Qualifying Exam Syllabus

Ning Tang

August 30th 2023, Time 2:00pm - 5:00pm, 961 Evans Hall

Committee: Daniel Tataru, Sung-Jin Oh (Advisor), Maciej Zworski (Chair), John Lott.

1 Major topic: Partial Differential Equations (Analysis)

References: L. C. Evans, *Partial Differential Equations*, second edition; F.G. Friedlander and M. Joshi, *Introduction to the Theory of Distributions*, second edition; Daniel Tataru, *Math 222A Notes*, Fall 2022; Daniel Tataru, *Math 222B Notes*, Spring 2023; Sung-Jin Oh, *Lecture notes for Math 222A*.

- **Distributions** (Friedlander Ch.1-3,5, 222A Notes and Oh Ch.5,8,9): Smooth functions with compact support, distributions, convolutions, tempered distributions, Fourier transform, fundamental solutions for ODEs, Laplace, heat, Schrödinger, wave and KdV equations

- **Sobolev Spaces** (Evans Ch.5, 222A Notes and 222B Notes): Definitions of Sobolev and Hölder spaces, Gagliardo-Nirenberg-Sobolev inequality, Morrey's inequality, Sobolev embeddings for BMO space, homogeneous Sobolev spaces, Hardy-Littlewood-Sobolev inequality, trace inequality, approximation theorems, extensions, Rellich-Kondrachov compactness theorem, Poincaré-type inequalities

- **Littlewood-Paley theory** (222A Notes): Characterizations of $W^{s,p}, \dot{W}^{s,p}$ for $1 < p < \infty$, Hormander-Mikhlin condition(statement), square functions, Bernstein inequalities

- **Second order Elliptic Equations** (Evans Ch.2.2, 6.1-6.4 and 222B Notes): L^2-based local elliptic regularity, maximum principle, mean value property of harmonic functions, Liouville theorem and comparison theorem for Laplacian operator, existence and uniqueness of solutions (L^2-based, energy estimates and adjoint method), Lax-Milgram theorem, variational methods, higher regularity on compact domains, Fredholm alternative, variational characterization of eigenvalues and eigenfunctions, spectrum of Laplacian on the sphere, Green functions for Laplacian equation, Hopf's lemma, Harnack's inequality, fixed point argument for specific semilinear equations

- **Second Order Parabolic Equations** (Evans 2.3, 7.1 and 222B Notes): L^2-based theory (energy estimates, existence, higher regularity, parabolic regularity), L^∞-based theory(weak maximum principle, strong maximum principle, mean value property for heat equations, Harnack's inequality)

- **Hyperbolic Equations** (Evans 2.4 and 222B Notes): Transport equations, energy-momentum tensor, finite speed of propagation, deformation tensor of wave equation, first order linear hyperbolic system and Hadamard type well-posedness

- **Miscellaneous topics** (222A Notes): Uncertainty principle for Schrödinger equations, Fourier series and local solvability of constant coefficient PDEs, Poisson summation formula

2 Major topic: Differential Geometry (Geometry)

References: Barrett O'Neill, *Semi-Riemannian geometry.*

- **Manifold Theory** (O'Neill Ch.1): Basic definitions and basic theorems in smooth manifolds
- **Tensors** (O'Neill Ch.2): Tensor algebra, tensor fields, tensor components, contraction, tensor derivations, Lie derivatives, symmetric bilinear forms, scalar products
- **Semi-Riemannian Manifolds** (O'Neill Ch.3) : The Levi-Civita connection, parallel translation, geodesics, the exponential map, type-changing and metric contraction, frame fields, some differential operators, curvatures, local isometries
- **Semi-Riemannian Submanifolds** (O'Neill Ch.4) : The induced connection, geodesics in submanifolds, totally geodesic submanifolds, semi-Riemannian hypersurfaces, hyperquadrics, the Codazzi equation, totally umbilic hypersurfaces
- **Riemannian and Lorentz Geometry** (O'Neill Ch.5) : The Gauss lemma, arc length, Riemannian distance, Riemannian completeness, Lorentz causal character, timecones, local Lorentz geometry, geodesics in hyperquadrics
- **Constructions** (O'Neill Ch.7) : Deck transformations, semi-Riemannian coverings, vector bundles, local isometries, warped products
- **Symmetry and Constant Curvature** (O'Neill Ch.8) : Jacobi fields, tidal forces, locally symmetric manifolds, simply connected space forms, transvections
- **Isometries** (O'Neill Ch.9) : Semiorthogonal groups, some isometry groups, time orientability, Killing vector fields, the Lie algebra $i(M)$, homogeneous spaces
- **Calculus of Variations** (O'Neill Ch.10) : First variation, second variation, the index form, conjugate points, local minima and maxima, some global consequences, the endmanifold case, focal points, a causality theorem
- **Causality in Lorentz Manifolds** (O'Neill Ch.14) : Causality relations, causality conditions, time separation, achronal sets, Cauchy hypersurfaces, Cauchy developments, spacelike hypersurfaces, Cauchy horizons, Hawking's singularity theorem and Penrose's singularity theorem

3 Minor topic: Microlocal analysis (Analysis)

Reference: Alain Grigis and Johannes Sjöstrand, *Microlocal analysis for differential operators.*

- **Fundamentals** (GS Ch.1 and 2) : Symbols(basic properties, asymptotic sums), phase functions, oscillatory integrals(basic regularity properties, Schwartz kernel theorem), method of stationary phase
- **Pseudodifferential operators** (GS Ch.1 and 3) : Mapping properties of pseudodifferential operators, smoothing operators, properly supported operators, the complete symbol, adjoints, compositions of pseudodifferential operators and changes of variables
- **Application to elliptic operators and L^2 continuity** (GS Ch.4) : Construction of a parametrix for an elliptic pseudodifferential operator, mapping properties between H^s, local solvability of an elliptic differential operator