  

How to Count Better Than a Three-Year-Old

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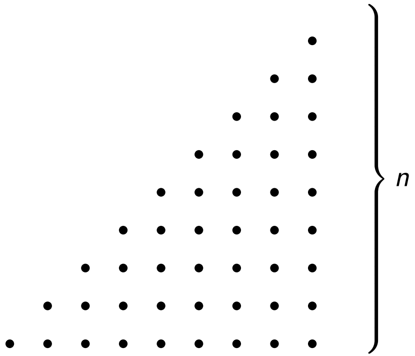
A famous counting problem

"Count this, young man!"

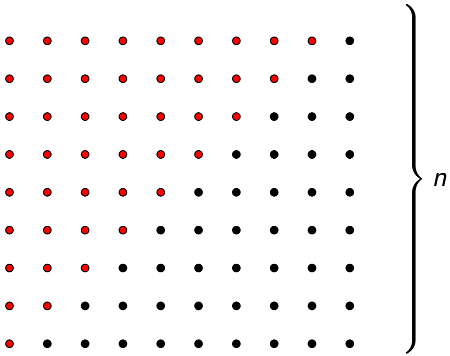
$$1 + 2 + 3 + \ldots + 98 + 99 + 100 = \quad ?$$

Abstraction and solution



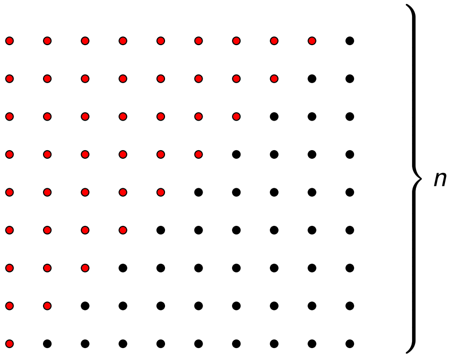
$$1 + 2 + 3 + \ldots + (n-1) + n \ = ?$$

Abstraction and solution



$$2 \cdot (1 + 2 + 3 + \ldots + (n-1) + n) = ?$$

Abstraction and solution



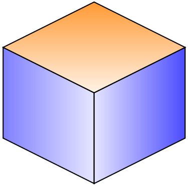
$$2 \cdot (1 + 2 + 3 + \ldots + (n-1) + n) = \; n(n+1)$$

Two principles

- Abstraction can lead to simpler solutions.
- It is easier to count things when cleverly grouped together.

A second counting problem

Playing with blocks is the best!



How many ways can we color the faces of a square block using the colors orange and blue?

An adequate solution

Group according to number of blue faces!

# of blue faces	# of possibilities
0	?
1	?
2	?
3	?
4	?
5	?
6	?
Total:	?

An adequate solution

Group according to number of blue faces!

# of blue faces	# of possibilities
0	1
1	?
2	?
3	?
4	?
5	?
6	?
Total:	?

An adequate solution

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1	1
2	2
3	2
4	?
5	?
6	?
Total:	?

An adequate solution

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# of blue faces	# of possibilities
0	1
1	1
2	2
3	2
4	2
5	?
6	?
Total:	?

An adequate solution

Group according to number of blue faces!

# of blue faces	# of possibilities
0	1
1	1
2	2
3	2
4	2
5	1
6	?
Total:	?

An adequate solution

Group according to number of blue faces!

# of blue faces	# of possibilities
0	1
1	1
2	2
3	2
4	2
5	1
6	1
Total:	?

An adequate solution

Group according to number of blue faces!

# of blue faces	# of possibilities
0	1
1	1
2	2
3	2
4	2
5	1
6	1
Total:	10

An adequate solution

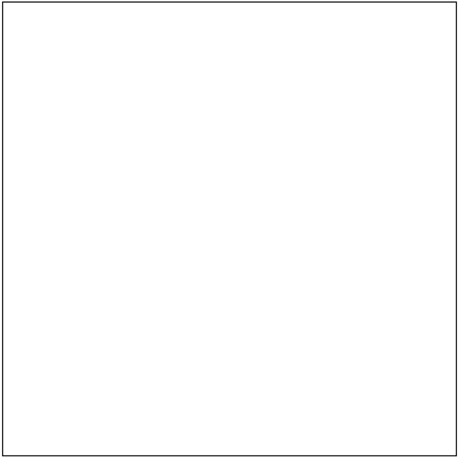
Group according to number of blue faces!

# of blue faces	# of possibilities
0	1
1	1
2	2
3	2
4	2
5	1
6	1
Total:	10

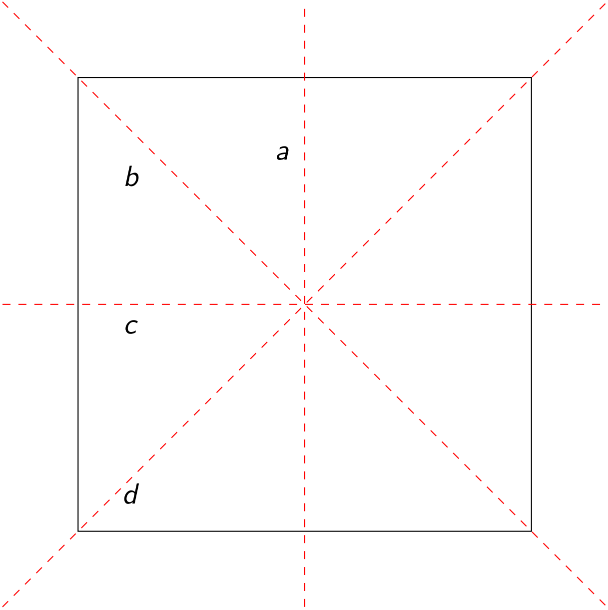
What about using the colors red, blue, and orange?

Symmetry and *group theory* give us a better way!

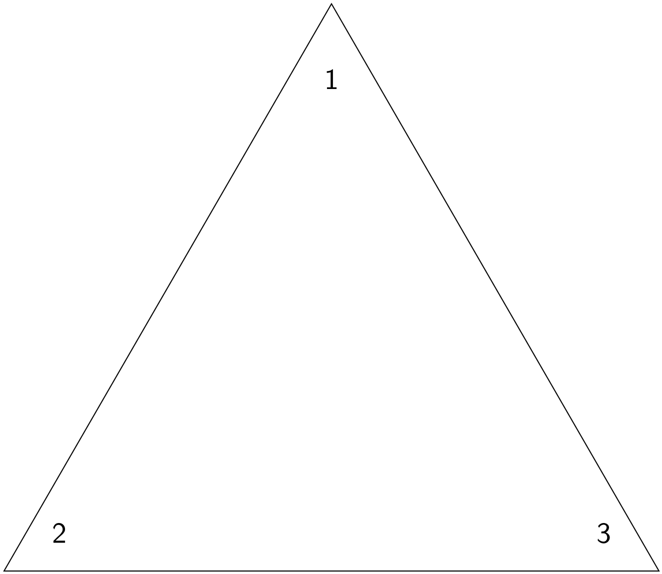
Symmetries of the square



Symmetries of the square



Symmetries of the triangle



Composing symmetries

Carrying out a symmetry x followed by a symmetry y results in a third symmetry, z! We write

$$z = y * x.$$

Example

In D_4,

$$s_a * r_1 = ?$$

Composing symmetries

Carrying out a symmetry x followed by a symmetry y results in a third symmetry, z! We write

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Example

In D_4,

$$s_a * r_1 = s_d$$

Composing symmetries in D_3

	e	σ_1	σ_2	τ_1	τ_2	τ_3
e						
σ_1						
σ_2						
τ_1						
τ_2						
τ_3						

Composing symmetries in D_3

	e	σ_1	σ_2	τ_1	τ_2	τ_3
e	e	σ_1	σ_2	τ_1	τ_2	τ_3
σ_1	σ_1	σ_2	e	τ_3	τ_1	τ_2
σ_2	σ_2	e	σ_1	τ_2	τ_3	τ_1
τ_1	τ_1	τ_2	τ_3	e	σ_1	σ_2
τ_2	τ_2	τ_3	τ_1	σ_2	e	σ_1
τ_3	τ_3	τ_1	τ_2	σ_1	σ_2	e

Composing symmetries in D_3

	e	σ_1	σ_2	τ_1	τ_2	τ_3
e	e	σ_1	σ_2	τ_1	τ_2	τ_3
σ_1	σ_1	σ_2	e	τ_3	τ_1	τ_2
σ_2	σ_2	e	σ_1	τ_2	τ_3	τ_1
τ_1	τ_1	τ_2	τ_3	e	σ_1	σ_2
τ_2	τ_2	τ_3	τ_1	σ_2	e	σ_1
τ_3	τ_3	τ_1	τ_2	σ_1	σ_2	e

- Neutral element: $e * x = x * e = x$.
- Inverse elements: $\forall x \; \exists y$ s.t. $x * y = y * x = e$.
- Associativity: $x * (y * z) = (x * y) * z$.

Definition of a group

A *group* is a set G together with a rule $*$ for taking pairs of elements of G and producing a third element satisfying the following properties:

- Associativity;
- Existence of a neutral element e;
- Existence of inverses elements.

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Example

- D_4, D_3
- The set of symmetries of any shape (e.g. the cube!!)
- The integers and addition
- Non-zero rational numbers and multiplication

Examples of non-groups

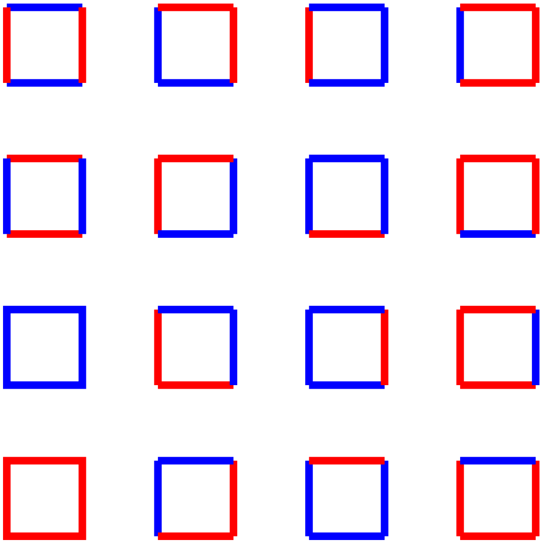
Discuss with your neighbor!

- Integers and multiplication
- Integers and subtraction
- All rational numbers and multiplication
- All integers larger than zero, and addition

Check: associativity, existence of neutral elements, and existence of inverses!

Groups often *transform* other objects!

Colorings of the square



Group actions

An *action* of a group G on a set X is a rule for taking a pair (g, x) for $g \in G$ and $x \in X$ and producing a new element $g \cdot x$ in X, satisfying the following:

- $e \cdot x = x$
- $(g * h) \cdot x = g \cdot (h \cdot x)$

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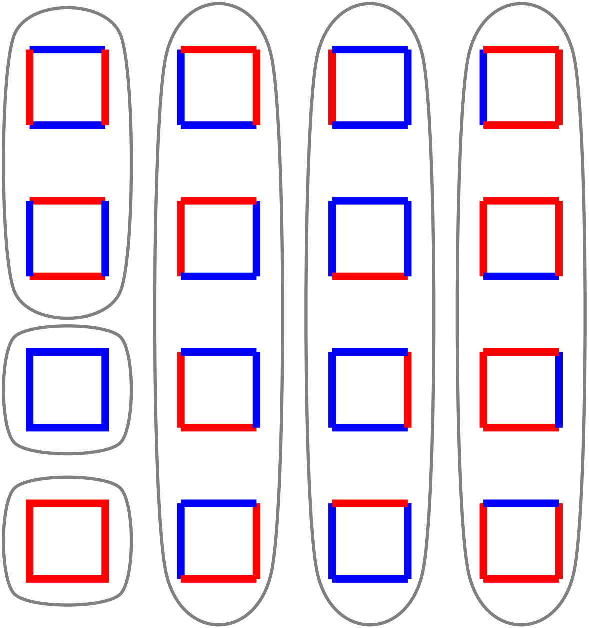
Example

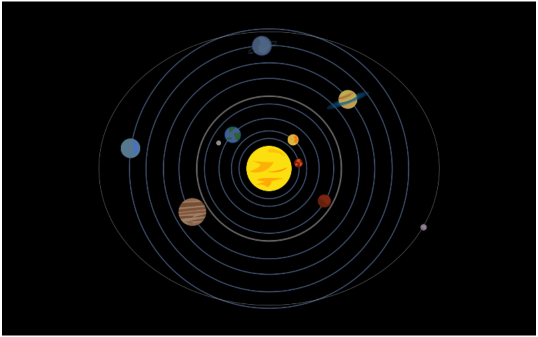
D_4 acts on

- Vertices of the square
- Edges of the square
- Pairs of vertices of the square
- ...

Colorings of the square revisited



Orbits



The *orbit* of x is the set

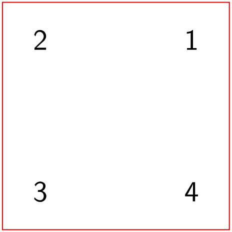
$$G \cdot x = \{g \cdot x \mid g \in G\}$$

Orbits: your turn

Let

$$X = \{(12), (13), \ldots, (34)\}$$

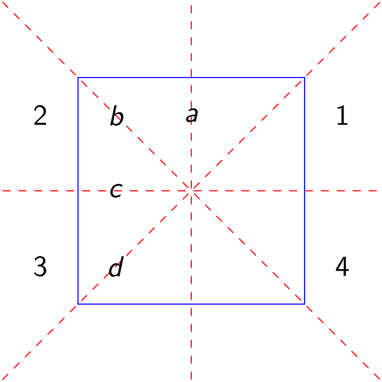
be the set of pairs of vertices of the square. Describe the orbits for the action of D_4 on X!



Symmetries doing nothing

Which elements of D_4 fix pairs of vertices?

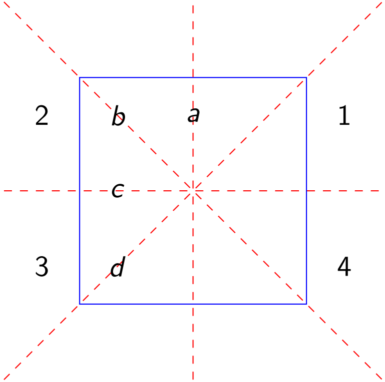
- (12): ?
- (23): ?
- (34): ?
- (14): ?
- (13): ?
- (24): ?



Symmetries doing nothing

Which elements of D_4 fix pairs of vertices?

- (12): e, s_a
- (23): ?
- (34): ?
- (14): ?
- (13): ?
- (24): ?



Symmetries doing nothing

Which elements of D_4 fix pairs of vertices?

- (12): e, s_a
- (23): e, s_c
- (34): ?
- (14): ?
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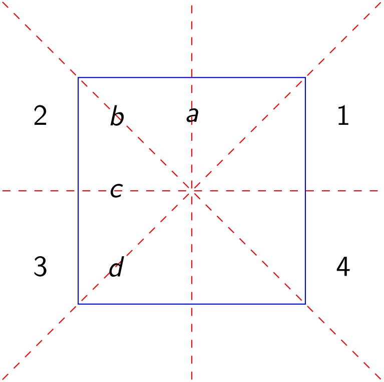
- (24): ?



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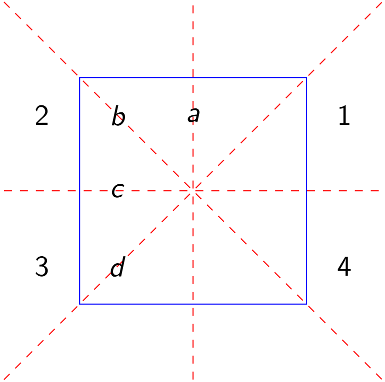

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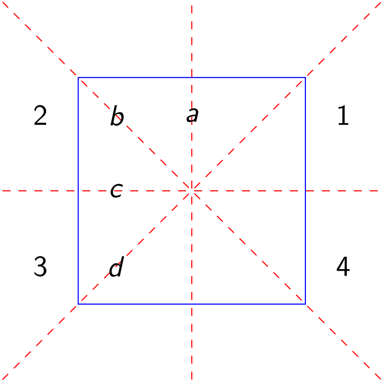

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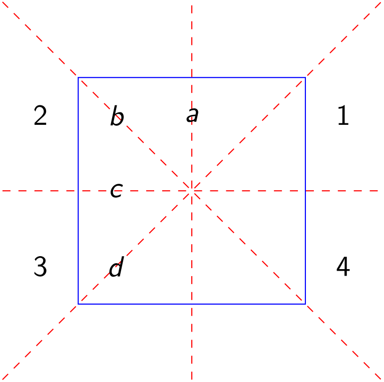

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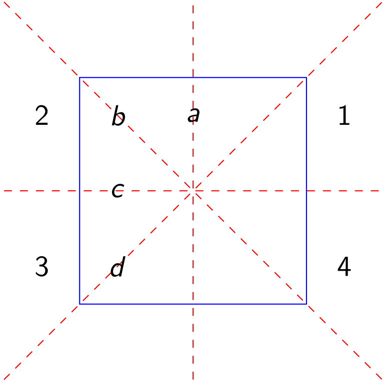

Stabilizers

The *stabilizer* of x is the set

$$G_x = \{g \in G \mid g \cdot x = x\}.$$

Symmetries doing nothing

Which elements of D_4 fix pairs of vertices?

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- (23): e, s_c
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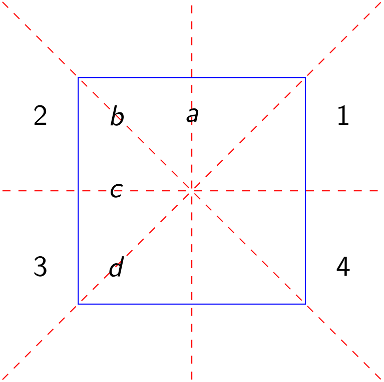

The Orbit-Stabilizer Theorem

Theorem

For any $x \in X$,

$$\#G = \#(G_x) \times \#(G \cdot x).$$

Proof.

Consider the map

$$G \to G \cdot x$$
$$g \mapsto g \cdot x$$

. . . □

Fixed sets

For any $g \in G$, let

$$X^g = \{x \in X \mid g \cdot x = x\}.$$

This is the *fixed set* of g.

Example

If X is the set of colorings of the square with 2 colors, what is the fixed set of r_1? What about s_a?

The Cauchy-Frobenius Fixed-Point Formula (aka Burnside's Lemma)

$$\#G \times \#\text{Orbits of } X = \sum_{g \in G} \#X^g$$

Burnside's Lemma: Proof

$$\#G \times \#\text{Orbits of } X = \sum_{\text{Orbits } Z} \#G$$

Burnside's Lemma: Proof

$$\begin{aligned}\#G \times \#\text{Orbits of } X &= \sum_{\text{Orbits } Z} \#G \\ &= \sum_{\text{Orbits } Z} \#G_z \times \#Z\end{aligned}$$

Burnside's Lemma: Proof

$$\begin{aligned}
\#G \times \#\text{Orbits of } X &= \sum_{\text{Orbits } Z} \#G \\
&= \sum_{\text{Orbits } Z} \#G_z \times \#Z \\
&= \sum_{\text{Orbits } Z} \sum_{z \in Z} \#G_z
\end{aligned}$$

Burnside's Lemma: Proof

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&= \sum_{\text{Orbits } Z} \sum_{z \in Z} \#G_z \\
&= \sum_{x \in X} \#G_x
\end{aligned}$$

Burnside's Lemma: Proof

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&= \sum_{x \in X} \#G_x \\
&= \#\{(g, x) \mid x \in X, g \in G, g \cdot x = x\}
\end{aligned}
$$

Burnside's Lemma: Proof

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&= \#\{(g, x) \mid x \in X, g \in G, g \cdot x = x\} \\
&= \sum_{g \in G} \#X^g
\end{aligned}
$$

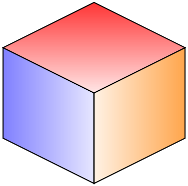
Back to our original problem!

The number of colorings of the cube with n colors is the number of *orbits* of the set of colorings of a fixed cube.

$$\#\text{Symmetries} \times \#\text{Colorings} = \sum_{g \in G} \#X^g$$

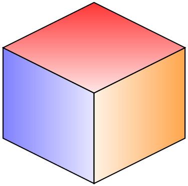
First step: determine the symmetries of the cube!

Symmetries of the cube



Type of symmetry	# of symmetries
$\pm 90^\circ$ face rotation	6
180° face rotation	3
180° edge rotation	6
$\pm 120^\circ$ diagonal rotation	8
Do nothing	1

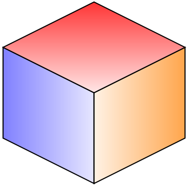
Invariant colorings



Second step: determine invariant colorings!

Type of symmetry	# invariant colorings
$\pm 90^\circ$ face rotation	
180° face rotation	
180° edge rotation	
$\pm 120^\circ$ diagonal rotation	
Do nothing	

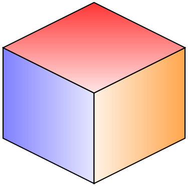
Invariant colorings



Second step: determine invariant colorings!

Type of symmetry	# invariant colorings
$\pm 90^\circ$ face rotation	n^3
180° face rotation	
180° edge rotation	
$\pm 120^\circ$ diagonal rotation	
Do nothing	

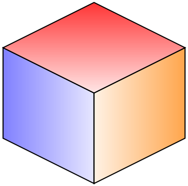
Invariant colorings



Second step: determine invariant colorings!

Type of symmetry	# invariant colorings
$\pm 90^\circ$ face rotation	n^3
180° face rotation	n^4
180° edge rotation	
$\pm 120^\circ$ diagonal rotation	
Do nothing	

Invariant colorings



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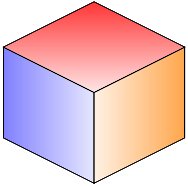
180° face rotation	n^4
180° edge rotation	n^3
$\pm 120^\circ$ diagonal rotation	
Do nothing	

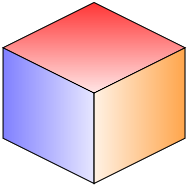
Invariant colorings



Second step: determine invariant colorings!

Type of symmetry	# invariant colorings
$\pm 90^\circ$ face rotation	n^3
180° face rotation	n^4
180° edge rotation	n^3
$\pm 120^\circ$ diagonal rotation	n^2
Do nothing	

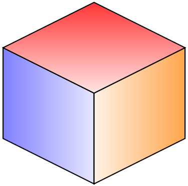
Invariant colorings



Second step: determine invariant colorings!

Type of symmetry	# invariant colorings
$\pm 90^\circ$ face rotation	n^3
180° face rotation	n^4
180° edge rotation	n^3
$\pm 120^\circ$ diagonal rotation	n^2
Do nothing	n^6

The answer



Type of symmetry	# of symmetries	# invariant colorings
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180° face rotation	3	n^4
180° edge rotation	6	n^3
$\pm 120^\circ$ diagonal rotation	8	n^2
Do nothing	1	n^6

$\Longrightarrow$ # of colorings using at most n colors is

$$\boxed{\frac{1}{24}\left(6n^3 + 3n^4 + 6n^3 + 8n^2 + n^6\right)}$$

Results for small n

n	Number of colorings
1	1
2	10
3	57
4	240
5	800
6	2226
97	34718692505

Thanks for your attention!