|f(t)\|_{L^2}\,dt \leq \|u\|_{L_t^\infty L_x^2}\|f\|_{L_t^1 L_x^2}.$$

On the other hand, if we use the second component of the bound for u, we derive

$$|I| \leq \int_0^T \|u(t)\|_{H_0^1}\|f(t)\|_{H^{-1}}\,dt \leq \|u\|_{L_t^2 H_0^1}\|f\|_{L_t^2 H^{-1}}.$$

Therefore, to combine these two estimates, we introduce $L_t^1 L_x^2 + L_t^2 H_x^{-1}$, where in general the space $X+Y$ is defined as follows :

$$X+Y = \{z = x+y : x \in X, y \in Y\}, \quad \|z\|_{X+Y} = \inf_{z=x+y}(\|x\|_X + \|y\|_Y).$$

Note that this definition does not for arbitrary Banach space X, Y since we require some compatibility conditions at least to make sure we can define addition for this. In practice, $X, Y \subset \mathcal{D}'$ ensures the compatibility.

To achieve this, we write $f = f_1 + f_2$ and hence

$$|I| \leq \|u\|_{L_t^\infty L_x^2}\|f_1\|_{L_t^1 L_x^2} + \|u\|_{L_t^2 H_0^1}\|f_2\|_{L_t^2 H^{-1}} \leq \|u\|_{L_t^\infty L_x^2 \cap L_t^2 H_0^1}\left(\|f_1\|_{L_t^1 L_x^2} + \|f_2\|_{L_t^2 H^{-1}}\right).$$

Then taking the infimum over all possibility of writing f into $f = f_1 + f_2$, we obtain

$$\|u\|^2_{L_t^\infty L_x^2 \cap L_t^2 H_0^1} \lesssim \|u_0\|^2_{L^2} + \|u\|^2_{L_t^\infty L_x^2 \cap L_t^2 H_0^1}\|f\|_{L_t^1 L_x^2 + L_t^2 H_x^{-1}}.$$

By applying Cauchy-Schwarz inequality on the last term, we would get

$$\|u\|^2_{L^\infty L_x^2} + \|u\|^2_{L_t^2 H_0^1} \lesssim \|u_0\|^2_{L^2} + \|f\|^2_{L_t^1 L_x^2 + L_t^2 H_x^{-1}}. \tag{18.4}$$

Remark 18.1. Finally, we make a comment that the full statement of our energy estimate (18.4) is as follows. Suppose $u_0 \in L^2$, $f \in L_t^1 L_x^2 + L_t^2 H^{-1}$ and $u \in L^\infty L_x^2 \cap L_t^2 H_0^1$ is a solution, then it satisfies the energy estimate. These assumptions are just what we are going to prove in the existence theory. With these assumptions, we can make the computation rigorous by approximation. See Remark 18.4 and Remark 18.5 for details.

18.3 L^2 theory for parabolic equations - existence

From the energy estimate (18.4), we expect the following existence theorem to be true.

Theorem 18.2. *For each $u_0 \in L^2$, $f \in L^1 L^2 + L^2 H^{-1}$, there exists a unique solution*

$$u \in L^\infty L^2 \cap L^2 H_0^1.$$

The energy estimate above gives the uniqueness part of this theorem. To prove the existence part, we set up a duality argument.

First, we need to identify the adjoint problem. For $(\partial_t + P)u = f$, by testing with v, we write

$$\int_0^T \int_\Omega v \cdot f\,dx\,dt = \int_0^T \int_\Omega (\partial_t + P)u \cdot v\,dx\,dt$$

$$= \int_0^T \int_\Omega u \cdot (-\partial_t v)\,dx\,dt + \int_\Omega u \cdot v\,dx\Big|_0^T + \int_0^T \int_\Omega u \cdot P^* v\,dx\,dt + \int_0^T \int_\Omega \left(u \cdot \frac{\partial v}{\partial \nu} - v \cdot \frac{\partial u}{\partial \nu}\right)\,d\sigma.$$

From the computation, we require $v|_{\partial\Omega} = 0$ so that

$$\int_0^T \int_\Omega v \cdot f \, dx \, dt = \int_0^T \int_\Omega u \cdot (-\partial_t + P^*) v \, dx \, dt + \int_\Omega u \cdot v \, dx \Big|_0^T,$$

where in the last term, $u(0)$ is known, so we view $t = T$ as the initial time for v and prescribe an initial data. Thus, the adjoint problem would be

$$\begin{cases} (-\partial_t + P^*) v = g, \\ v(T) = v_T, \\ v = 0 \text{ in } \partial\Omega. \end{cases}$$

Note that this is a heat equation coming back in time so it is also natural to require an initial condition for $v(T)$. We compute

$$\int_0^T \int_\Omega f \cdot v \, dx \, dt + \int_\Omega u(0) v(0) \, dt = \int_0^T \int_\Omega g \cdot u \, dx \, dt + \int_\Omega u(T) v(T) \, dx.$$

By duality, u solves the equation if the relation above holds for all $v \in C^\infty$ satisfies the dual problem. For v given, the left hand side is known and we want to think of the right hand side as a linear functional on $(v(T), g) \in L^2 \times (L^1 L^2 + L^2 H^{-1})$. Note that the linear functional is only defined on a subspace only (if we do not know the existence). The map

$$T(v(T), g) = \int_0^T \int_\Omega f \cdot v \, dx \, dt + \int_\Omega u(0) v(0) \, dt$$

is defined on $T : X \subset L^1 L^2 + L^2 H^{-1} \times L^2 \to \mathbb{R}$. Under the assumptions for u, f in Theorem 18.2, we compute

$$T(v(T), g) \le \|f\|_{L^1 L^2 + L^2 H^{-1}} \cdot \|v\|_{L^\infty L^2 \cap L^2 H^1} + \|u(0)\|_{L^2} \|v(0)\|_{L^2} \lesssim \|v(T)\|_{L^2} + \|g\|_{L^1 L^2 + L^2 H^{-1}},$$

where the last step follows from the energy estimate (18.4) for the adjoint problem with (v, g), which implies the map T is bounded. By Hahn-Banach theorem, we can extend T so that there exists $u \in (L^1 L^2 + L^2 H^{-1})^* = L^\infty L^2 \cap L^2 H_0^1$ such that

$$T(v(T), g) = \int_0^T \int_\Omega g \cdot u \, dx + \int_\Omega v(T) u(T) \, dx,$$

which completes the proof. This is the general duality argument for evolution equations.

Remark 18.3. When the duality method applies to linear wave equation, after showing existence of u, one also needs to argue a bit more on regularity of u. See [20, Theorem I.3.2].

Remark 18.4. Given the existence theorem, Theorem 18.2, one can make the proof of the energy estimate (18.4) rigorous in the case $\mathbb{R}^{1+d}$ by introducing the following approximation : We choose $\varphi \in \mathcal{D}(\mathbb{R}^{1+d})$ such that $\int \varphi = 1$ and $\mathrm{supp}\varphi \in (-1, 1) \times B(0, 1)$, $\varphi_\varepsilon(t, x) = \varepsilon^{-2-d} \varphi(\varepsilon^{-2} t, \varepsilon^{-1} x)$. (To make the following convergence in ε easier to prove, one may also assume $\varphi(t, x) = \varphi_1(t) \varphi_2(x)$ with desired support property.) Another sequence of cutoff $\chi(\varepsilon x)$ is given by $\chi \in \mathcal{D}(\mathbb{R}^d)$ which is 1 near 0 and 0 away from a large region. On the subset $[\varepsilon^2, T - \varepsilon^2] \times \mathbb{R}^d$, we define

$$u^\varepsilon := (\varphi_\varepsilon *_{t,x} (\chi_\varepsilon(x) u(t, x)))|_{[\varepsilon^2, T - \varepsilon^2] \times \mathbb{R}^d}.$$

In this approximation, we first do a cut-off so that we can restrict in a compact subset and then convolve with a mollifier to make it smooth. Instead, we can also change the brutal cut-off near the boundary of the time interval to a furthermore smooth cut-off $\tilde{\chi}(t)$ such that $\tilde{\chi} \equiv 1$ in the interval and $\tilde{\chi} = 0$ at the boundary point $0, T$, which make the boundary term of time interval vanish. However, for instance, $\int_0^\varepsilon \tilde{\chi}'(t) \int |u|^2 \, dx \, dt$ can be viewed as an average and converges to $\int |u(0)|^2 \, dx$, which make the boundary term appear again after taking the limit.

Since $u^\varepsilon \in C^\infty([\varepsilon^2, T - \varepsilon^2] \times \mathbb{R}^d)$, we can justify the interchangability between ∂_t and integration. Moreover, due to the compact support property, we can justify the integration by parts in the proof of the energy estimates for $(\partial_t - \Delta)u^\varepsilon = f^\varepsilon$. Moreover, for any compact interval $J \subset (0, T)$, we have

$$\begin{aligned} &u^\varepsilon \to u \text{ in } C_t(J; L^2(\mathbb{R}^d)); \quad \nabla u^\varepsilon \to \nabla u \text{ in } L^2(J \times \mathbb{R}^d); \\ &u^\varepsilon(\varepsilon^2) \to u_0 \text{ in } L^2(\mathbb{R}^d); \qquad \partial_t u^\varepsilon - \Delta u^\varepsilon \to f \text{ in } L^1(J; L^2(\mathbb{R}^d)). \end{aligned}$$

Therefore, the energy estimates follows in the general case.

Remark 18.5. Unfortunately, the approximation sequence selected above needs some modification in the bounded region case.

We choose $\chi_\varepsilon \in \mathcal{D}(\Omega)$ to be $\chi_\varepsilon \equiv 1$ in $\Omega_{2\varepsilon}$ and $\operatorname{supp}\chi_\varepsilon \subset \Omega_\varepsilon$. On the subset $[\varepsilon^2, T - \varepsilon^2] \times \Omega_\varepsilon$, we define

$$u^\varepsilon := (\varphi_\varepsilon *_{t,x} (\chi_\varepsilon(x) u(t,x)))|_{[\varepsilon^2, T-\varepsilon^2] \times \Omega_\varepsilon},$$

where $\Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \Omega) \geq \varepsilon\}$. Then when computing Δu^ε, a term like

$$\nabla \chi_\varepsilon(x) \cdot \nabla u(x)$$

will appear and finally converges to $\delta_{\partial\Omega}(\nabla u)$, which corresponds to a Neumann boundary term. However, we do not know any property about this term. Therefore, for the term $\partial_t u^\varepsilon - \Delta u^\varepsilon$, we need to consider its convergence in $L^2 H^{-1}$ instead of $L^1 L^2$. Due to the fact that $u^\varepsilon \to u$ in $L^2 H_0^1$, we know that $\Delta u^\varepsilon \to \Delta$ in $L^2 H^{-1}(\Omega)$ and hence the result follows. This is kind of similar to the elliptic case.

We can also choose the cut-off only in the tangential direction $\chi_\varepsilon(x')$ so that it does not produce the normal derivative at the boundary to avoid the appearance of the Neumann boundary term.

Remark 18.6. To provide the rigorous argument for

$$t \|\nabla u\|_{L^2}^2 \lesssim \|u_0\|_{L^2}^2,$$

we still do the approximation u^ε above. First of all, the estimate above is true for $u_0 \in \mathcal{D}$. Then one would notice that the initial data of $u^\varepsilon - u^\delta$ is compactly supported and convergent to 0 in L^2 so that we know

$$\nabla u^\varepsilon - \nabla u^\delta \to 0 \text{ in } C_t(J \times K).$$

On the other hand, $\nabla u^\varepsilon \to \nabla u$ in $L^2(J \times K)$ by the property of mollifiers. Therefore, we know the limit coincides, that is,

$$\nabla u^\varepsilon \to \nabla u \text{ in } C_t(J \times K)$$

and hence the desired estimate follows for any $t > 0$ by approximation.

Second order parabolic equations - L^2 regularity

Date: March 21, 2023

Compared to the existence theorem that we introduced last time, one can upgrade the regularity. Before we state the theorem, we give the following definition of functional spaces.

Definition 19.1. We say $u \in H^1L^2$ if $u \in L^2L^2$ and $\partial_t u \in L^2L^2$.

Remark 19.2. This definition is not related to the differentiablity of $\|u(t)\|_{L^2_x}$.

19.1 Higher regularity - energy estimates and existence

Theorem 19.3. *Suppose $u_0 \in H^1_0$, $f \in L^1H^1_0 + L^2L^2$, then the solution $u \in C(H^1) \cap L^2(H^2 \cap H^1_0) \cap H^1L^2$.*

Remark 19.4. Note that we have the Dirichlet boundary condition so we use H^1_0 instead of H^1 at all the places where H^1 appears in the theorem above. A new idea here is the compatibility condition for the data : connecting initial data with boundary data.

This sounds trivial but if we want to consider solutions with more regularity $\partial_t u = 0$ on $\partial\Omega$, then we need more compatibility $Pu_0 - f(0) = 0$ on $\partial\Omega$, which is a second order condition.

Remark 19.5. Later, we would simply denote the mixed norm space $L^2_tL^2_x$ by L^2.

Instead of giving a proof directly, we try to provide some ideas.

Via energy estimates - Naive try : We try to develop an energy estimate as before. Suppose $f = 0$, we compute

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx = 2\int_\Omega \nabla u \cdot \nabla u_t\,dx &= 2\int_\Omega \partial_m u \partial_m \partial_j a^{jk}\partial_k u + l.o.t.\\ &= -2\int_\Omega \partial_j\partial_m u \cdot \partial_m a^{jk}\partial_k u + l.o.t. = -2\int_\Omega \partial_j\partial_m u \cdot a^{jk}\partial_m\partial_k u + l.o.t.\\ &\le -C\|\partial_j\partial_m u\|^2_{L^2} + C\|u\|_{H^2}\|u\|_{H^1},\end{aligned}$$

where the last step follows from the uniform positive definiteness of a^{jk}. Finally, we get

$$\frac{d}{dt}\|u\|^2_{H^1} \le -c\|u\|^2_{H^2} + C\|u\|^2_{H^1}$$

and hence

$$\|u\|^2_{CH^1} + \|u\|^2_{L^2H^2} \lesssim \|u_0\|^2_{H^1}.$$

Via energy estimates - adapted energy : However, we need to fix the problem that we implicitly use the vanishing of boundary terms produced from integration by parts though we actually cannot ensure the boundary term vanishes since it contains first order derivatives. To fix the boundary issue, we use the adapted energy

$$\int_\Omega a^{lm}\partial_l u \partial_m u\,dx$$

instead. If we do the computation

$$\begin{aligned}\frac{d}{dt}\int_\Omega a^{lm}\partial_l u\partial_m u &= 2\int_\Omega a^{lm}\partial_l u\cdot\partial_m(\partial_j a^{jk}\partial_k u)\,dx + \int_\Omega(\partial_t a^{lm})\partial_l u\partial_m u\,dx\\ = -2\int_\Omega |Pu|^2 &+ 2\int_{\partial\Omega}\nu_m a^{lm}\partial_l u\cdot\partial_j a^{jk}\partial_k u\,d\sigma + \int_\Omega(\partial_t a^{lm})\partial_l u\partial_m u\,dx,\end{aligned}$$

then it turns out that we still have the boundary terms. Instead of trying to integrate by parts, we multiply the equation by Pu, which gives

$$\int\partial_t u\cdot Pu\,dx + \int|Pu|^2\,dx = \int Pu\cdot f\,dx,$$

where the right hand side can be bounded by $\|f\|_{L^2}\|Pu\|_{L^2}$ or $\|f\|_{H^1_0}\|u\|_{H^1}$ and the first term can be written as

$$\int\partial_t u\cdot Pu\,dx = \frac{1}{2}\partial_t\int B(u,u)\,dx - \int(\partial_t a^{jk})\partial_j u\partial_k u\,dx,$$

where the integration by parts

$$\int B(u,u)\,dx = \int a^{jk}\partial_j u\partial_k u = -\int u\partial_k a^{jk}\partial_j u$$

can be justified since we know $u|_{\partial\Omega}=0$.

Remark 19.6. Compared this with the preceding computations, we notice that there is merely zeroth order term u involved in the boundary terms so that it vanishes and this is the advantage of our method here. However, u does not appear in the boundary terms for the previous two computations.

Upgrading regularity : The second issue is that the computation requires knowing that we have solutions in H^2. For duality, we have to do energy estimates at the level of H^{-1}, which is hard to manage. An alternative method of proving this is just to upgrade the L^2 solution we already obtained to H^1_0. We can replace derivatives ∂_j by D^h_j, which is given by

$$D^h_j u = \frac{u(x+he^j)-u(x)}{h}$$

and then conclude by passing to the limit $h\to 0$. This allows us to prove the existence of higher regularity solutions. See [7], [4, Chapter 9.6] for details.

Viewing it as a system : In $\mathbb{R}^n$, there is another idea. The reason why this only works for $\mathbb{R}^n$ since we do not need to worry about the boundary condition. We write equations for $\partial_j u := u_j$ and try to think of this as a new function :

$$\partial_t u_j + Pu_j = \partial_j f + [P,\partial_j]u.$$

The operator $[P,\partial_j]$ is second order but we can think of it as $\partial\nabla u$ to make it to be a first order term when we are seeking for a solution to the following system

$$\begin{cases}\partial_t u_j + Pu_j = \partial_j f + \partial\nabla u + u,\\ \partial_t u + Pu = f,\end{cases}$$

where the reason why we need to put the original equation in is that $[P, \partial_j]u$ might have zeroth order terms. For this $n+1$ order system, we can apply L^2-solvability for this system in $(\nabla u, u)$.

Now the only thing we need to prove that the solutions satisfy the constraints $u_j = \partial_j u$. We know that the constraints hold at the initial time and want to propagate it in time. We write $v_j = u_j - \partial_j u$ and write down the equation for v_j, which is a parabolic equation

$$\partial_t v_j - P v_j = 0, \quad v_j(0) = 0$$

for the same parabolic operator P. Then one would like to conclude that $v_j = 0$ by energy estimates and therefore, this proves Theorem 19.3.

Remark 19.7. Note that, since we only know $v_j = u_j - \partial_j u \in L^2L^2 \cap L^\infty H^{-1}$, we need to prove a lower regularity energy estimate to conclude $v_j = 0$ in the last step. This is easy to prove in the $\mathbb{R}^d$ case while $P = -\Delta$ since one can apply Fourier transform to show

$$\|v\|^2_{L^\infty H^{-1}} + \|v\|^2_{L^2L^2} \lesssim \|u_0\|^2_{H^{-1}} + \|f\|^2_{L^2H^{-1}}.$$

In the variable coefficient case, we need to prove a commutator estimate for $[|D|^{-1}, a_{ij}(x)]$, which expoits the positive definiteness of $a^{ij}(x)$, then the L^2 result implies the H^{-1} result as what it is like how we upgrade from L^2 to H^1. Also, if we already know result for H^1, then by duality, the energy estimates can be obtained from results in H^{-1}.

Remark 19.8. For the last method, when you are in a bounded domain, your vector fields need to be tangential to the boundary of the domain to preserve the boundary condition. So we can only do this for tangential derivatives when we are in a bounded domain, which does not help a lot.

Remark 19.9. When the setting is for bounded domains, an alternative way is to work with $(u, \partial_t u)$. Since the initial data of $\partial_t u$ is just H^{-1}, we would only expect $\partial_t u \in C_t H^{-1}$ and therefore by the elliptic theory for $Pu + f = \partial_t u \in C_t H^{-1}$, we know $u \in C_t H^1$, which is the desired result. First we prove for the case f nice so that the initial data $(\partial_t u)(0) = \Delta u_0 + f(0, \cdot)$ is well-defined and then use an approximation argument to conclude.

Remark 19.10. Another method is to use Galerkin approximation. Compared to the approximation before, this is to use a finite dimensional approximation while ours are an application of compactness.

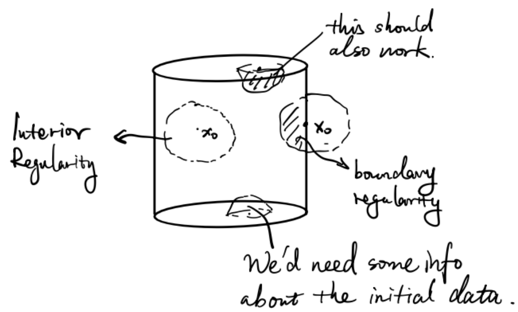
We record the higher regularity theorems without giving a proof.

Theorem 19.11. *Given $m \in \mathbb{N}$, $\partial\Omega \in C^m$ and $a^{ij}, b, c \in C^m$. Suppose $u_0 \in H^m$, $f \in L^1H^m + L^2H^{m-1}$ and f satisfies all the compatibility conditions. Then $u \in CH^m \cap L^2H^{m+1}$.*

19.2 Parabolic regularity

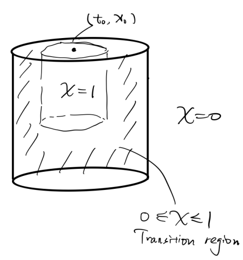
In this subsection, we introduce the parabolic regularity theorem, which is an analogy of Theorem 5.3.

Theorem 19.12 (Parabolic regularity)**.** *Suppose u solves locally $\partial_t u + Pu = f$. If $u \in (L^2H^1_0)_{loc}$ and $f \in L^2_{loc}$, then $u \in (L^2H^2)_{loc} \cap (H^1L^2)_{loc}$.*



Remark 19.13. Note that the localized region can be at the boundary of Ω. However, if the region has intersection with $t = 0$, then we need information for initial data, otherwise it would be false.

For simplicity, we only consider the interior regularity. Without loss of generality, we assume (t_0, x_0) lies on the top of the cylinder and choose a cutoff function as below.



By truncating u to a neighborhood of x_0 : $v = \chi u$, where

$$\chi = \begin{cases} 1 \text{ near } x_0, \\ 0 \text{ away from } x_0, \end{cases}$$

we write an equation for v :

$$\partial_t v + Pv = \chi f + (\partial_t \chi)u + \nabla^2 \chi u + \nabla \chi \cdot \nabla u,$$

where the right hand side is in L^2. Moreover, the initial data for v is zero since we can start from where $\chi = 0$.

By the higher regularity theorem for v, we know that $v \in C(H^1) \cap L^2 H^2 \cap H^1 L^2$. Moreover, we conclude that

$$f \in L^2 H^m_{loc} \Rightarrow u \in L^2 H^{m+2}_{loc} \cap H^1 H^m_{loc}.$$

Finally, we remark that second parabolic equations share the same kind of properties as second order elliptic equations including this higher regularity even though one of them is evolutionary while the other is stationary. However, for wave type equations, we do not have such local regularity statements because things would propagate. If one tries to mimic this proof, then one would find we end up with no improvements on regularity.

Second order parabolic equations - L^∞ theory

Date: March 23, 2023

We discuss the maximum principle for parabolic equations

$$Lu = \partial_t u - a^{ij}\partial_i\partial_j u + b^i\partial_i u + cu = 0$$

in a cylinder $C = \Omega \times [0,T]$. We define the subsolution (resp. supersolution) to be those u's such that $Lu \le 0$(resp. $Lu \ge 0$). In addition, we require $c \ge 0$. Heuristically, this assumption can be viewed as to be a decay assumption since $c \ge 0$ makes $\partial_t u + cu = 0$ to decay at infinity.

The maximum principle relates what happens inside the domain with the behavior at the boundary. Note that the boundary of the cylinder $C = \Omega \times [0,T]$ has 3 pieces :

- bottom (initial data) ;
- lateral boundary (Dirichlet boundary condition) ;
- top (final data).

Heuristically, the final data will depend on the Dirichlet data and the initial data, so it is a derived quantity. Because of this, we introduce the parabolic boundary of the cylinder C to be

$$\partial C = (\{0\} \times \Omega) \cup ([0,T] \times \partial\Omega).$$

20.1 Weak maximum principle (in parabolic cylinders) for subsolutions, comparison principle

20.1.1 The case when $c \ge 0$

Theorem 20.1. *Assume $c = 0$, a^{ij} is positive definite (a^{ij} does not need to be strictly positive definite) and $a, b \in C$, $u \in C^2(C) \cap C(\overline{C})$ is a subsolution. Then $\max_{\bar{C}} u = \max_{\partial C} u$.*

Remark 20.2. We do not impose any boundary condition in the assumption.

Proof. For strict subsolutions, suppose we have a maximum point

$$(t_0, x_0) \in C \setminus \partial C = (0,T] \times \Omega.$$

Then

$$\partial_t u \ge 0, \quad \nabla_x u = 0, \quad \nabla_x^2 u \le 0,$$

which implies $Lu(t_0,x_0) \ge 0$ and this contradicts to the assumption.

For subsolutions, we want to penalize u by setting

$$u_\varepsilon = u - \varepsilon t$$

and $Pu_\varepsilon = Pu - \varepsilon < 0$. Thus $\max_C u_\varepsilon = \max_{\partial C} u_\varepsilon$. Since we are in a compact region, $u_\varepsilon \to u$ uniformly. Let $\varepsilon \to 0$, this proves the theorem. $\square$

Remark 20.3. The penalization for parabolic equations is much simpler than the elliptic case since we can easily take advantage of the time dependence.

Example 20.4. The equation $\partial_t u - \Delta(u^p) = 0$ is usually called the porous medium equation. for $1 \le p < \infty$. For positive solutions, we also have maximum principle, where the same proof applies.

When we discuss the maximum principle for elliptic equation, if $c \geq 0$, then we need to consider non-negative maximum, that is,

$$\max_C u \leq \max\{0, \max_{\partial C} u\}.$$

Moreover, this also implies the comparison theorem when $c \geq 0$. If u^- is subsolution, u^+ is supersolution and $u^- \leq u^+$ on ∂C, then $u^- \leq u^+$ in C. In the parabolic setting, these two results still hold.

20.1.2 The case when we do not assume $c \geq 0$

Furthermore, we introduce another simple trick which works for the parabolic setting even if we do not assume $c \geq 0$. We replace u by $v = e^{-\alpha t} u$ and we write the equation for v :

$$Pv = P(e^{-\alpha t} u) = e^{-\alpha t} Pu - \alpha e^{-\alpha t} u,$$

which is equivalent to

$$P_\alpha v = e^{-\alpha t} Pu, \qquad P_\alpha = \partial_t - a^{ij}\partial_i\partial_j + b^i\partial_i + (c + \alpha).$$

Therefore, one observe that if u is a subsolution for P, then v is a subsolution for P_α. Moreover, if $c \in L^\infty$, then we choose α such that $c + \alpha \geq 0$ and thus

$$\max_C e^{-\alpha t} u \leq \max_{\partial C} e^{-\alpha t} u, \qquad \text{if } u \geq 0,$$

which follows from the maximum principle in the case $c + \alpha > 0$. The advantage of this is that the comparison theorem remains true (independent of α) for u^-, u^+ subsolutions and supersolutions since the comparison between two functions remain unchanged after multiplying a positive function.

Corollary 20.5 (Uniqueness of solutions). *The equation*

$$\begin{cases} Pu = f, & \\ u = u_0 & \text{at } t = 0, \\ u = g & \text{on } \partial\Omega \times [0, T] \end{cases}$$

has a uniqueness solution in the class $u \in C^2(C) \cap C(\partial C)$.

Remark 20.6. One can also define viscosity sub/super-solutions for u USC/LSC. Moreover, comparison property holds for sub/super-solutions, which is good for C^0 coefficients and nonlinear equations. Finally, one can prove existence using Perron's method in this context. The last step is easy to implement in this case while the hardest thing in the viscosity context is to prove the comparison, especially in the nonlinear setting. Furthermore, one can also prove higher regularity. From $u \in C$, one can prove $u \in C^\alpha$ and then $u \in C^{1,\alpha}$.

20.2 Strong maximum principle (in parabolic cylinders) and a mean value property for heat equation

An obvious formulation is : if a solution u to $Pu = 0$ attains maximum inside at (t_0, x_0), then u is constant. However, if we consider a mug with coffee, we can cool it off by adding some boundary condition as time goes by. Therefore, u can be smaller after t_0 without changing the maximum point (t_0, x_0). In other words, the solution cannot respond to the changes in boundary until these changes happen.

Theorem 20.7 (Strong maximum principle)**.** *Assume $c = 0$. If (t_0, x_0) is a maximum point in $C \setminus \partial C$, then $u(t, x)$ is constant for all $t \le t_0$.*

The easiest proof is to use the mean value property when the coefficients are constant, that is the heat equation $\partial_t u - \Delta u = 0$, as we can write down the fundamental solution in this case.

As the proof of the mean value property in the elliptic case, we write for $\varphi \in \mathcal{D}$ that

$$\int_C (\partial_t - \Delta) u \cdot \varphi \, dx \, dt = \int_C u \cdot (-\partial_t - \Delta)\varphi \, dx \, dt.$$

If we change C to other domains, then we need to consider boundary terms. If we want the appearance of $u(0, 0)$, the natural way is to expect $(-\partial_t - \Delta)\varphi = \delta_0$. Also, note that

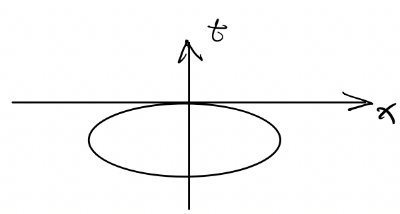
$$K(x, t) = (4\pi|t|)^{-n/2} e^{x^2/4t} \cdot 1_{t \le 0}$$

is the backward fundamental solution to heat equation, that is, the fundamental solution for $-\partial_t - \Delta$.

Let us observe that in the mean value property for harmonic functions, the sphere is actually the level set of the fundamental solution. This suggest us looking at the level sets

$$E_r = \{(t, x) : K(t, x) \ge r^{-n}\},$$

where $r > 0$ is the spatial scale. We first roughly draw a picture for the region E_r. Note that $E_r \subset \{t \le 0\}$ and the equivalent condition for $(t, x) \in E_r$ is that $e^{x^2/4t}(-4\pi t)^{-n/2} \ge r^{-n}$. For fix x, r, the left hand side goes to 0 as $t \to 0^-$. On the other hand, if $(t, x) \in E_r$, then $e^{x^2/4t} \le 1$ and hence $-4\pi t \le r^2$. This tells us how deep E_r is. Now we think of how wide E_r is. As $x \to \infty$, $e^{x^2/4t} \to 0$, which violates the inequality. The best case scenario for t is $|t| \simeq r^2$ and therefore $x^2 \simeq r^2$ is the width we can expect. We sketch the picture below.



Though it looks like an ellipse, it is not. One can notice this from the fact that it is not a quadratic touching at $(0, 0)$ even if it looks like it does. By taking the natural log to $e^{x^2/4t}(-4\pi t)^{-n/2} \ge r^{-n}$, there is an extra logarithm term on t.

Theorem 20.8 (Mean value property for heat equation)**.** *For each $r > 0$,*

$$u(0, 0) = \int_{E_r} \omega_r(t, x) u(t, x) \, dx \, dt,$$

where the weight function satisfies $\omega_r \ge 0$, $\int_{E_r} \omega_r \, dx \, dt = 1$.

Remark 20.9. The integral of ω_r is equal to 1 comes for free since if this mean value property works for all solutions, then in particular, it works for constant solutions $u \equiv 1$ and hence

$$\int_{E_r} \omega_r \, dx \, dt = 1.$$

Proof. First, we force φ vanishes at the boundary ∂E_r by selecting $\varphi(t,x) = K(t,x) - r^{-n}$.

However, we need to change the normal derivative by adding another term

$$\varphi(t,x) = K(t,x) - r^{-n} - c\ln(K(t,x)r^n),$$

which is still 0 on ∂E_r but we furthermore require the property that $\nabla_{t,x}\varphi \cdot \nu = 0$ on ∂E_r. Note that the normal derivative of the level set ∂E_r is given by

$$\nu = \frac{\nabla_{t,x} K}{|\nabla_{t,x} K|}$$

and hence we require

$$0 = \partial_j \varphi \nu_j = \frac{|\partial_j K(t,x)|^2}{|\nabla_{t,x} K|}\left(1 - c\frac{1}{K(t,x)}\right), \quad \forall (t,x) \in \partial E_r, \ \forall j = 0, 1, 2, \cdots, n,$$

where $j = 0$ corresponds to t. However, $K(t,x) = r^{-n}$ on ∂E_r so that the constant $c = K(t,x) = r^{-n}$ satisfies the requirement.

From the construction of K, we compute

$$(-\partial_t - \Delta)\varphi = \delta_{0,0} - c(-\partial_t - \Delta)\ln K = \delta_{0,0} - c\frac{|x|^2}{4t}.$$

Thus $\omega_r(t,x) = c\frac{|x|^2}{4t} = r^{-n}\frac{|x|^2}{4t}$, which completes the proof. □

Remark 20.10. From the computational result

$$\omega_r(t,x) = c\frac{|x|^2}{4t} = r^{-n}\frac{|x|^2}{4t},$$

it is natural to view r as a scaling parameter.

From Theorem 20.8, we give a proof for the strong maximum principle, Theorem 20.7.

Proof of Theorem 20.7 : Recall in the assumption we assume u reaches the maximum M at $(x_0, t_0) \in C$. For any $(x_1, t_1) \in C \cap \{t \le t_0\}$, we connect it with $(x_0, t_0) \in C$ by a line segment L. It is obvious that the set

$$S = \{s \ge t_1 : u(x,t) = M \text{ for all points } (x,t) \in L, s \le t \le t_0\}$$

is closed in L. On the other hand, for any $(y,s) \in S$, we can form a heat ball E_r with (y,s) as its top for some r sufficiently small so that $E_r \subset C$. Then thanks to the mean value property, we know S is open. Therefore, $S = L$ and hence $u \equiv M$ in C, which completes the proof.

Second order parabolic equations - L^∞ theory

Date: April 4, 2023

Last time, we proved the mean value property for the heat equation. The mean value property also exists for variable coefficients. Specifically, we take $\partial_t - \partial_i a^{ij}\partial_j + b^j\partial_j$, where we do not take the zeroth term since we want constant functions solve the equation. Therefore, we can expect the weight function in the mean value property to have average 1. The proof for the heat case applies to the variable coefficient case.

21.1 Harnack's inequality - Statements

In the elliptic case, we proved that if

$$-\partial_i a^{ij}\partial_j u = 0,$$

$u \geq 0$ in D, then for any $K \subset\subset D$,

$$\max_K u \leq C_{D,K,a} \min_K u.$$

In the parabolic case, we have to take into account the causality as what we did in the mean value property. Given any parabolic cylinder C, for any parabolic cylinder K such that $K \subset\subset C$ and $K_{top} \subset C_{top}$. We consider

$$\partial_t u - Lu = 0, \quad c = 0.$$

Then

$$\max_K u \leq \tilde{C} \min_{K_{top}} u$$

for $u \geq 0$ in C, where the constant $\tilde{C}$ again depends on our configuration.

The essential idea of using the positivity of the Green's function in C in the elliptic case also works but we may need to compute the Green's function carefully first. In [7, Chapter 7.1], they use another way to prove the Harnack's inequality. It illuminates a way of computations for bounded measurable coefficients, in which case we cannot use the Green's function. Basically, by taking the natural logarithm, it suffices to prove this $\ln u(x_1, t_1) \leq \ln u(x_2, t_2) + \gamma$ for all $x_1, x_2 \in V$, that is, we want to show the log of the function lives in the strip.

Moreover, Harnack's inequality would give a simple alternative proof of strong maximum principle. Also, one can upgrade the regularity of C^0 solutions to C^σ for bounded measurable coefficients and Green's parabolic coefficients as well, where the proof for the elliptic counterpart can be found in [11].

21.2 Regularity of solutions for the heat equation

In the elliptic case, we proved that for any local solution u to $-\Delta u = 0$, u is smooth and furthermore analytic. We consider

$$\begin{cases} (\partial_t - \Delta)u = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^n, \\ u(t=0) = u_0 \in \mathcal{E}', \end{cases}$$

where it can be solved by $u(t,x) = K(t,x) *_x u_0$. Recall that the fundamental solution $K(t,x) = (4\pi t)^{-n/2} e^{-|x|^2/4t} 1_{t>0}$ is smooth for any $t > 0$ and hence $u(t,x) \in C^\infty$ in $(0,\infty) \times \mathbb{R}^n$

since the regularity of a convolution inherited from the best guy. Another observation is that $K(t,x)$ is analytic in both t and x in $(0,\infty)\times\mathbb{R}^n$ and is not analytic in t at $t=0$ and uniformly analytic in x as $t\to 0$, by which we mean the coefficients in the series expansion are uniform in x as $t\to 0$. Therefore, u is analytic for all $t>0$.

Indeed, to prove u is analytic, we use the Taylor's inequality, that is, we need to show

$$\frac{\sup_{|x|<R}\int K^{(j)}(x-y)f(y)\,dy}{j!}|x|^j \to 0 \tag{21.1}$$

as $j\to\infty$. For $f\in\mathcal{E}'$, by the definition of distributions, we know that there exists some N which depends on $\mathrm{supp} f$ such that for any $|x|<R$,

$$\int K^{(j)}(x-y)f(y)\,dy \lesssim \sum_{|\alpha|\le N}\sup_{y\in\mathrm{supp} f}|\partial^\alpha K^{(j)}(x-y)| \lesssim \sum_{|\alpha|\le N}\sup_{|x-y|\in B(0,R)+\mathrm{supp} f}|\partial^\alpha K^{(j)}(x-y)|.$$

Then (21.1) holds as $j\to\infty$.

However, the solution above is not a local solution. To consider a local solution in an parabolic cylinder C. We define $v=\chi u$, where χ localized to C. Then

$$(\partial_t-\Delta)v = f := [(\partial_t-\Delta),\chi]u,$$

where $\mathrm{supp} f$ is inside the transition region. The picture is like the one we have in Section 19.2. This time, the inhomogeneous equation can be solved by $v=K(t,x)*_{t,x}f$. For $f\in\mathcal{E}'$, we only use K away from $(0,0)$ thanks to the support property of f, that is,

$$u(t,x)=\iint f(s,y)K(t-s,x-y)\,ds\,dy.$$

Since K is analytic away from 0 for x, then u is analytic in x. However, u is not necessarily analytic in t because each point in the transition region will have an impact for its future, so it starts to affect u right away. One can think of this by adding sources away from the cylinder region to change it. To make it work, we need to consider a boundary value problem instead of a local solution.

There is no difference for the variable coefficient case as long as the coefficients are analytic.

Remark 21.1. It is inappropriate to apply the Cauchy-Kowalevski theorem here even if the coefficients are all analytic since we need the assumption that the boundary condition is non-characteristic and it is only suitable for initial value problem.

21.3 Behavior at infinity in space - Maximum principle in $[0,T]\times\mathbb{R}^n$ for the heat equation

We discuss what happens as $x\to\infty$. Note that the energy estimates require decay at ∞ and the maximum principle requires boundedness at ∞. However, we do not expect these properties for u in general since if we consider the initial value problem, then

$$u(t,x)=\int u_0(y)K(t,x-y)\,dy$$

is a global in time solution as long as u_0 satisfies the property that for any $\varepsilon>0$, there exists C_ε such that

$$|u_0(y)|\le C_\varepsilon e^{\varepsilon|y|^2}.$$

For u_0 such that

$$|u_0(y)| \lesssim e^{C|y|^2},$$

one can solve up to $t < \frac{1}{4C}$. A trivial estimate for the fundamental solution of heat equation shows it can be defined up to $t < \frac{1}{16C}$. More precisely, when $2|x| < |y|$, so $e^{-|x-y|^2/4t} \leq e^{-|y|^2/16t}$ and hence $\int K(x-y,t)u_0(y)\,dy$ is well-defined up to $t < \frac{1}{16C}$. Given this, it is natural to consider the following formulation for the maximum principle in the whole region:

Proposition 21.2. *Suppose*

$$(\partial_t - \Delta)u = 0 \text{ in } [0,T) \times \mathbb{R}^n,$$

then

$$\sup_{[0,T]\times\mathbb{R}^n} u = \sup_{x\in\mathbb{R}^n} u(0,x),$$

provided $|u| \leq Ae^{C|x|^2}$.

Proof. We want to penalize u

$$u_\varepsilon = u - \varepsilon v,$$

where the function v at least $e^{2C|x|^2}$ at infinity, which kills the growth of u at infinity and allows us the apply the classical maximum principle in cylinders. We choose v by replacing $t \mapsto t - T$ in the fundamental solutions, that is,

$$v = \varepsilon(4\pi(T-t))^{-n/2} e^{|x|^2/4(T-t)}.$$

Thus, set

$$u_\varepsilon(x,t) := u - \varepsilon(4\pi(T-t))^{-n/2} e^{|x|^2/4(T-t)},$$

then we have

$$\max_{C_r} u_\varepsilon \leq \max_{\partial C_r} u_\varepsilon,$$

where C_r is the cylinder centered at $x = 0$ with radius r. Now we compute

$$\max_{\partial C_r} u_\varepsilon(x,t) \leq Ae^{Cr^2} - \varepsilon(4\pi T)^{-n/2} e^{r^2/4T} \leq \sup_{\mathbb{R}^n} u(0,x)$$

if $C < 1/4T$ and r is sufficiently large. So $\sup_{[0,T_1]\times\mathbb{R}^n} u \leq \sup_{x\in\mathbb{R}^n} u(0,x)$ when $T_1 < 1/4C$. One can do an induction on the time interval to view T_1 as the initial time again to conclude for general $T < \infty$. $\square$

21.4 Unique continuation

Obviously, we have two aspects, time and space. In terms of time behavior, by solvability, if the initial data is zero, then the solution is zero. Given a solution u in $[0,T]$, if we know $u(T)$, can we find $u(0)$, $u[0,T]$? This goes by the name backward unique continuation.

Theorem 21.3. *Suppose* $u_t - \Delta u + Vu = 0$ *in* $\Omega \times [0,T]$, $u|_{\partial\Omega} = 0$, *if* $u(T) = 0$, *then* $u([0,T]) = 0$.

Proof. We still prove some type of Carleman estimate first. We claim

$$\tau^{1/2}\|e^{\tau\phi}u\|_{L^2} \le \|e^{\tau\phi}(\partial_t - \Delta)u\|_{L^2}, \tag{21.2}$$

where ϕ large on the top of the cylinder and ϕ small in the bottom. We try $\phi = \phi(t)$ and set $v = e^{\tau\phi}u$. Then it suffices to show $\|v\|_{L^2} \le \|P_\tau v\|_{L^2}$. We compute

$$P_\tau = e^{\tau\phi}(\partial_t - \Delta)e^{-\tau\phi} = \partial_t - \Delta - \tau\phi'(t),$$

where $P_\tau^a = \partial_t$ is anti-symmetric and $P_\tau^s := -\Delta - \tau\phi'(t)$ is symmetric. We write

$$\|P_\tau v\|_{L^2}^2 = \|P_\tau^s v\|_{L^2}^2 + \|P_\tau^a v\|_{L^2}^2 + 2\left\langle [P_\tau^s, P_\tau^a]v, v\right\rangle_{L^2_{t,x}},$$

where $[P_\tau^s, P_\tau^a] = \tau\phi''$. Thus we want ϕ increasing and convex. By choosing the simplest function $\phi(t) = t^2 - M$ for some constant M,

$$\left\langle [P_\tau^s, P_\tau^a]v, v\right\rangle_{L^2_{t,x}} \ge \tau\|v\|_{L^2}.$$

which completes the proof of the Carleman estimate.

Now we turn to the proof of the unique continuation property. We select $\chi(t)$ compactly supported in $[0,T]$ and $\chi \equiv 1$ near T. Then we select an M properly such that $\phi(t)$ is positive near T but negative in the transition region $\operatorname{supp}\chi' \times \Omega$. Let $w = \chi u$, then we have

$$\partial_t w - \Delta w + Vw = -\chi'(t)u,$$

which implies

$$\tau^{1/2}\|e^{\tau\phi}w\|_{L^2} \lesssim \|e^{\tau\phi}Vv\|_{L^2} + \|e^{\tau\phi}\chi'(t)u\|_{L^2}.$$

By absorbing the first term on the right hand side to the left, we reach the estimate

$$\tau^{1/2}\|e^{\tau\phi}w\|_{L^2(\{\phi>0\}\times\Omega)} \lesssim \|\tau^{1/2}\|e^{\tau\phi}w\|_{L^2} \lesssim \|e^{\tau\phi}\chi'(t)u\|_{L^2} \lesssim \|u\|_{L^2(\operatorname{supp}\chi'(t)\times\Omega)} \lesssim 1.$$

However, let $\tau \to \infty$, then the left hand side blows up unless $w \equiv 0$ in $\{\phi > 0\} \times \Omega$, which proves the theorem. □

Remark 21.4. In this proof, we use $-\Delta$ symmetric, where we implicitly use the boundary condition for u.

Theorem 21.5 (Strong unique continuation). *Suppose u solves $(\partial_t - \Delta)u - Vu = 0$ in Ω. If u vanishes of infinite order at (t_0, x_0), then u vanishes at $\{t = t_0\}$.*

We cannot further apply the weak unique continuation to conclude anything backward in time since we do not know boundary conditions. The Carleman estimate in this case is

$$\|t^{-\tau-1}e^{-|x|^2/8t}u\|_{L^2} \le \|t^{-\tau}e^{-|x|^2/8t}(\partial_t - \Delta)u\|_{L^2},$$

where τ is away from integers. The τ integers come from a change of variable $t = e^s$ and $y = x/\sqrt{t}$ and the spectrum of the Hermite operator $H = -\Delta_y + |y|^2$.

Hyperbolic equations

Date: April 6, 2023

First, we provide some examples of hyperbolic equations :

- The simplest case of hyperbolic equation is
$$\partial_t u = b^j \partial_j u + cu, \quad u(0) = u_0.$$
- Wave equation :
$$\begin{cases} \Box u = 0, \\ u(0) = u_0, \partial_t u(0) = u_1, \end{cases}$$
where $\Box = -\partial_t^2 + \Delta$ is called the d'Alembertian. The variable coefficient case is given by
$$g^{\alpha\beta}\partial_\alpha \partial_\beta u = f, \tag{22.1}$$
where the Greek notation α, β means the summation over space and time $(t, x) = (x^0, x^1, \cdots, x^n)$.
- Hyperbolic systems : Suppose $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$, then $\partial_j u$ is a vector in $\mathbb{R}^m$. Suppose also A^j is an $m \times m$ matrix for any $1 \le j \le n$. The equation
$$\partial_t u = A^j \cdot \partial_j u \tag{22.2}$$
is called a hyperbolic system.

Remark 22.1. Note that not all equations of the forms (22.1) and (22.2) are wave equations and hyperbolic systems, respectively. We need to impose some conditions on $g^{\alpha\beta}$ and A^j.

Remark 22.2. We care about hyperbolic systems since many physical problems are in a system instead of a single equation.

(1) If we study a gas, we need to consider velocity and density together.
(2) In electromagnetics, we want to treat electric field and magnetic field at the same time.
(3) Imagine you have a solid that have elasticity, the motion of the solid's oscillation has three directions to move so we need to study three equations as a system.

Sometimes the system is first order while sometimes is second order.

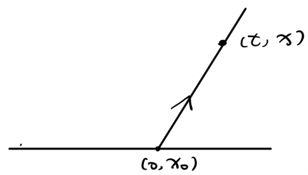
22.1 Transport equation

The transport equation is a baby version of a hyperbolic system.
We consider
$$(\partial_t - b^j \partial_j) u = c \cdot u, \tag{22.3}$$
where $\partial_t - b^j \partial_j$ can be thought as a directional derivative along $v = (1, -b^1, \cdots, -b^n)$ with b^j constants. Then the equation is equivalent to
$$\nabla_{x,t} u \cdot v = c \cdot u.$$

22.1.1 Constant coefficient case

For $(0, x_0) \in \mathbb{R}^{1+n}$ with $x_0 \in \mathbb{R}^n$, we consider

$$x_t = x_0 - tb.$$



If we are in the constant coefficient case, (22.3) becomes an ODE :

$$\frac{d}{dt}u(t, x_t) = c \cdot u(t, x_t),$$

which is able to write down the solution $u(t, x_t) = e^{ct}u(0, x_0)$. By setting $y = x_t = x_0 - tb$, we start from (t, y) and go backward to initial time, then we derive

$$u(t, y) = e^{ct}u_0(y + bt).$$

22.1.2 Variable coefficient case

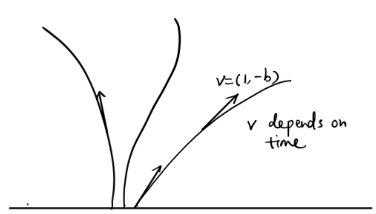
Now we consider the variable coefficient case. Suppose $v = (1, -b(t, x))$ depends on time, then we need to work with curves instead of lines such that our vector field v tangent to the curves. Then it is natural to introduce the notion of characteristics.

Definition 22.3. Characteristic curves are those curves that are tangent to v at every point.

For any t, x_t is given by

$$\frac{d}{dt}x_t = -b(t, x_t),$$

which is a system of ODEs.



We couple these with our equation, which gives an ODE in $n + 1$ components

$$\begin{cases} \frac{d}{dt}x_t = -b(t, x_t), \\ \frac{d}{dt}u(t, x_t) = cu(t, x_t). \end{cases} \tag{22.4}$$

This system is actually decoupled since in principle, we can solve the first n nonlinear equations first and then solve the last one. Later, when we discuss nonlinear hyperbolic scalar equation such as the Burger's equation, we still have a similar system. However, it is not possible to decouple them in such a simple way.

Remark 22.4. The uniqueness of solutions to ODEs ensures that the characteristics cannot intersect or merge at some point.

22.2 Recap of the wave equation

22.2.1 Fundamental solutions

For $\Box = -\partial_t^2 + \Delta_x$, the forward fundamental solution is given by

$$\Box K(t,x) = \delta_{(0,0)}, \quad \mathrm{supp} K \subset \{t \geq 0\}.$$

In fact,

$$\mathrm{supp} K \subset \{(t,x) : t^2 \geq x^2, t > 0\} := C$$

and this relates to the finite speed of propagation. Thanks to homogeneity, we know K is homogeneous of $1-n$. (Note that the fundamental solution of Laplacian equation is homogeneous of $2-n$.)

In 1 dimension,

$$K(t,x) = \begin{cases} \frac{1}{2}, & \{(t,x) : t^2 \geq x^2, t > 0\}, \\ 0, & \{(t,x) : t^2 < x^2, t > 0\}, \end{cases}$$

where $\Box = (\partial_t - \partial_x)(\partial_t + \partial_x)$ is a product of transport operators. In 2 dimensions,

$$K(t,x) = \frac{1}{2\pi} \frac{1}{\sqrt{t^2 - x^2}} |_C$$

and for higher dimensions, K cannot be viewed as a function. In 3 dimensions, we have

$$K(t,x) = c_3 \delta_{t=|x|},$$

where the support is on the cone. This is called the Huygens principle and this is a phenomenon only true for constant coefficient case. When n is even,

$$K(t,x) = c_n \cdot (t^2 - x^2)_+^{\frac{1-n}{2}}.$$

When n is odd,

$$K(t,x) = c_n \cdot t^{\frac{1-n}{2}} \delta_{t=|x|}^{(\frac{n-3}{2})}.$$

22.2.2 Duhamel's formula

For

$$\begin{cases} \Box u = f, \\ u(0) = u_0, \partial_t u(0) = u_1, \end{cases}$$

we can solve by the fundamental solution for $t \geq 0$:

$$u = K *_{t,x} f + K(t) *_x u_1 + \partial_t K(t) *_x u_0.$$

Since $K(t) \in \mathcal{E}'(\mathbb{R}^n)$, so we can solve by this formula for any $u_0, u_1 \in \mathcal{D}'$. Moreover, due to the support property of K, if we choose $f \in \mathcal{D}'$ with $\mathrm{supp} f \in \{t > 0\}$, then $K *_{t,x} f$ is also well-defined. One can also relate this property with the finite speed propagation. Due to the finite speed propagation, we don't need to care about what happens at $x \to \infty$ at initial time if we want to solve in a finite time, so we can choose $\mathcal{D}'$ initial data.

22.2.3 Symmetries of the wave equation

In the end, we recall the symmetries of the wave equation. First, the rigid spatial rotations are symmetries for the wave equation. On the other hand, we can also mix space and time together and consider the transform $y^\alpha = A^\alpha_\beta x^\beta$. Those A's such that $\Box$ is unchanged are such that

$$AMA^T = M,$$

where $\Box = m^{\alpha\beta}\partial_\alpha\partial_\beta$. These A's are called Lorentz group, whose generators are space rotations and Lorentz boosts $L = x_j\partial_0 + t\partial_j$. In 1 dimension, we consider

$$u = \frac{1}{2}(t-x), v = \frac{1}{2}(t+x),$$

then

$$\partial_t^2 - \partial_x^2 = \partial_u \cdot \partial_v,$$

where $u \mapsto \lambda u$, $v \mapsto \lambda^{-1} v$ leaves the $\Box$ unchanged. The level sets

$$u \cdot v = t^2 - x^2 = const$$

are hyperbolas. Lorentz boosts leaves these unchanged.

Conserved energies : The energy is given by

$$E[u](t) = \int \frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla_x u|^2 \, dx, \quad \frac{d}{dt}E(u) = 0,$$

where the first one is the kinetic energy and the second one is the potential energy. The momentum is given by

$$P_j(u) = \int \partial_t u \cdot \partial_j u \, dx, \quad \frac{d}{dt}P_j(u) = 0.$$

Conserved quantities for wave equations - 1

Date: April 11, 2023

In today's lecture, we use the convention $\Box = \partial_t^2 - \Delta$. We forget the fundamental solution for a moment and want to use the energy estimates to study

$$\begin{cases} \Box u = f \text{ in } \mathbb{R} \times \mathbb{R}^n, \\ u(0) = u_0, \quad \partial_t u(0) = u_1. \end{cases}$$

We define

$$E[u] = \frac{1}{2} \int |\partial_t u|^2 + |\partial_x u|^2 \, dx,$$

and last time we showed that $\partial_t E[u] = 0$, which gives us a conservation law. But a conservation law only tells us that if an integral quantity is conserved, then the energy at each time is the same. However, we don't know how much energy goes there and how much goes to the other place. To describe this, we want to study the density-flux relations

$$\partial_t \rho = \partial_j f_j,$$

which describes how the density ρ of some gas is moving around. Here, $F = (f_1, \cdots, f_n)$ is a vector field, representing the flux, which tells us how much gas is incoming/outgoing of the domain we are interested in. Combined with the divergence theorem, this relation implies that

$$\frac{d}{dt} \int \rho = 0.$$

23.1 Energy-momentum tensor

Given this philosophy, we want to see how the energy moves around. We define the energy density of a function u as

$$e(u)(x) = \frac{1}{2} \left(|\partial_t u(x,t)|^2 + |\partial_x u(x,t)|^2 \right)$$

and E is the integral of this quantity. We compute

$$\partial_t e = \partial_t u \cdot \partial_t^2 u + \partial_t \partial_j u \cdot \partial_j u = \partial_t u \cdot \Box u + \partial_t u \partial_j \partial_j u + \partial_t \partial_j u \cdot \partial_j u = \partial_j (\partial_t u \cdot \partial_j u) + \Box u \cdot \partial_t u \quad (23.1)$$

Inspired from the result produced by this computation, we also define the momentum density by

$$p_j(u) = \partial_t u \cdot \partial_j u$$

and the momentum is given by

$$P_j(u) = \int \partial_t u \cdot \partial_j u \, dx.$$

The density flux relation for the momentum is analogous. Indeed,

$$\begin{aligned} \partial_t(\partial_j u \cdot \partial_t u) &= \partial_j \partial_t u \cdot \partial_t u + \partial_j u \cdot \partial_t \partial_t u = \partial_j u \cdot \Box u + \partial_j u \cdot \partial_k \partial_k u + \partial_j \partial_t u \cdot \partial_t u \\ &= \partial_j u \cdot \Box u + \frac{1}{2} \partial_j |\partial_t u|^2 + \partial_k(\partial_j u \cdot \partial_k u) - \partial_j \partial_k u \cdot \partial_k u \end{aligned}$$

and hence

$$\partial_t p_j(u) = \partial_k(\partial_j u \cdot \partial_k u) + \frac{1}{2}\partial_j \left(|\partial_t u|^2 - |\partial_x u|^2\right) + \partial_j u \cdot \Box u. \tag{23.2}$$

Recall that the Lagrangian interpretation for the Laplacian equation is to minimize $\int |\nabla u|^2$. The reason why we group $|\partial_t u|^2 - |\partial_x u|^2$ together is that this is just the Lagrangian for the wave equation.

Now we introduce a matrix, which has some geometric flavor - energy momentum tensor $T_{\alpha\beta}$. The reason why it is called a tensor is that it is invariant under a change of coordinates. For now, we do not need to worry about this and can think of this as a matrix. We want to use this to rewrite our $n+1$ conservation laws. We seek $T^{\alpha\beta}$ such that

$$\partial_\alpha T^{\alpha\beta} = \Box u \cdot \partial^\beta u,$$

where $\partial^\beta u = m^{\alpha\beta}\partial_\alpha u$. We define

$$T^{\alpha\beta} = \begin{pmatrix} e & p^1 & p^2 & \cdots & p^n \\ p^1 & |\partial_1 u|^2 + \frac{1}{2}(|\partial_t u|^2 - |\partial_x u|^2) & \partial_1 u \partial_2 u & \cdots & \partial_1 u \partial_n u \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p^n & \partial_n u \partial_1 u & \partial_n u \partial_2 u & \cdots & |\partial_n u|^2 + \frac{1}{2}(|\partial_t u|^2 - |\partial_x u|^2) \end{pmatrix}, \tag{23.3}$$

where the derivative of the first column can be written into a combination of the following columns thanks to the computation we did in (23.1) and (23.2). This is not a good way to represent the matrix and we want to write it in a more condensed way. Notice that for $\alpha = \beta \neq 0$,

$$T^{\alpha\beta} = \partial^\alpha u \partial^\beta u + \frac{1}{2}(|\partial_t u|^2 - |\partial_x u|^2)$$

and the energy density e can be written into a similar form

$$e = \frac{1}{2}(|\partial_t u|^2 + |\partial_x u|^2) = |\partial_t u|^2 - \frac{1}{2}(|\partial_t u|^2 - |\partial_x u|^2).$$

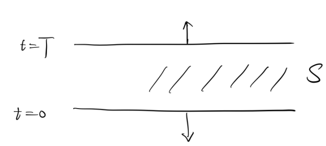
At the end of the day, we write the whole matrix in a geometric flavor way by

$$T^{\alpha\beta} = \partial^\alpha u \partial^\beta u - \frac{1}{2} m^{\alpha\beta}\partial^\gamma u \partial_\gamma u. \tag{23.4}$$

Last time, the d'Alembertian is $\Box = \partial_t^2 - \partial_x^2$. However, going forward, we use $\Box = -\partial_t^2 + \partial_x^2 = m^{\alpha\beta}\partial_\alpha\partial\beta = \partial^\alpha\partial_\alpha$ as the definition. We redo the computations by using this geometric notation :

$$\begin{aligned} \partial_\alpha T^{\alpha\beta} &= \partial_\alpha\partial^\alpha u \cdot \partial^\beta u + \partial^\alpha u \cdot \partial_\alpha \partial^\beta u - m^{\alpha\beta}\partial_\alpha\partial_\gamma u \cdot \partial^\gamma u \\ &= \partial_\alpha\partial^\alpha u \cdot \partial^\beta u + \partial^\alpha u \cdot \partial_\alpha \partial^\beta u - \partial^\beta\partial_\gamma u \cdot \partial^\gamma u = \Box u \cdot \partial^\beta u. \end{aligned} \tag{23.5}$$

Example 23.1 (Energy conservation)**.** Suppose we have a strip S in the spacetime region between $t = T$ and $t = 0$.



From $\partial_\alpha T^{\alpha\beta} = 0$, we know

$$\int_S \partial_\alpha T^{\alpha\beta} = 0.$$

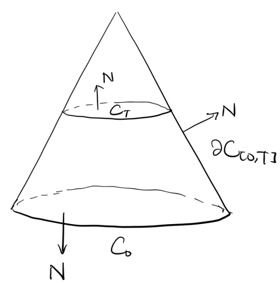
This implies

$$\int_{t=T} T^{0\beta} = \int_{t=0} T^{0\beta}.$$

If $\beta = 0$, then we get the conservation of energy and if $\alpha = 0$, then we get the conservation of momentum.

23.2 Finite speed of propagation

Now we want to use the energy momentum tensor to prove the finite speed of propagation, that is, data in I determines solution in C.



Furthermore, this is equivalent to say if the initial data is zero in I, then $u = 0$ in C. We consider the truncated cone $C_{[0,T]}$ and compute

$$\int_{C[0,T]} \partial_\alpha T^{\alpha 0} = 0,$$

which gives

$$\int_{C_T} T^{\alpha 0} N_\alpha + \int_{\partial C[0,T]} T^{\alpha 0} N_\alpha + \int_{C_0} T^{\alpha 0} N_\alpha = 0$$

and hence

$$\int_{C_0} e\,dx = \int_{C_T} e\,dx + \int_{\partial C[0,T]} (T^{00}, T^{10}, \cdots, T^{n0}) \cdot \frac{1}{\sqrt{2}}(1, \vec{e_r}) = \int_{C_T} e\,dx + \frac{1}{\sqrt{2}} \int_{\partial C[0,T]} (e + \vec{e_r} \cdot p).$$

By our assumption, the left hand side is zero and $\int_{C_T} e\,dx$ is positive, so it suffices to show the positiveness of the last term. We compute

$$e + \vec{e_r} \cdot p = \frac{1}{2}(|\partial_t u|^2 + |\partial_x u|^2) - e_r \cdot \nabla u \cdot \partial_t u \geq 0$$

where the positiveness follows from the Cauchy-Schwartz inequality since e_r is a unit vector. This completes the proof of finite speed of propagation. In fact, one can actually do some algebra to show that

$$e + \vec{e_r} \cdot p = \frac{1}{2} |\nabla_T u|^2,$$

where ∇_T is the tangential derivative. This is easy to prove since one can just assume $e_r = (0, 1, 0, \cdots, 0)$ without loss of generality thanks to the rotational symmetry.

Now we want to discuss a more general case where we change the cone to a region between any two hypersurfaces Σ and $\{t = 0\}$. In fact, the previous computation is pretty robust and we want to find the hypotheses we need to make the finite speed propagation work in this region. It turns out that we need Σ to be spacelike.

We want to show that we can determine $u|_\Sigma$ from $u|_{\{t=0\}}$. Thanks to the divergence theorem, we obtain

$$\int_{t=0} e\,dx = \int_\Sigma N_\alpha T^{\alpha 0}\,dx.$$

As what we did in the previous computation, if

$$N_0^2 \geq \sum N_j^2, \tag{23.6}$$

that is, the size of the temporal part of the normal controls the size of the spatial part of the normal, then we know that the right hand side is positive definite and thus implies the finite speed of propagation. However, the relation (23.6) written in this way not elegant. In fact, it is equivalent to

$$m^{\alpha\beta} N_\alpha N_\beta \leq 0$$

and it motivates the following definition.

Definition 23.2. We say Σ is spacelike if $|N|^2 < 0$ and time-like if $|N|^2 > 0$. We say Σ is null or characteristic if $|N|^2 = 0$.

Remark 23.3. In fact, we need to distinguish N and X since in geometry, the normal N is a co-vector and X is a vector. The vector perpendicular to the $\partial C[0, T]$ is actually inside this surface since it is a null surface.

Moreover, the previous computation goes through by applying divergence theorem in the Euclidean space. Instead, we can try to compute in a more geometric way. However, this may involve a slight issue that we cannot induce a measure on the null surface uniquely since the metric is degenerate when it is restricted to the null surface. However, the area form $n_\beta\,dS$ remains well-defined in the limit and one can refer to [21, Section 2.5].

In this setting, the constant t slices do not make so much sense. We can state a better local well-posedness. We compute

$$\int_{\Sigma_0} N_\alpha T^{\alpha 0}\,d\sigma = \int_{\Sigma_1} N_\alpha T^{\alpha 0}\,d\sigma.$$

If Σ_1, Σ_0 are spacelike, then we have these two positive definite quantities. We will discuss in detail next time. There is also no point of multiplying $\partial_t u$, where ∂_t is a pretty specific choice. Instead, we can use Xu as a multiplier as long as the vector field X can be transformed to be ∂_t by a Lorentz transform, which means that X is time-like.

Definition 23.4. We say X is time-like if $|X|^2 < 0$ where we have forward and backward. X is null if $|X|^2 = 0$ and X is spacelike if $|X|^2 > 0$.

Conserved quantities for wave equations - 2

Date: April 13, 2023

Last time, we discussed the energy-momentum tensor

$$T^{\alpha\beta} = \partial^\alpha u \partial^\beta u - \frac{1}{2} m^{\alpha\beta} \partial_\gamma u \partial^\gamma u, \tag{24.1}$$

where $\partial_\alpha T^{\alpha\beta} = \Box u \cdot \partial^\beta u$, which is verified in (23.5). Here $\beta = 0$ corresponds to the energy conservation while $\beta \neq 0$ corresponds to momentum conservation.

24.1 Deformation tensor of the wave equation

24.1.1 Deformation tensors

Since ∂^0 is not invariant under the Lorentz transformation, so we consider more general vector field. Suppose X is a vector field $X = X^\alpha \partial_\alpha$, then we define a co-vector $X_\beta = m_{\alpha\beta} X^\alpha$. Multiplying (24.1) by X_β, we get

$$\partial_\alpha T^{\alpha\beta} \cdot X_\beta = \Box u \cdot X_\beta \partial^\beta u = \Box u \cdot X^\beta \partial_\beta u, \tag{24.2}$$

and we can see from the symmetry of $T_{\alpha\beta}$ that

$$\partial_\alpha (T^{\alpha\beta} X_\beta) = \Box u X^\beta \partial_\beta u + T^{\alpha\beta} \partial_\alpha X_\beta = \Box u X^\beta \partial_\beta u + T^{\alpha\beta} \frac{1}{2} (\partial_\alpha X_\beta + \partial_\beta X_\alpha),$$

where we symmetrize $\partial_\alpha X_\beta$ in the last step and it has a geometric meaning.

Definition 24.1. The deformation tensor of X is given by

$$\pi^X_{\alpha\beta} = \partial_\alpha X_\beta + \partial_\beta X_\alpha.$$

In geometry, by the definition of Lie derivatives $\mathcal{L}_X$ along X of the Minkowski metric g, we have

$$\mathcal{L}_X (g(\partial_\alpha, \partial_\beta)) = (\mathcal{L}_X g)(\partial_\alpha, \partial_\beta) + g(\mathcal{L}_X \partial_\alpha, \partial_\beta) + g(\partial_\alpha, \mathcal{L}_X \partial_\beta),$$

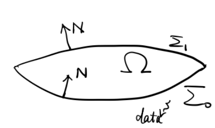
and thus

$$(\mathcal{L}_X g)_{\alpha\beta} = \partial_\alpha X^\mu g_{\mu\beta} + \partial_\beta X^\mu g_{\mu\alpha} = \partial_\alpha X_\beta + \partial_\beta X_\alpha.$$

The geometry interpretation of the deformation tensor is as follows. We have a vector field X, which determines a flow, like what we discussed for the transport equation. One can look at the flow lines $x(s)$ of the vector field X given by

$$\dot{x}^\alpha = X^\alpha(x)$$

and the deformation tensor measures to what extent the Minkowski space is left invariant with respect to the flow of X. See the definition of Lie derivatives in [18, Chapter 1] . One can view the flow as a mapping $x \to S(t)x$ from $\mathbb{R}^n \to \mathbb{R}^n$, where we map g correspondingly.



Given data on the initial surface Σ_0 and suppose that we can solve up to Σ_1. Then we integrate both sides to obtain

$$\int_\Omega \partial_\alpha(T^{\alpha\beta}X_\beta)\,dx = \int_\Omega \Box u X^\beta \partial_\beta u + \pi^X_{\alpha\beta}T^{\alpha\beta}\,dx, \tag{24.3}$$

where the left hand side, by the divergence theorem, is

$$\int_{\Sigma_1} n_\alpha T^{\alpha\beta}X_\beta\,d\sigma - \int_{\Sigma_0} n_\alpha T^{\alpha\beta}X_\beta d\sigma,$$

where we choose n as forward normal to the surface.

We can think of $\int_{\Sigma_i} n_\alpha T^{\alpha\beta}X_\beta\,d\sigma$ as the energy measured with respect to the vector field X for $i=0,1$ since for $X=\partial_t$, it just gives the energy while $X=\partial_j$, it corresponds to the momentum. We would like it to have it to be positive definite so that the size of the solution can be controlled by this quantity. Though we want the energy to be conserved, but the right hand side of (24.3) is probably not zero. It naturally reminds us two questions to ask :

(1) Is $n_\alpha T^{\alpha\beta}X_\beta$ positive definite?
(2) When does X give an exact conservation law?

From the previous discussion, $X=\partial_t$ gives the real energy, so it is positive definite. However, for $X=\partial_j$, it is the momentum and hence it is not positive definite. The first one can be addressed by the following proposition.

Proposition 24.2. *Assume X is forward time-like and Σ is spacelike, then*

$$n_\alpha T^{\alpha\beta}X_\beta$$

is positive definite, where n is a forward normal.

Proof. One can think of a geometric proof (using the Lorentz transform) or a purely algebraic proof. We leave the proof of this as a HW problem. □

Remark 24.3. This proposition is not true for the corresponding energy-momentum tensor for Schrodinger equation and many other equations. However, it is already general enough since it also works for variable coefficients wave equations.

24.1.2 Killing vector fields in Minkowski space

For the second question, if the source term $\Box u = 0$ and $\pi^X = 0$, then we know that the energy is conserved thanks to (24.3). We want to find some flows such that the d'Alembertian remains unchanged under such flows. Recall that $\pi^X_{\alpha\beta} = \partial_\alpha X_\beta + \partial_\beta X_\alpha$. The simplest examples are $X=\partial_t, \partial_j$, which correspond to a translation forward in time and a translation flows in the j direction, respectively. For the rest of the examples, let's think this backwards. We start from our symmetries and then try to find what is the corresponding vector field. One example of symmetry is the rigid rotation, where $X = x_i\partial_j - x_j\partial_i$ corresponds the rotation in the plane determined by i,j. Now recall that in the Lorentz group, besides rigid rotations, we have the Lorentz boosts (Lorentz rotations), where the picture looks like squeezing in one direction and expanding in the other, which are given by $X = x_j\partial_0 + x_0\partial_j$. Since all these symmetries leave the metric $-dt^2+dx^2$ the same, X are all Killing. However, for the vector field $S = t\partial_t + x\partial_x$ generated by the scaling symmetry, it is not Killing since the corresponding change on the metric $-dt^2 + dx^2$ is conformal. One can verify that S is conformal Killing, that is, $L_S g = cg$ with some c.

Recall that $\pi^X = \mathcal{L}_X g$ for the Minkowski metric g, so $\pi^X = 0$ implies that X is Killing.

24.2 Variable coefficient wave equations

If we proceed as the second order elliptic operators, then we write

$$P = -g^{\alpha\beta}\partial_\alpha\partial_\beta + b^\alpha\partial_\alpha + c.$$

The first thing we need to impose conditions such that it is a wave equation. We say P is a wave operator if $g^{\alpha\beta}$ has signature $(n, 1)$. The principal symbol is $p(x, \xi) = g^{\alpha\beta}\xi_\alpha\xi_\beta$. Moreover,

$$(g^{\alpha\beta})^{-1} = (g_{\alpha\beta}),$$

where $g_{\alpha\beta}$ is the associated metric $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$.

We define the energy-momentum tensor

$$T^{\alpha\beta} = \partial^\alpha u\partial^\beta u - \frac{1}{2}g^{\alpha\beta}\partial_\gamma u\partial^\gamma u.$$

Recall that in the constant coefficient case, the equation (24.2) we have is not true anymore. Instead, in the variable coefficient case, we have extra terms like $\nabla g \cdot (\nabla u)^2$. Moreover, if we take a vector field X and contract $T^{\alpha\beta}$ with X, we get

$$\partial_\alpha(T^{\alpha\beta}X_\beta) = \partial^\alpha\partial_\alpha u \cdot Xu + \frac{1}{2}(\partial_\alpha X_\beta + \partial_\beta X_\alpha)T^{\alpha\beta} + \nabla g(\nabla u)^2. \tag{24.4}$$

As before, we consider exactly the same picture and integrate both sides over Ω to get

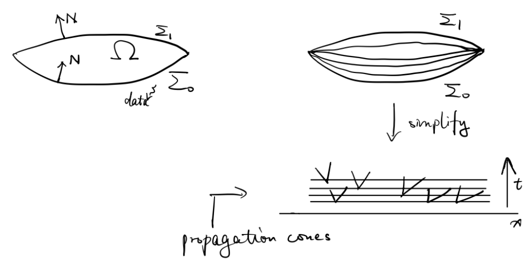
$$\int_{\Sigma_1} n_\alpha T^{\alpha\beta}X_\beta - \int_{\Sigma_0} n_\alpha T^{\alpha\beta}X_\beta = \int_\Omega \partial^\alpha\partial_\alpha u \cdot Xu + \frac{1}{2}(\partial_\alpha X_\beta + \partial_\beta X_\alpha)T^{\alpha\beta} + \nabla g(\nabla u)^2.$$

Now we need to focus on the same two questions again. Note that even though we are in the variable coefficient case, nothing changes in Proposition 24.2 and it still holds since it is just a point-wise property and does not involve how the metric changes. The proof is exactly the same after we make a linear change of coordinates to change it back to Minkowski metric at one single point you are interested in. We classify surfaces as follows :

(1) Σ is spacelike if $g^{\alpha\beta}N_\alpha N_\beta < 0$;
(2) Σ is null (characteristic) if $g^{\alpha\beta}N_\alpha N_\beta = 0$;
(3) Σ is time-like if $g^{\alpha\beta}N_\alpha N_\beta > 0$;

Remark 24.4. The reason why we call null also by characteristic is that somehow a plane is characteristic means that the solution has a jump at this plane. For example, for $N = \nabla u$, $N_\alpha m^{\alpha\beta}N_\beta = 0$ is exactly the condition which leads to the nonexistence of δ'_Σ when we try to compute $\Box u$ at the plane when u has a jump from the left to the right. Therefore, the case when P is null is the only case that $\Box u = 0$ can have a solution with a plane P separating the solution at which u has a jump.

Remember that even if we have an energy estimate, it is not good enough to derive a local well-posedness theory. Suppose we have time-slices as foliations, which corresponds to the t-slices in the Minkowski case, where $\Sigma_{t_0} = \{t = t_0\}$ are spacelike. We want to start with spacelike surfaces and all the way through spacelike surfaces.



At every point, we still have a propagation cone. To prove a good energy estimate, we require our vector field X to be in this cone. The property Σ_{t_0} is spacelike requires the cones are pointing up. We also need to choose X and this time ∂_t may not be a good choice this time.

We just do a heuristic computation. Instead of integrating over space-time region, we integrate both sides of (24.4) purely in x without integration in time involved so that we obtain

$$\partial_t \int T^{0\beta} X_\beta \, dx = \int f \cdot Xu \, dx + \int (\nabla g, \nabla X)(\nabla u)^2 \, dx \tag{24.5}$$

where we assume $g^{\alpha\beta}\partial_\alpha\partial_\beta u = f$ for simplicity. Since $\int T^{0\beta} X_\beta \, dx \simeq \int |\nabla_{t,x} u|^2$ is comparable to $E(u)(t)$ and hence

$$RHS \leq \|f\|_{L^2}\|\nabla u\|_{L^2} + (\|\nabla g\|_{L^\infty} + \|\nabla X\|_{L^\infty})\|\nabla u\|_{L^2}^2.$$

Finally, we write

$$\frac{d}{dt}E(u) \lesssim \|f\|_{L^2}(E(u))^{1/2} + (\|\nabla g\|_{L^\infty} + \|\nabla X\|_{L^\infty})E(u).$$

We assume furthermore g, X are Lipschitz. If $f = 0$, then we can apply Gronwall's inequality directly. If $f \neq 0$, we compute

$$\frac{d}{dt}\sqrt{E(u)} \leq \|f\|_{L^2} + C\sqrt{E(u)}$$

and hence

$$E(u)(t) \leq E(u)(0)e^{Ct} + \|f\|_{L^1_t L^2_x} e^{Ct}.$$

Remark 24.5. Recall that the energy estimates for the elliptic case, we require the coefficients to be bounded. Here, we need to have $g \in Lip$.

In the case for bounded region, we make a rigorous computation to present the preceding idea. This is in fact a homework problem.

Proposition 24.6. *Consider a wave equation*

$$g^{\alpha\beta}\partial_\alpha\partial_\beta u = 0$$

with a Lipschitz metric in a cylinder $\Omega \times [0, T]$ *, with initial data*

$$u(t = 0) = u_0, \qquad \partial_t u(t = 0) = u_1$$

and Dirichlet boundary condition

$$u(t, x) = 0 \qquad \text{in } [0, T] \times \partial\Omega.$$

Assume that the surfaces $t = const$ are space-like, and that the lateral boundary $[0,T] \times \partial\Omega$ is time-like. Then prove that energy estimates hold for this problem. [Hint: Think of what vector field X you would use as a multiplier.]

Proof. First, we notice that all the computations work out as in (24.5). The only thing we need to prove is to select some X bounded and Lipschitz such that

$$\int_\Omega T^{0\beta} X_\beta \, dx \simeq \int_\Omega |\nabla_{t,x} u|^2 \, dx.$$

Note that if we want some Y such that $\langle Y, \partial_j \rangle = 0$ for $j = 1, 2, 3$ on the initial surface $\Omega \times \{t = 0\}$, then Y is timelike since $\partial_1, \partial_2, \partial_3$ are spacelike. This requires $Y^\alpha g_{\alpha j} = 0$. On the other hand, $Y^\delta = Y^\alpha g_{\alpha\beta} g^{\beta\delta} = Y^\alpha g_{\alpha 0} g^{0\delta} = Y_0 g^{0\delta}$. So we know that $Y = g^{0\delta}\partial_\delta = \partial^0$ is timelike at $\Omega \times \{t = 0\}$. Then $\int_\Omega T^{0\beta} X_\beta \, dx = \int_\Omega T(\partial^0, X) \, dx$ is positive definite and is comparable to $\int_\Omega |\nabla_{t,x} u|^2 \, dx$ as long as we choose X to be forward timelike as well.

Since $\partial_\alpha (T^{\alpha\beta} X_\beta) = f X u + (\nabla g, \nabla X) |\nabla u|^2$, we first naively want to choose X such that

- X forward timelike;
- $T^{j\beta} X_\beta N_j \geq 0$ with N the conormal in the Euclidean sense for the boundary $\partial\Omega \times [0,T]$.

Note that we have the Dirichlet boundary condition, so if we choose X tangent to the boundary, then $Xu = 0$. Since $\partial\Omega \times [0,T]$ is timelike, so we can choose a forward timelike vector field X near the parabolic boundary satisfying the requirements at least.

However, we do not want the Euclidean conormal. So we use the divergence theorem in the Lorentz setting directly by

$$\begin{aligned}
\int_\Omega \partial_j (T^{j\beta} X_\beta) \, dx &= \int_\Omega \mathrm{div}\, \Big(\frac{1}{\sqrt{|g|}} T^{\cdot\beta} X_\beta\Big) \sqrt{|g|} \, dx = \int_{\partial\Omega} T^{j\beta} X_\beta N_j \frac{1}{\sqrt{|g|}} dS \\
&= -\frac{1}{2} \int_{\partial\Omega} \langle N, N \rangle \langle X, N \rangle \partial_\gamma u \partial^\gamma u \frac{1}{\sqrt{|g|}} dS,
\end{aligned}$$

where N is the Lorentz normal of $\Omega \times \{t\}$. So $\langle X, N \rangle = 0$. ∎

Theorem 24.7. *Assume $g \in Lip$ with $(n,1)$-signature and we have nice spacelike surface foliations. Then the wave equation $\Box_g u = f \in L^1_t L^2_x$ is uniquely solvable in $u \in C_t H^1$ and $u_t \in C_t L^2$. To avoid the cumbersome two step notations, we introduce the notation*

$$u[t] = (u(t), \partial_t u(t)),$$

and then we require $u[\cdot] \in C(H^1 \times L^2)$.

We will discuss the proof of this in next class, which is a basic theorem about the local well-posedness of second order hyperbolic variable coefficient equations.

One can also refer to [16].

Wrap-up for second order hyperbolic equation and set-up for hyperbolic systems

Date: April 18, 2023

Last time, we introduced the notation $\Box_g = g^{\alpha\beta}\partial_\alpha\partial_\beta$, where g has signature $(n,1)$, which is a pseudo-Riemannian metric. Also, we assume $g \in Lip$ and we consider $\Box_g u = f$. The basic setup is to assume the initial data is

$$u(t=0) = u_0, \quad \partial_t u(t=0) = u_1.$$

Furthermore, we make the simplifying assumption that the surface $t = const$ are spacelike.

25.1 Local theory for second order variable coefficient hyperbolic equations

Theorem 25.1. *Assume $(u_0, u_1) \in H^1 \times L^2$ and $f \in L^1_t L^2_x$, then there exists a unique local solution u such that $u \in C_t H^1$ and $\partial_t u \in C_t L^2$.*

Remark 25.2. This is a local result and we do not need to worry about the behavior as $x \to \infty$ because of finite speed of propagation.

Moreover, we have the energy estimates

$$\|(u, u_t)(t)\|_{\dot{H}^1 \times L^2} \lesssim \|(u_0, u_1)\|_{H^1 \times L^2} + \|f\|_{L^1 L^2}.$$

Recall that in order to use the duality method to show the local existence, one needs to prove an energy estimates in $L^2 \times H^{-1}$, which can be found in [20]. There are also ways to go around this such as the one in [7] or one of the book in the series by Michael Taylor. We would not prove it but we will discuss the proof of hyperbolic system later.

Here is also an extension of this result is the higher regularity :

Theorem 25.3. *Assume $g \in C^k$, $(u_0, u_1) \in H^k \times H^{k-1}$, $f \in L^1_t H^{k-1}_x$. Then $u \in CH^k, \partial_t u \in CH^{k-1}$.*

Proof. For the corresponding energy estimates, we differentiate the equation $k-1$ times and apply the $H^1 \times L^2$ estimates. $\Box$

For more about variable coefficient wave equation, one can also refer to [2].

25.2 Comments on boundary value problems

Next, we briefly talk about the boundary value problem. Suppose we have the space-time cylinder $C = \Omega \times [0, T]$ and we focus on the wave equation. The bottom and top should be spacelike and the lateral boundary should be a time-like surface, which is a requirement for the causality.

Imagine the wave emits from the bottom and when the wave hits the boundary, it reflects. The boundary condition tells you how it reflects. The phase shift in the Dirichlet case is different the one in the Neumann case.

There is another way to formulate an initial value problem which lets it look like a boundary value problem. One can look it up on Wikipedia by searching "Goursat problem".

25.3 Differential geometry setup

Now we briefly discuss how to set up a wave equation in differential geometry, which also works for the Laplacian operator. Here, geometric means that it does not depend on the coordinates. In the elliptic case, the operator is the Laplace-Beltrami operator $\Delta_g = \frac{1}{\sqrt{|g|}}\partial_i\sqrt{|g|}g^{ij}\partial_j$ if g is Riemannian. While g is Lorentz, we say $\Box_g$, which is defined in the same way, is the covariant d'Alembertian.

One can compute

$$\nabla_\alpha T^{\alpha\beta} = \Box_g u \cdot \partial^\beta u,$$

where the ∇_α is the covariant derivative with respect to the Levi-Civita connection. This means that there is no lower order terms if one phrase the problem in geometry. We compute by using the properties listed in [21, Exercise 6.5] as follows :

$$\begin{aligned}\nabla_\alpha T^{\alpha\beta} &= \nabla_\alpha(\partial^\alpha u\partial^\beta u - \frac{1}{2}g^{\alpha\beta}\partial_\gamma u\partial^\gamma u) = \Box_g u\partial^\beta u + \partial^\alpha u\nabla_\alpha\partial^\beta u - \frac{1}{2}g^{\alpha\beta}\nabla_\alpha\partial_\gamma u\partial^\gamma u - \frac{1}{2}g^{\alpha\beta}\nabla_\alpha\partial^\gamma u\partial_\gamma u\\ &= \Box_g u\partial^\beta u + \nabla^\alpha u\nabla_\alpha\nabla^\beta u - \nabla^\beta\nabla_\gamma u\nabla^\gamma u = \Box_g u\partial^\beta u,\end{aligned}$$

where in the last step, we use the zero torsion property in [21, Exercise 6.5]. Here, $\Box_g u = \nabla_\alpha\nabla^\alpha u$ instead.

Since the divergence $\nabla_\alpha X^\alpha$ of any vector field X satisfies

$$\nabla_\alpha X^\alpha = \partial_\alpha X^\alpha$$

when $\sqrt{g} = const$. However, it does not work for 2-tensors $\nabla_\alpha T^{\alpha\beta} \neq \partial_\alpha T^{\alpha\beta}$. Instead, we contract it with X first and consider

$$\nabla_\alpha(T^{\alpha\beta}X_\beta).$$

25.4 Nonlinear wave equations

For $\Box u = u^p$, it is a semi-linear wave equation. For $\Box_{g(u)}u = \cdots$, it is a quasi-linear equation. One can also prove local well-posedness for these equations.

As an example, we discuss the Einstein equation. The Einstein equation in general relativity is supposed to describe our spacetime. In other words, the unknown here is the metric itself. To write the equation, we write an equation to each component. For each α, β, we have the equation of the form

$$\Box_g g^{\alpha\beta} = \Gamma(g) \cdot (\nabla g)^2.$$

The catch here is you need to first choose a coordinate and then formulate the equation. This goes by the name gauge choice.

25.5 First order hyperbolic systems

Suppose $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$, we consider

$$\partial_t u = A^j \partial_j u + f, \tag{25.1}$$

where A^j is a $m \times m$ matrix for each j. The easiest case is the constant coefficient case, that is, A^j's are all constants. Moreover, if $A^j = A^j(x,t)$, then it is the variable coefficient case while $A^j = A^j(x,t,u)$ corresponds to the quasi-linear case. Note that this is first order in time, so we do not need the initial velocity to formulate an initial value problem.

25.5.1 Definition of hyperbolic systems

First, we need to determine when (25.1) is hyperbolic. To motivate, we look at the constant coefficient ones with real coefficients so that we can apply the Fourier transform in x and obtain an ODE system

$$\partial_t \widehat{u}(t,\xi) = iA^j \xi_j \widehat{u}(t,\xi).$$

Recall that in ODE theory, to solve this equation, we need to look at the eigenvalues of A^j. For each ξ, $\xi_j A^j$ has the same family of eigenvalues multiplied by A^j. For instance, if λ_m, v_m's are eigenvalues and eigenvectors of A^j, then the solution is of the form

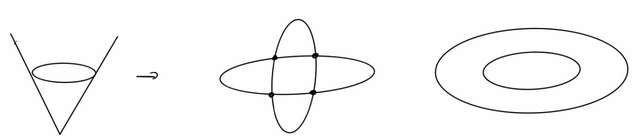
$$\widehat{u}(t,\xi) = \sum c_m(\xi) e^{it\lambda_m(\xi)} v_m(\xi)$$

provided that $A^j \xi_j$ is diagonalizable. (Here, it is a sum over j.)

If $\mathrm{Im}\lambda_m < 0$, then there are no solutions since it has an exponential growth in the exponent, which is like to solve a heat equation backwards. However, if we change $\xi \to -\xi$, λ_m will be turned into $-\lambda_m$ and therefore, the case $\mathrm{Im}\lambda_m > 0$ is also a bad case scenario.

Definition 25.4. We say that the system is hyperbolic if the matrix $A^j \xi_j$ only have real eigenvalues.

Unfortunately, this requirement cannot ensure the well-posedness of (25.1). Also, the existence of Jordan blocks is also bad since we want it to be diagonalizable. One also should pay attention to the phenomenon that polynomials have multiple roots and it corresponds to the non-smoothness of eigenvalues and eigenfunctions of a matrix.



sections of different cones

If the eigenvalues are different, then the speed of propagation are different, so their cones are disjoint (the last picture). When the eigenvalues are multiple, they might intersect and then it will be very hard to write out the fundamental solutions where we need to use the stationary phase method.

A naive definition is that we can say that the system is strictly hyperbolic if the eigenvalues A^j's are real and distinct.

Recall that a real symmetric matrix only has real eigenvalues, so it motivates the following definition.

Definition 25.5. We say that the system is symmetric hyperbolic if all A^j's are real symmetric.

Remark 25.6. The same definition works for higher order hyperbolic systems with A^j's are the coefficients of all the derivatives. To distinguish the wave equations and the hyperbolic equations, we need to study and view the determinant $|\tau I - A_j \xi_j|$ as the principal symbol if one want to view the system as an equation. If the symbol is like the wave symbol (signature $n, 1$), then we say it is a wave equation. So hyperbolic systems may not be a wave equation while wave equations are hyperbolic systems.

Remark 25.7. In the case of Maxwell equation, if we choose Coulomb gauge and put $A_r = 0$, it even loses the finite speed of propagation since some ∂_t^2 terms are missing and some equations are more like an elliptic equation.

25.5.2 Well-posedness of first order linear hyperbolic systems

Theorem 25.8. *Suppose $A^j = A^j(x) \in Lip$ and symmetric. Then the hyperbolic system (25.1) is well-posed in L^2 : for each $u_0 \in L^2$, $f \in L_t^1 L_x^2$, there exists a unique solution $u \in C_t L_x^2$.*

Energy estimates : We prove the energy estimates first. Let

$$E[u] = \|u\|_{L^2}^2 = \int |u|^2 \, dx$$

and we compute

$$\frac{d}{dt} E[u] = 2 \int u \partial_t u \, dx = 2 \int u(A^j(x)\partial_j u + f) \, dx.$$

Now we take advantage of the symmetric property of A^j, we have

$$2uA^j(x)\partial_j u = \partial_j(u \cdot A^j u) - u \cdot (\partial_j A^j)u.$$

By integrating both sides, we have

$$\frac{d}{dt} E[u] = -\int u^T (\partial_j A^j) u + uf \, dx \leq C \int |u|^2 \, dx + \|u\|_{L^2} \|f\|_{L^2},$$

where we use $\partial_j A^j \in L^\infty$. Then we write the inequality with respect to $\frac{d}{dt}\sqrt{E(u)}$, then by Gronwall's inequality,

$$\sup_{t \in [0,T]} \|u(t)\|_{L^2} \lesssim \|u_0\|_{L_x^2} + \|f\|_{L_t^1 L_x^2}.$$

From the energy estimates, the uniqueness follows immediately.

Duality method : Moreover, we can also replicate the proof of parabolic equation to show existence. We denote $P = \partial_t - A^j \partial_j$. Then we compute

$$\int_0^T \int_{\mathbb{R}^n} Pu \cdot v \, dx \, dt = \int u \cdot v \, dx|_0^T + \int \int u(-\partial_t + \partial_j A^j) v \, dx \, dt,$$

where the adjoint operator $P^* = -\partial_t + \partial_j A^j$ appears in the step above. For the term $\partial_j(A^j v)$ in the adjoint problem, it is fine since $\partial_j A^j \in L^\infty$. Another thing we need to be careful is that one is forward-in-time and the other one is backward-in-time. However, this does not affect anything since one only need to change the symmetric matrix A to another symmetric matrix $-A$. The only difference between the proof for this and the proof for the parabolic case is that one needs to change the space $L^1L^2 + L^2H^{-1}$ in the parabolic case to purely L^1L^2. Then one can just mimic the proof to obtain existence in $L^\infty L^2$. Moreover, $\partial_t u$ is $L^\infty L^2$, so $u \in C_t L^2_x$. One can also see [19, Chapter 7].

First order hyperbolic systems - Finite speed of propagation, Quasilinear systems, Hadamard type well-posedness

Date: April 20, 2023

Last time we proved that a symmetric hyperbolic equation

$$\partial_t u = A^j \partial_j u, \quad u(0) = u_0,$$

is L^2 well-posed by using energy estimates. The essence is to prove

$$\frac{d}{dt}\|u\|_{L^2}^2 \leq \|\partial A\|_{L^\infty}\|u\|_{L^2}^2$$

and to use Gronwall's inequality if $A^j \in Lip$.

26.1 Finite speed of propagation for symmetric hyperbolic systems

The following theorem indicates why we call this system hyperbolic.

Theorem 26.1. *Finite speed of propagation holds for symmetric hyperbolic system.*

Proof. In other words, we want to prove that if the data is known in a compact region in $\{t = 0\}$, then the solution u is determined in D, where D is a region enclosed by $\{t = 0\}$ and Σ.

Step 1 : If A^j's are all constant

First, we assume A^j's are all constant functions. Then we can compute

$$0 = 2\int_D u(\partial_t - A^j\partial_j)u \, dx\, dt = \int_D \partial_t|u|^2 - \partial_j(A^j u \cdot u)\, dx\, dt$$

We denote the normal to the surface Σ by $N = (N_0, N')$ to be chosen (Σ to be chosen). Then it follows from divergence theorem that

$$\int_{t=0} |u|^2\, dx = \int_\Sigma N_0|u|^2 - N_j \cdot (A_j u \cdot u)\, dx.$$

We want the property that $u = 0$ at $t = 0$ would imply $u = 0$ on Σ. This requires the right hand side to be positive definite. We need $(N_j A^j)u \cdot u < N_0|u|^2$. We need $\lambda_m \leq N_0$, where λ_m's are eigenvalues of $N_j A^j$. It is true if $|N_j| \ll N_0$, that is, by choosing the slope of Σ small enough.

Step 2 : If $A^j \in Lip$ Now we penalize u by e^{-ct} with $c > 0$ to be chosen. Then we compute

$$\partial_t v = e^{-ct}\partial_t u - cv = A^j(x)\partial_j v - cv.$$

Since

$$2\int_D v(\partial_t - A^j(x)\partial_j)v\, dx\, dt = -2c\int_D v^2\, dx\, dt = \int_D \partial_t|v|^2 - \partial_j(A^j v \cdot v) + \partial_j A^j v \cdot v,$$

it follows from the divergence theorem that

$$\int_{t=0} |u|^2\, dx = \int_{t=0} |v|^2\, dx = \int_\Sigma N_0|v|^2 - (N_j A^j)v \cdot v\, dx + \int_D (\partial_j A^j + c)v^2\, dx\, dt.$$

Now we choose $c > \|\partial_j A^j\|_{L^\infty}$, then

$$0 = \int_{t=0} |u|^2 \, dx \geq \int_\Sigma N_0 |v|^2 - (N_j A^j) v \cdot v \, dx.$$

By the same argument as the case for A^j constants, we know that $v \equiv 0$ when we choose Σ ideally. So $u \equiv 0$ in D. (We can also use Gronwall's inequality to get around this instead of a penalty like what we did for the wellposedness of first order hyperbolic system.) $\square$

Remark 26.2. Suppose Σ is a plane, which moves in a direction ω with velocity v and we write it into the form

$$vt = x \cdot \omega.$$

Then $N = (v, -\omega)$. This gives $\omega_j A^j \leq v \cdot I_m$ and $v = v(\omega)$ is the largest eigenvalue of $\omega_j A^j$.

26.2 Examples of quasilinear hyperbolic system

We consider

$$\begin{cases} \partial_t u = A^j(u) \cdot \partial_j u, \\ u(0) = u_0. \end{cases} \tag{26.1}$$

We want to answer two questions.

Q1 Is this system well-posed for initial data $u_0 \in H^s(\mathbb{R}^n)$?

Q2 What is well-posedness?

We first examine some examples and address these questions in the next subsection.

Example 26.3. The first example is that $u_t = u \cdot u_x$, which is the Burgers equation. The way this solution behave is an important part of the behavior of more complicated systems.

Example 26.4. The compressible Euler equation : Suppose ρ denotes density and v denotes velocity. Gas particles are moving with velocity v and then $\rho \cdot v$ is the flux, which further implies the density flux relation $\partial_t \rho + \text{div } (\rho v) = 0$. To compute the acceleration of a particle, we write

$$v_t + v \cdot \nabla v = (\partial_t + v \cdot \partial_x) v,$$

which is the directional derivative of v in the direction v. Therefore, by Newton's law

$$\rho(v_t + v \cdot \nabla v) + \nabla p = 0,$$

where p is the pressure and describes how the force pushes it from higher pressure to lower pressure region. Naturally, we add an assumption that $p = p(\rho)$ and $p' > 0$, which is called the constitutive law.

Therefore, we derived

$$\begin{cases} \partial_t \rho + \text{div } (\rho v) = 0, \\ \rho(v_t + v \cdot \nabla v) + \nabla p = 0, \end{cases}$$

where the first one is like a conservation law. For the second one, we can use the first one to write into a similar form

$$\partial_t(\rho v) + \frac{1}{2} \nabla(\rho \cdot v \otimes v) = \nabla \tilde{p},$$

where $\tilde{p}$ is a new function depending on ρ. The last two terms are called the momentum flux.

Example 26.5. Fluids tend to be incompressible than gas. The incompressible Euler equation can derived as follows. We assume ρ_0 is the usual density of the water and we suppose $\rho \simeq \rho_0 + o(\varepsilon)$ and $p(\rho) = \frac{1}{\varepsilon}(\rho - \rho_0)$.

The incompressibility condition $\nabla \cdot v = 0$ can be obtained from the fact ρ is almost constant and the first equation in the compressible case. Now we derive another equation from the second one $v_t + (v \cdot \nabla)v + \nabla p = 0$ in the compressible case. Since the density is fixed, this time the p is fixed. We need to apply divergence to this equation and use the incompressiblity condition to get

$$\nabla \cdot ((v \cdot \nabla)v) + \Delta p = 0,$$

where the first one is $\partial_k((v_j\partial_j)v_k) = \partial_k v_j \cdot \partial_j v_k$ thanks to the incompressibility conditions.

The Burgers equation is obviously symmetric hyperbolic with u as coefficients. However, the incompressible Euler equation is not symmetric hyperbolic. Since you want to solve a Laplacian equation, even if v is localized, we still have p supported everywhere, which means that we do not have finite speed propagation.

Note that the finite speed of propagation still holds for quasilinear hyperbolic system.

26.3 Hadamard type well-posedness

For a quasilinear system, we want to start with $u \in H^s$ with a suitable choice of s. As in the linear case, we still want $(A^j(u))' = (A^j)'(u)\partial u \in L^\infty$. Since $\partial u \in H^{s-1}$, if we want $\partial u \in L^\infty$, then we would expect $s = \frac{n}{2} + 1 + \varepsilon$. ($\frac{n}{2}$ is the boundary case for Sobolev embedding, so we need ε more regularity.)

We look for scaling symmetry. Due to the appearance of the term $A^j(u)$, we need to ensure the value of u does not change when we do the scaling. Otherwise, we cannot expect the same property for different A^j's. So if we take $\tilde{u}(t,x) = u(\lambda t, \lambda x)$, then it is obvious that $u, \tilde{u}$ both satisfy (26.1). Obviously, $\|u\|_{L^\infty}$ is scale invariant. Moreover, in order to let $\|u\|_{\dot{H}^s} = \|\tilde{u}\|_{\dot{H}^s}$, we need to have $s = n/2$.

So in principle, we hope to prove a well-posedness theory with $s = \frac{n}{2} + \varepsilon$. However, this cannot be achieved for any given quasilinear hyperbolic system. We will see that what we can prove (Theorem 26.7) in general is for $s = \frac{n}{2} + 1 + \varepsilon$. And this result turns out to be optimal for incompressible Euler equations.

Definition 26.6 (Hadamard type well-posedness)**.** Given $u_0 \in H^s$, there exists a time $T = T(\|u_0\|_{H^s})$ and a solution $u \in C(0,T; H^s)$ which satisfies

- existence
- uniqueness
- continuous dependence on data : if $u_0^k \to u_0$ in H^s, then there exists $T > 0$ such that u_k exists in $[0,T]$ and $u^k \to u$ uniformly in H^s.

Then we say this equation/system is well-posed in the sense of Hadamard.

Theorem 26.7. (26.1) *is locally well-posed in H^s for $s > \frac{n}{2} + 1$.*

The general approach is to prove energy estimates for solutions. On the other hand, there is another building block of our approach, which is a construction of solutions. These two steps are usually interlaced with each other.

Recall that we used the fixed point argument to construct solutions earlier in this course. If we still want to apply the fixed point argument to the hyperbolic equations, we write $\partial_t u = A^j(0)\partial_j u + (A^j(u) - A^j(0))\partial_j u$. If we start with $u \in H^s$, the nonlinear term (the last term) contains $\partial_j u$, which means that we lose one derivative. This cannot be recovered. Moreover, if we use the fixed-point method, then we get the Liptshitz dependence of initial data, but we do not expect this. Therefore, we need to introduce an interation scheme to tackle this problem. We will discuss this in the next lecture. We start with $u^{(0)}(t) = u_0$ and write

$$u^{k+1}(0) = u_0, \partial_t u^{(k+1)} = A^j(u^{(k)})\partial_j u^{(k+1)}.$$

We can prove a uniform bound for the energy.

Remark 26.8. For Navier-Stokes equations, we can use a fixed point method instead. The heuristic reason is that Navier-Stokes equation is more like a heat equation and the operator Δ does not cause any loss of derivatives due to the solvability of Δ.

Hadamard type well-posedness for first order quasilinear hyperbolic systems

Date: April 25, 2023

Today, we focus on a symmetric quasilinear hyperbolic system

$$\begin{cases} \partial_t u = A^j(u)\partial_j u, \\ u(0) = u_0 \in H^s, \end{cases} \tag{27.1}$$

where A^j's are symmetric matrices and smooth.

Theorem 27.1. *The problem (27.1) is locally well-posed in H^s for $s > \frac{n}{2} + 1$. In other words, given $u_0 \in H^s$, there exists $T = T(\|u_0\|_{H^s})$ such that (27.1) has a unique solution $u \in C([0,T]; H^s)$ and u depends continuously on the data, that is, the map $u_0 \mapsto u$ which maps $H^s \to C([0,T], H^s)$ is continuous.*

Proof. We want to construct a sequence of iterations $u^{(k)} \in C([0,T]; H^s)$. The initial step can be simply chosen to be $u^{(0)} = 0$ and we would like to define $u^{(k+1)}$ recursively by

$$\begin{cases} \partial_t u^{(k+1)} = A^j(u^{(k)})\partial_j u^{(k+1)}, \\ u^{(k+1)}(0) = u_0, \end{cases}$$

which is a linear system and we want to show that this converges. To remedy for the regularity, we can replace the initial data at each step $u_0^{(k+1)} \in \mathcal{D}$ which satisfies $u_0^{(k+1)} \to u_0$ in H^s and hence in each step, the solution we solved for is a smooth function.

Step 1 : We show that $u^{(k)}$ remains bounded in H^s

Suppose $\|u_0\|_{H^s} = M$. We prove by induction that $\|u^{(k)}\|_{L^\infty([0,T],H^s)} \leq CM$. We want to use the energy estimates for the induction step. Here, we make a simplification assumption that s is integer.

First, we apply the linear energy estimates in L^2 we obtained in Theorem 25.8 :

$$\frac{d}{dt}\|u^{(k+1)}\|_{L^2}^2 \leq \|\nabla A\|_{L^\infty}\|u^{(k+1)}\|_{L^2}^2,$$

where $\nabla(A(u^{(k)})) = DA(u^{(k)})\nabla u^{(k)}$. Since $u^{(k)}$ is bounded, $DA(u^{(k)})$ is bounded by some universal constant. Moreover, $\|\nabla u^{(k)}\|_{L^\infty} \leq \|u^{(k)}\|_{H^s} \leq CM$, which implies

$$\frac{d}{dt}\|u^{(k+1)}\|_{L^2}^2 \leq C\tilde{C}M\|u^{(k+1)}\|_{L^2}^2$$

and thus we obtain by Gronwall's inequality that

$$\|u^{(k+1)}(t)\|_{L^2}^2 \leq e^{C\tilde{C}Mt}\|u^{(k+1)}(0)\|_{L^2}^2 \leq Me^{C\tilde{C}Mt}.$$

We want $e^{C\tilde{C}MT} \leq C$, which is true if $T \ll M^{-1}$.

Next, we want to show the energy estimates in H^s, which is equivalent to an energy estimate in L^2 for $\partial^s u^{(k)}$. By using the chain rule, we obtain that

$$\partial_t(\partial^s u^{(k)}) = A^j(u^{(k)})\partial_j\partial^s u^{(k+1)} + f_s.$$

If we replicate what we did just now, we get

$$\frac{d}{dt}\|\partial^s u^{(k+1)}\|_{L^2}^2 \leq CC_2M\|\partial^s u^{(k+1)}\|_{L^2}^2 + \|f_s\|_{L^2}\|u^{(k+1)}\|_{L^2}$$

and it suffices to show that $\|f_s\|_{L^2} \lesssim \|u^{(k+1)}\|_{H^s}$. We compute

$$f_s := \partial^s(A^j(u^{(k)})\partial_j u^{(k+1)}) - A^j(u^{(k)})\partial_j\partial^s u^{(k+1)} = \sum_{|\alpha|+|\beta|=s}^{|\beta|\le s-1} \partial^\alpha A^j(u^{(k)})\partial^\beta\partial_j u^{(k+1)}$$

$$= \sum_{|\alpha_1|+\cdots+|\alpha_j|=|\alpha|,|\alpha|+|\beta|=s} D^* A(u^{(k)})\partial^{\alpha_1}u^{(k)}\cdots\partial^{\alpha_j}u^{(k)}\partial^\beta\partial_j u^{(k+1)},$$

where there is no need to write out the exact formula by generalizing the chain rule using Faa di Bruno's formula since we only focus on the terms itself. We estimate

$$\|\partial^{\alpha_1}u^{(k)}\cdots\partial^{\alpha_j}u^{(k)}\partial^\beta u^{(k+1)}\|_{L^2} \le \|\partial^{\alpha_1}u^{(k)}\|_{L^{p_1}}\cdots\|\partial^{\alpha_j}u^{(k)}\|_{L^{p_j}}\|\partial^\beta u^{(k+1)}\|_{L^q},$$

where $|\alpha_1| + \cdots + |\alpha_j| + |\beta| = s+1$ and $\frac{1}{2} = \frac{1}{p_1} + \cdots + \frac{1}{p_j} + \frac{1}{q}$.

For the simplest case $j = 1$, $|\alpha_1| = 1, |\beta| = s$, we have

$$\|\partial u^{(k)}\|_{L^\infty} \cdot \|\partial^s u^{(k+1)}\|_{L^2} \lesssim \|u^{(k)}\|_{H^s}\|u^{(k+1)}\|_{H^s}.$$

For the other cases, we just choose $q = \infty$ and $p_1, \cdots, p_j$ are like an arithmetic progression such that $\frac{1}{2} = \frac{1}{p_1} + \cdots + \frac{1}{p_j}$. For $\|\partial^s u\|_{L^2} \le \|u\|_{H^s}$, $\|\partial u\|_{L^\infty} \le \|u\|_{H^s}$ when $s > \frac{n}{2}+1$ and the Gagliardo-Nirenberg interpolation ([19, Chapter 6]) tells us $\|\partial^\alpha u\|_{L^{p_\alpha}} \le \|\partial u\|_{L^\infty}^{\theta_\alpha}\|\partial^s u\|_{L^2}^{1-\theta_\alpha}$ for some $\theta_\alpha \in (0,1)$, which implies $\|\partial^\alpha u\|_{L^{p_\alpha}} \le \|u\|_{H^s}$.

We remark that we can also use Moser estimates instead to prove this : if $u \in H^s$, $A(0) = 0$, then $A(u) \in H^s$ with some moderate assumption on A.

Step 2 : Convergence in a weaker topology (L^2)

To show convergence, we want to write down the energy estimates for differences. First, we notice that

$$\begin{aligned}\partial_t(u^{(k+1)} - u^{(k)}) &= A(u^{(k)})\partial u^{(k+1)} - A(u^{(k-1)})\partial u^{(k)}\\ &= A(u^{(k)})\partial(u^{(k+1)} - u^{(k)}) + (A(u^{(k)}) - A(u^{(k-1)}))\partial u^{(k)}\\ &= A(u^{(k)})\partial(u^{(k+1)} - u^{(k)}) + (u^{(k)} - u^{(k-1)})B(u^{(k)}, u^{(k-1)})\partial u^{(k)},\end{aligned}$$

where B is some smooth functions with two variables. Then the energy estimate goes as follows :

$$\frac{d}{dt}\|u^{(k+1)} - u^{(k)}\|_{L^2}^2 \le CC_2M\|u^{(k+1)} - u^{(k)}\|_{L^2}^2 + C_3\|u^{(k+1)} - u^{(k)}\|_{L^2}\|u^{(k)} - u^{(k-1)}\|_{L^2}$$

and hence

$$\frac{d}{dt}\|u^{(k+1)} - u^{(k)}\|_{L^2} \le CC_2M\|u^{(k+1)} - u^{(k)}\|_{L^2} + C_3\|u^{(k)} - u^{(k-1)}\|_{L^2}, \tag{27.2}$$

where C_3 only depends on CM due to the smoothness of B and the Sobolev embedding. This further implies

$$\|(u^{(k+1)} - u^{(k)})(t)\|_{L^\infty L^2} \le e^{CC_2MT}\left(\|(u^{(k+1)} - u^{(k)})(0)\|_{L^2} + \|(u^{(k)} - u^{(k-1)})(t)\|_{L^1L^2}\right)$$

and the last term satisfies $\|(u^{(k)} - u^{(k-1)})(t)\|_{L^1L^2} \le T\|(u^{(k)} - u^{(k-1)})(t)\|_{L^\infty L^2}$. Since $u^{(k)}$ has the same initial data for each k, we can conclude that

$$\|u^{(k+1)} - u^{(k)}\|_{L^\infty([0,T],L^2)} \le e^{CT}T\|u^{(k)} - u^{(k-1)}\|_{L^\infty([0,T],L^2)},$$

where we want $e^{CT}T \leq L < 1$. Therefore, $u^{(k+1)} = u^{(k+1)} - u^{(k)} + \cdots + u^{(1)} - u^{(0)}$ is a geometric series, and hence converges in L^2.

Step 3 : Convergence in higher norms So we have $u^{(k)}$ bounded in $L^\infty H^s$ and $u^{(k)} \to u$ in $L^\infty L^2$, which implies $u \in L^\infty H^s$. Indeed, one can argue by interpolation that all $s' < s$, $u^{(k)}$ is Cauchy in $H^{s'}$ norms. Then by taking the limit $s' \uparrow s$, we can conclude that $u \in L^\infty H^s$. See [19, Proposition 9.12].

Alternatively, $u^{(k)} \to u$ in $\mathcal{D}'$ and from a homework problem, we know that given a bounded sequence in H^s, which convergences in $\mathcal{D}'$, then the limit is in H^s as well.

Step 4 : Continuous dependence Moreover, one also need to show $u \in C(H^s)$ and this requires extra work. For the continuous dependence, we refer to [14] and the references therein.

Step 5 : Uniqueness Suppose u, v are two solutions with $u(0) = v(0)$. It suffices to show u and v are equal in L^2. We write

$$\partial_t(u - v) = A(u)\partial(u - v) + (u - v)B(u, v)\partial v$$

and mimic the computation for (27.2), we conclude that

$$\|u - v\|_{L^\infty L^2} \leq e^{CT}\|(u - v)(0)\|_{L^2},$$

which gives uniqueness. It also gives a bound of $\|u - v\|_{L^\infty L^2}$ by the initial data, which is called a weak Lipschitz dependence, that is, $\|u - v\|_{L^\infty L^2} \lesssim \|u(0) - v(0)\|_{L^2}$. Here, weakness means that it is not in H^s, the space in which the solution lives.

As a remark, for quasilinear hyperbolic systems, we can only achieve weak Lipschitz dependence. However, in the semilinear model, we can expect strong Lipschitz dependence. $\square$

And next time, we discuss the Burgers euqation $u_t = uu_x$, which is well-posed in H^s for $s > 3/2$ thanks to today's theorem.

Burgers equation

Date: April 27, 2023

We focus on the study of a scalar equation $u_t + uu_x = 0$, which is called the Burgers equation. The same kind of analysis also works in higher dimensions for $u_t + \sum_j a_j(u)\partial_j u = 0$.

28.1 Solving Burgers equation using characteristics method

This is a quasilinear transport equation. By thinking of it as a transport equation, one needs to look at integral curves of this vector field $v = (1, u)$. By denoting the integral curves by $(t, x(t))$, we have

$$\dot{x} = \frac{d}{dt}x(t) = u(t, x(t)).$$

Then we compute

$$\frac{d}{dt}u(t, x(t)) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

So, to solve the equation, one needs to consider the characteristic system

$$\begin{cases} \dot{x} = u(t, x(t)), \\ \dot{u}(t, x(t)) = 0 \end{cases} \tag{28.1}$$

with $x(0) = x_0, u(0) = u_0(x_0)$. In order to use the system (28.1) to solve the original equation, we can first solve (28.1) for each x_0 to obtain $x(t, x_0)$ and $u(t, x(t, x_0))$. But what we want is $u = u(t, x)$, so the remaining problem is whether we can invert the map $(t, x_0) \mapsto (t, x(t, x_0))$.

The easy part is to invert it locally in time since we have the local inversion theorem : if the map $(t, x_0) \to (t, x(t, x_0))$ has a nonsingular Jacobian, then we can invert it locally.

Therefore, we compute

$$\frac{\partial(t, x(t, x_0))}{\partial(t, x_0)} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

and hence if we are given $x(0, x_0) = x_0$ and we know (t, x), then we can determine (t, x_0).

Then we can conclude the general strategy as follows :

(1) solve the characteristic equation;
(2) do the local inversion;
(3) $u = u_0(x_0)$, $x = x_0 + tu_0(x_0)$

For Burgers equation, we have

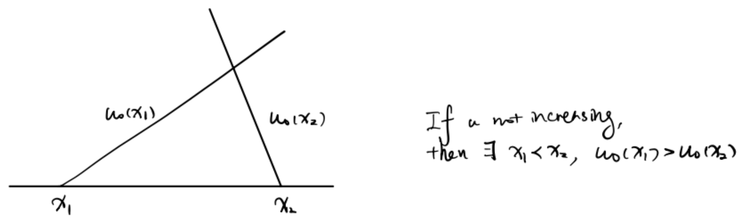
$$u(t, x_0 + tu_0(x_0)) = u_0(x_0).$$

The map $x_0 \mapsto x_0 + tu_0(x_0)$ is invertible as long as $tu_0' + 1 > 0$, which is true for short time if $u_0 \in C^1$.

Theorem 28.1. *Suppose u_0 is C^1 (u_0' is bounded), then Burgers equation has a unique C^1 solution for a short time.*

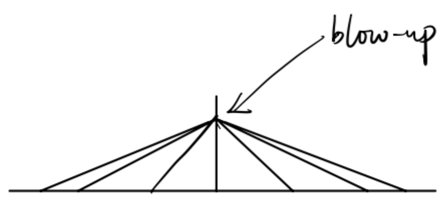
28.2 Examples of intersecting characteristics, shocks, Rankine–Hugoniot conditions

Now we want to know for how long we can solve the equation. Algebraically, we need $1 + tu_0' > 0$, if $u_0' \geq 0$, then we obtained a global solution and if u' has some negative values, then $T_{max} = \frac{1}{\max(-u_0')}$. If u is not increasing, then there exists $x_1 < x_2$ such that $u_0(x_1) > u_0(x_2)$.



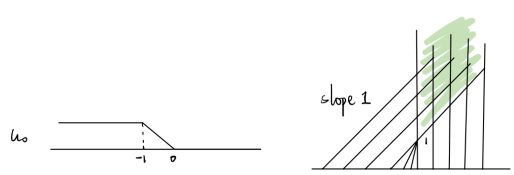
Characteristics can intersect. So at the intersection point, then values of u cannot be defined since it is different from the view of different starting point. The conclusion then goes as follows : If u_0 is not increasing, then there is no global C^1 solution.

Example 28.2. If $u_0(x) = -x$, which means that $T_{max} = 1$. The picture for the characteristics will be as follows.



If one thinks of this for a moment, one will find that such bad examples can be cooked is due to the unboundedness of the initial data. So at very far away region, one can have extremely large speed.

Example 28.3. Now we give another example where u_0 is bounded. We keep u_0 to be 1 until -1 and go down to 0 along a straight line and then keep 0. This is shown in the following picture.



In this example, there is a green region so that the solutions may have intersection within that region. In practical terms, when two waves coming from two sides, they might collide and there might be a balance between them. We expect a line, which characterize this balance. The line shall separate the solutions into two regions, where in the left, $u = 1$ while

$u = 0$ in the right. In other words, the line is the interface where the solution jump. We shall call this line a shock.

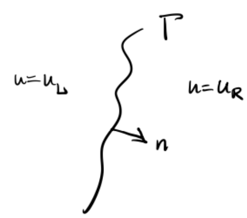
Definition 28.4. If the solution has a jump discontinuity, then we say it is a shock.

Suppose we indeed find a solution which has a jump. Then at where it has a jump, it is δ_0. However, we do not know how to make sense of $u \cdot u_x$ if we try to multiply δ_0 by a discontinuous function. So this is not well-defined. Now it seems like trying to understand the equation as it wrote is hopeless. We need to first rewrite the equation as

$$\partial_t u + \partial_x \left(\frac{1}{2}u^2\right) = 0$$

before we start to study it. The strategy is that we need to ask for the equation to be satisfied in the sense of distributions.

Suppose $\Gamma = \{x = \sigma(t)\}$ is the shock curve.



For $\phi \in \mathcal{D}$, we compute

$$0 = \left(\partial_t u + \frac{1}{2}\partial_x(u^2)\right)(\phi) = -u(\partial_t\phi) - \frac{1}{2}u^2(\partial_x\phi).$$

We separate this integral into two parts and we have

$$\begin{aligned}
0 =& (\int_L + \int_R)(-u(\partial_t\phi) - \frac{1}{2}u^2(\partial_x\phi)) \\
=& \int_L (\partial_t u_L + \frac{1}{2}\partial_x u_L^2)\phi + \int_\Gamma (-n_t u_L \phi - \frac{1}{2}n_x u_L^2 \phi) + \int_R (\partial_t u_R + \frac{1}{2}\partial_x u_R^2)\phi + \int_\Gamma (n_t u_R \phi + \frac{1}{2}n_x u_R^2\phi) \\
=& \int_\Gamma \phi(n_t(u_R - u_L) + n_x(\frac{1}{2}u_R^2 - \frac{1}{2}u_L^2)\, d\sigma
\end{aligned}$$

We denote $[u] = u_R - u_L$, then this implies

$$n_t[u] + n_x[\frac{1}{2}u^2] = 0.$$

The tangent vector of Γ is $T = (1, \dot\sigma(t))$ and then $N = (\dot\sigma(t), -1)$ is the normal. Therefore, we derive the shock speed

$$\dot\sigma = \frac{[\frac{1}{2}u^2]}{[u]},$$

which is called the Rankine–Hugoniot condition.

For general equations written in the density-flux form (meaning that the equation is a kind of conservation law) $\partial_t u + \partial_x F(u) = 0$, the Rankine–Hugoniot condition is also given by

$$\dot{\sigma} = \frac{[F(u)]}{[u]}.$$

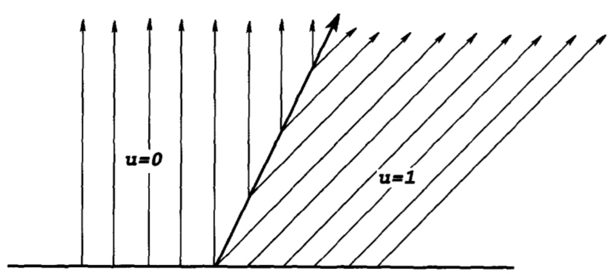
We just proved for Burgers equation as a special case simply by integration by parts. The shock type solutions are a meaningful physical object for general conservation law equations and is interesting to study.

Example 28.5. We revisit Example 28.3. In this example, the shock speed is $\dot{\sigma} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$. Note that though we do not know the expression for the shock curve Γ, we have the Rankine–Hugoniot condition which helps us to determine σ and hence we know σ will just be a straight line with slope $\frac{1}{2}$.

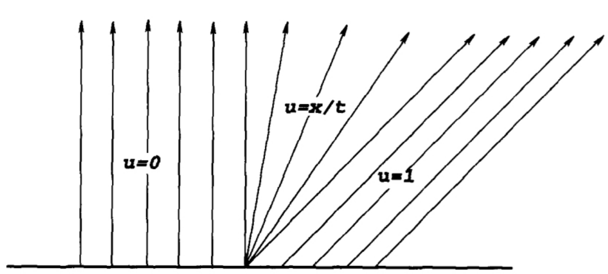
28.3 Riemann-Hilbert problem

We still start with Example 28.3. Note the solution is well-defined up to time $t = 1$, it motivates our study below. We view the data of u in this example at time $t = 1$ as the new initial data, which is just 1 in the left, 0 in the right, then we would get a specific problem. We say if $u_0 = \begin{cases} u_L, & x < 0 \\ u_R, & x > 0 \end{cases}$, then this is a Riemann-Hilbert problem.

We have two different cases, if $u_L > u_R$, then we are in the case that waves might collide. On the other hand, if $u_L < u_R$, then there is an empty region in the middle. We have two candidates. The first one is to compute the shock speed in the middle and have a shock line contained while the second one is filled by a fan starting from the origin. The second one it obtained by regularizing the initial data a little bit.



A "nonphysical" shock



Rarefaction wave

The shock picture is nonphysical since one cannot expect something is produced from nothing. There are waves emanating from the shock, and it's more like producing something from nothing, so from the physical view, the second one is more correct. We call this rarefaction wave and the second solution is Liptshitz continuous. We have a name for this, which is the entropy condition. See [7, Section 3.4] for the picture and further discussion.

A remark is that we cannot reverse these choices. Even though we start by motivating from Example 28.3, the information of the little triangle in this example is already lost after we view $t = 1$ as the new initial time. Moreover, if one tries to solve backward in time, it may not be unique. So the choice of physical solution here does not violate our previous discussion.

Theorem 28.6. *Given "decent" data, there exists a unique Rankine–Hugoniot + entropy solution.*

This theorem is true for scalar equation in $\mathbb{R}^n$ and systems in $\mathbb{R}$ and two dimensional systems in $\mathbb{R}^2$. The $1+1$ dimension setting can be found in [7].

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