

Poisson 2006: Workshop Notes

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1 Alan Weinstein, *Group(oid)-like objects in Poisson geometry and algebra I*

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The goal of these lectures is to see if there is a completely general notion of a group to encompass all of the ideas we like to think of associated to a group. Of course, we will not reach this goal. Some are spaces, and others will be more algebraic, but will still take on a geometric flavor. Originally we think of a space as a set with a topological structure, then we moved on to a scheme or variety, but now even a category can be taken as a model for a space.

Historically, a group was not abstract, but a transformation group, usually selected because they had a certain property, closed under composition and inverse. Then there's Cayley's theorem that a group can be realized as a transformation. One can put an abstract structure on these groups to get an idea of Lie group, topological group, etc. but still a group in the usually sense. Especially with Lie groups, we can localize to simple linear algebra. However, going from Lie algebras to Lie groups, doesn't always work, such as moving from a Lie algebra over a Banach space.

Suppose for simplicity, we'll work in \mathbb{R}^n and we have an infinite sequence of subsets such that

$$\mathbb{R}^n \supseteq \underbrace{U_1 \supseteq U_2 \supseteq \cdots \supseteq \{0\}}_U$$

and $\bigcap U_m = \{0\}$. Similarly a chain of V_i 's with $\{f_i\} : U_i \rightarrow v_i$, with $f_i(0) = 0$. Then $\{f_i\} \sim \{g_i\}$ if $g_i = f_i$ for i sufficiently large, and $f_j = f_i|_{U_j}$ if $j \geq i$. The maps don't even have to be defined for all i . We can think of a map on $U \times U = U_i \times U_i \subseteq \mathbb{R}^{2n}$ and the diagram:

$$\begin{array}{ccc}
\mathcal{U} \times \mathcal{U} \times \mathcal{U} & \xrightarrow{m \times \text{id}} & \mathcal{U} \times \mathcal{U} \\
\text{id} \times m \downarrow & & \downarrow m \\
\mathcal{U} \times \mathcal{U} & \xrightarrow{m} & \mathcal{U}
\end{array}$$

With this construction, one can produce the *germ* of a Lie group. Even in finite dimensions, this has the advantage of being unique. However, they aren't groups in the usual sense because there are no points, except 0. Therefore we have an example of a group object in the category. A group in the category of differential manifolds is a Lie group, but here we don't have any underlying structure.

So a group in a category has the notion of multiplication $X \times X \xrightarrow{m} X$, $x \xrightarrow{e} X$, and $X \xrightarrow{i} X$.

Exercise 1. *When can the group axioms be weakened (for example by changing things such as 2-sided to 1-sided)?*

So for example, what is a Poisson Lie group? You might think it's a group in the category of Poisson manifolds, but there's a problem. Most things work out fine, and you don't even need a Poisson map, but inversion is not even a Poisson map, but an *anti*-Poisson map. So there's another operation, think of it as a functor, where the manifold stays the same but the structure is multiplied by -1 . Possibly it's an involution, i.e. $X \rightarrow X^{\text{op}}$. So what does it mean in the context of $g \cdot g^{-1} = e$. This can be expressed as:

$$\begin{array}{ccccc}
g & \xrightarrow{\Delta} & (g, g) & \xrightarrow{\text{id} \times i} & (g, g^{-1}) \\
\downarrow & & & & \downarrow m \\
\text{pt} & \xrightarrow{e} & & & gg^{-1} = e
\end{array}$$

which should commute. But in the following diagram, $\text{id} \times i$ could never be a Poisson map!

$$\begin{array}{ccccc}
X & \xrightarrow{\Delta} & X \times X^{\text{op}} & \xrightarrow{\text{id} \times i} & X \times X \\
\downarrow & & & & \downarrow m \\
\text{pt} & \xrightarrow{e} & & & X
\end{array}$$

Set theoretically, this gives us the notion of what inverse of the identity looks like. But this diagram doesn't work in the case of Poisson manifolds, or in a category. We can try something that is a little better and works in some other situations but is not quite completely satisfactory. Consider the diagram:

$$\begin{array}{ccc}
 G \times G & \xrightarrow{m} & G \longleftarrow e & \text{pt} \\
 & \swarrow & \searrow & \\
 & G \times G \times \text{pt} & &
 \end{array}$$

where we have $\{(g, h) : gh = e\} = m^{-1}(e) \subset G \times G \times \text{pt}$.

The graph of the inversion map is not a Poisson submanifold, but it is a coisotropic submanifold. This means that if it's a graph, it's the graph of a Poisson map from $G^{op} \rightarrow G$, which is good, i.e. what we want the graph of an inverse map to be. Much of this is explain in a paper called *Coisotropic Calculus and Poisson submanifolds*.

Exercise 2 (Open Question). *How to “categorize” the notion of Poisson-Lie group.*

Staying within Poisson geometry a little longer, there's a geometric analogue of the categorical representation of the objects of a Poisson manifold. We have a “complete Poisson map” from a symplectic manifold to a Poisson manifold

$$\begin{array}{c}
 S \\
 \downarrow J \\
 P
 \end{array}$$

If we're given a multiplication, we can compose it with the product, to get:

$$\begin{array}{ccc}
 S_1 \times S_2 & & \\
 J_1 \times J_2 \downarrow & \curvearrowright & \\
 P \times P & & J_1 \otimes J_2 \\
 m \downarrow & \searrow & \\
 P & &
 \end{array}$$

Also the unit element is a particular representation because the point is a 0-dimensional Poisson manifold. Inversion corresponds to some kind of dual but it's already tricky enough. With respect to this product, we then take pt to act as a unit.

$$\begin{array}{c}
 \text{pt} \\
 \downarrow \\
 P
 \end{array}$$

Exercise 3 (Question). *How is inversion “coded” as a “duality” in the representations of P if P is a Poisson Lie group?*

The place to go for guidance is in algebra where there's the notion of *Rigid Tensor Category*. Here the objects form abelian groups, and there is an extra functor that takes two objects and produces a new one. All this is quite precise in the algebraic sense.

A more abstract way of describing a group-like thing is as a category. In general one can look at a tensor category and construct a more group-like object, or one can simply take the tensor category itself as the group object.

So this is one way that categories play a role in our story, but categories themselves can be exotic objects. They are essentially group-like, but the compositions are different and not always defined. In a category we have morphisms and objects with sources and targets, referred to as l and r here. Then $(l, r)^{-1}(A, B) = \text{Hom}(A, B)$. Sometimes $\text{Hom}(A, B)$ will also mean as maps from B to A as well. Furthermore, we have composition as:

$$\text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$$

with

$$X_1 \times_{a_{X_0 b}} X_1 \rightarrow X_1$$

We can sort of enrich the structure by requiring these to be bilinear maps, but they're still sets. However, we can find examples in differential geometry where they aren't even sets. Let's instead take them to be symplectic manifolds and we can define it to be

$$\text{Hom}(A, B) = \text{Lag}(A \times B^{op})$$

This turns out to be nice morphism between symplectic manifolds but they don't form a category because they don't compose nicely. We can go a step further and make the following definition:

$$\text{Hom}(A, B) \stackrel{\text{def}}{=} A \times B^{op}$$

Now we need maps like:

$$(A \times B^{op}) \times (B \times C^{op}) \rightarrow A \times C^{op}$$

However, remember these are symplectic manifolds (i.e. $A \times B^{op}$), so look at the lagrangian submanifolds of

$$A^{op} \times B \times B^{op} \times C \times A \times C^{op}$$

We can get the triple diagonal $\Delta_A \times \Delta_B \times \Delta_C$. So then the morphisms themselves are Lagrangian submanifolds. All the compositions are then well-defined. So what's the unit in this category? It should be a map from $x \rightarrow \text{Hom}(A, A) = A \times A^{op}$ with $(\text{pt})^{op} \times A \times A^{op} \supseteq \Delta_A$. We can also take $\text{Hom}(A, B) = A \times B^{op}$ to $\text{Hom}(B, A) = B \times A^{op}$, but this is an *anti*-symplectic map. We therefore run into problems of inversion.

Next time, we'll look at groupoids and stacks which will allow us to consider notions such as the quotient of a circle by a dense subgroup ($S^1 \setminus \mathbb{Z}$) to be some sort of Lie group in the world of differentiable stacks. A paper recently appeared on the arxiv by Behrend-Xu.

Question: How do you compose the Lagrangian submanifolds that are supposed to be morphisms?

Answer: Take $L \subseteq A \times B, M \subseteq B \times C$, with $\{(a, c) | \exists b \text{ such that } (a, b) \in L, (b, c) \in M\}$. Then you compose as

$$L \times M \subseteq A \times B^{op} \times B \times C^{op} \subseteq A \times \Delta_B \times C^{op}$$

Then

$$L \times M \cap A \times \Delta_B \times C^{op} \rightarrow A \times C^{op}$$

2 Yannick Voglaire, *Some aspects of non-formal deformation quantization I*

1. Introduction to quantization, Invariant WKB quantization
2. Phase functions
 - (a) two examples
 - (b) Weyl triples and associativity
 - (c) Admissible functions on symplectic symmetric spaces
3. Universal Deformation Formulae
4. WKB quantization of solvable symmetric spaces
 - (a) Formal invariant associative star products
 - (b) Strictness of the quantization
5. Examples: $A \ltimes B$, AN of $SU(1, n)$

2.1 Introduction to quantization, Invariant WKB quantization

We want a correspondence between operators in the classical and quantum spaces.

WKB invariant quantization:

Let (M, ω, ∇) be a symplectic manifold with a torsion-free linear connection ∇ such that $\nabla\omega = 0$

$$\text{Aut}(M, \omega, \nabla) = \text{Aff}(M, \nabla) \cap \text{Symp}(M, \omega)$$

Definition 2.1. G fixed subgroup of $\text{Aut}(M, \omega, \nabla)$. A WKB G -invariant quantization of (M, ω, ∇) is a triple

$$(\{\mathcal{E}_\nu, *_\nu\}_{\nu \geq 0}, S, \{a_\nu\}_{\nu \geq 0})$$

such that:

1. $\{\mathcal{E}_\nu, *_\nu\}_{\nu \geq 0}$ 1-parameter family of associative $*$ -algebras with

- (a) $(\mathcal{E}_0, *_0)$ Poisson subalgebra of $C^\infty(M)$
 - (b) $D(M) \subset \mathcal{E}_0 \subset \mathcal{E}_\nu \quad \forall \nu \geq 0$
2. $S \in C^\infty(M \times M \times M, \mathbb{R})$
- (a) $L_g^* S = S$
 - (b) $\forall x_0 \in M, S(x_0, x_0, x_0)$ has a non-degenerate critical point at (x_0, x_0) .
3. $a_\nu \in C^\infty(M \times M \times M, \mathbb{R}^+)$ with $L_g^* a_\nu = a_\nu$
4. On $D(M) \subset \mathcal{E}_\nu$, we have

$$(u *_\nu v) = \int_{M \times M} a_\nu e^{\frac{i}{\nu} S} u \otimes v$$

5. $(u *_\nu v)(x) = u(x) \cdot v(x) + \frac{\nu}{i} c_1(u, v)(x) + o(\nu^2)$

$$c_1(u, v) - c_1(v, u) = \{u, v\}$$

2.2 Phase functions

Example 2.2 (Weyl product).

$$u * v(x) = \int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} \frac{1}{\hbar^{2n}} \exp \frac{2i}{\hbar} S_0(x, y, z) u(y) v(z) dy dz$$

where $S_0(x, y, z) = \Omega(x, y) + \Omega(y, z) + \Omega(z, x)$, $\mathcal{E}_\hbar = S(\mathbb{R}^{2n})$ (Schwartz functions).

Example 2.3 (Weinstein phase). (insert picture here)

with t such that $s_x s_y s_z(t) = t$ and $S_w = \int_{\mathcal{E}} w$

Weyl triples:

Let (M, μ) , μ a volume form, be an integral manifold. Need S such that

1. antisymmetric: $S(x, y, z) = S(y, z, x) = -S(y, x, z)$
2. $S(x, y, z) = S(m, x, y) + S(m, y, z) + S(m, z, x)$
3. $\forall x \in M, \exists s_x \mu$ -preserving diffeomorphism such that $S(x, y, z) = -S(x, s_x y, z)$ with $s_x(y) = 2x - y$.

We therefore have

$$v*v = \int K v \otimes v \text{ is } * \text{-associative} \iff \int_M K(a, b, t) K(t, c, d) dt = \int_M K(a, \tau, d) K(\tau, c, d) d\tau$$

$$S(x, y, z) = (\text{picture}), \dots$$

Theorem 2.4. For $K = e^{iS}$, $*$ is associative.

Proof.

$$* \text{ associative} \iff \int_M \exp(i(\text{picture})) dt = \int_M \exp(i(\text{picture})) d\tau$$

We want to find $\varphi : M \rightarrow M$, μ -preserving such that (picture). Remainder of proof to be added with diagrams... \square

Symplectic symmetric spaces

Definition 2.5. A *symplectic symmetric space* (SSS) denoted (M, ω, s) where $s : M \times M \rightarrow M$ such that

1. $\forall x \in M, s_x : M \rightarrow M$ such that $y \mapsto s_x y = s(x, y)$ is an involutive symplectomorphism
2. s_x has an isolated fixed point at x .
3. $s_x s_y s_x = s_{s_x y}$ (picture)

Definition 2.6. A function S is *admissible* on (M, ω, s) if:

1. antisymmetric
2. $s_x^* S = S$
3. $S(x, y, z) = -S(x, s_x y, z)$

Proposition 2.7. *Weinstein's phase is admissible.*

2.3 Universal Deformation Formulae

If G is a group for which we've found a WKB invariant (under left action of group with itself) quantization, with kernel K , and if we have an action on this manifold, $\alpha : G \times A \rightarrow A$ (A is a topological algebra), then the formula

$$a *^A b = \int_{G \times G} K(e, g, h) \alpha_g(a) \alpha_h(b)$$

where $\alpha_g(a) = \alpha(g, a)$ and $a, b \in A$.

3 Eckhard Meinrenken, *Pure spinors and moment maps I*

Joint work with A. Alekseev and H. Bursztyn

3.1 Motivation

Assume G is a simply connected Lie group with a bi-invariant pseudo-Riemannian metric. (i.e. G semi-simple)

Claim 3.1. Every conjugacy class $\mathcal{C} \subset G$ carries a *canonical* invariant volume form.

Remark 3.2. • the assumption $\pi_0(G) = \pi_1(G) = \{0\}$ can be weakened

- without any assumptions on π_0 or π_1 still there exists an invariant measure

Let $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ be the bilinear form on $\mathfrak{g} = Lie(G)$. Invariant 2-form on conjugacy classes $\mathcal{C} \subset G$

$$\omega(\xi^*, \eta^*) = \mathcal{B} \left(\frac{Ad_g - Ad_{g^{-1}}}{2} \xi, \eta \right), \xi, \eta \in \mathfrak{g}$$

Theorem 3.3. If $\pi_0(G) = \pi_1(G) = \{0\}$ the formula

$$\psi_g = \det \left(\frac{Ad_g + 1}{2} \right) \exp \left(\frac{1}{2} B \left(\frac{Ad_g - 1}{Ad_g + 1} \theta^L, \theta^L \right) \right)$$

defines an invariant form $\psi \in \Omega(G)$.

Here $\theta^L \in \Omega^1(G, \mathfrak{g})$ is the left-Maurer Cartan form “ $g^{-1}dg$ ”

3.2 Pure spinors

V vector space, $\mathcal{V} = V \oplus V^*$. inner product on $\mathcal{V} : \langle x, x \rangle = 2\langle \alpha, v \rangle (x = v \oplus \alpha)$

Definition 3.4. $E \subset \mathcal{V}$ is *Lagrangian* if $E = E^\perp$.

Equivalently, $\dim(E) = \frac{1}{2} \dim \mathcal{V}$ and $\forall x \in E : \langle x, x \rangle = 0$.

Example 3.5. $V, V^*, Gr_\omega = \{(v, \omega(v, \cdot))\}, Gr_\pi = \{(\pi(\alpha, \cdot), \alpha)\}$ with $\omega \in \Omega^2(V), \pi \in \Lambda^2(V)$.

Consider contravariant spinor representation $\rho : \mathcal{V} \rightarrow \text{End}(\wedge V^*)$ with $\rho(x) \cdot \varphi = 2(v)\varphi + \alpha \wedge \varphi (x = v \oplus \alpha)$. Note $\rho(x)^2 \varphi = \langle \alpha, v \rangle \varphi = \frac{1}{2} \langle x, x \rangle \varphi$. Hence all veo

Definition 3.6. $\varphi \in \wedge V^* \setminus \{0\}$ is a pure spinor $\iff N_\varphi$ is Lagrangian.

Lemma 3.7. *For every pure spinor $\varphi \in \Lambda V^*$ there exists a unique*

$$S \subset V, \omega \in \Lambda^2 S^*, \theta \in \Lambda^{\text{top}}(\text{Ann}(S)) \setminus \{0\}$$

such that

$$\varphi = e^{-\omega} \wedge \theta$$

Note:

1. $\varphi = e^{-\omega} \wedge \theta$ is a pure spinor defining

$$E = \{v \oplus \alpha | v \in S, \alpha|_S = \omega(v, \cdot)\}$$

2. For any $E \in \text{Lag}(\mathcal{V})$ get $S = \text{pr}_v(E)$,

Consequences of $\varphi = e^{-\omega} \wedge \theta$.

- Every pure spinor is *even* or *odd*
- $\varphi^{\text{top}} \neq 0 \iff N_\varphi \cap V = 0 \iff N_\varphi = \text{Gr}_\pi$
- $A : V \rightarrow V'$ linear, $\varphi' \in \Lambda(V')^*$ pure $\implies A^*\varphi' =: \varphi$ is pure (unless $\varphi = 0$)
- $\varphi_1, \varphi_2 \in \Lambda V^*$ pure $\implies \varphi_1 \wedge \varphi_2 =: \varphi$ is pure (unless $\varphi = 0$).

Theorem 3.8 (Chevalley). *If φ, ψ are pure spinors representing $E, F \in \text{Lag}(\mathcal{V})$ then*

$$E \cap F = \{0\} \iff (\varphi^t \wedge \psi)^{\text{top}} \neq 0$$

Remark 3.9. There is a parallel theory using the covariant spinor representation.

$$\tilde{\rho} : \mathcal{V} \rightarrow \text{End}(\Lambda V)$$

$$\tilde{\rho} \cdot \chi = v \wedge \chi + 2(\alpha)\chi$$

Proposition 3.10. *Suppose $(V, E), (V', E')$ Dirac spaces and*

$$A : V \rightarrow V'$$

is strong Dirac map. Suppose $F' \in$

3.3 Almost Dirac structures

Definition 3.11. An *almost Dirac structure* on a manifold M is a Lagrangian subbundle $E \subset \mathcal{T}$

Definition 3.12. $\varphi \in \Omega(M)$ is a *pure spinor* if each $\varphi_x \in \Lambda T_x^* M$ is a pure spinor on $T_x M$.

E.g.: $\omega \in \Omega^2(M)$...

Back to our Lie group G :

$\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ Maurer-Cartan form

$\xi^L, \xi^R \in X(G)$ left, right invariant vector fields corresponding to $\xi \in \mathfrak{g}$.

Define section $e(\xi), f(\xi) \in \Gamma(\mathcal{T}G)$ by

$$E = \text{span}\{e(\xi), \xi \in \mathfrak{g}\}$$

$$e(\xi) = (\xi^L - \xi^R) \oplus B \left(\frac{\theta^L + \theta^R}{2}, \xi \right)$$

Note that $pr_{TG}(E) \in TG$...

Hence, if $\psi \in \Omega(G)$ is a pure spinor defining $F = \text{span}\{f(\xi), \xi \in \mathfrak{g}\}$ then

$$2e^* \psi \in \Omega(G)$$

defines a complement F_C to $E_C = Gr_\omega$. Hence, by Chevalley's theorem,

Finally, where do these come from?

If (M, B) is a pseudo-Riemannian manifold there is an isometry

$$\kappa : TM \oplus \overline{TM} \longrightarrow \mathcal{T}$$

by $B \oplus (-B)$ to $\langle \cdot, \cdot \rangle$ given by

$$\kappa(v \oplus w) = (v + w) \oplus B^b \left(\frac{v - w}{2} \right)$$

Hence, for $C \in \Gamma(O(TM))$ get transverse Lagrangians E, F

$$E = \text{Ad}_\kappa \left(\begin{array}{cc} C & 0 \\ 0 & I \end{array} \right) \cdot TM$$

$$F = \dots$$

For $M = G$ take the transformation

$$TG \leftrightarrow G \times \mathfrak{g} \leftrightarrow TG$$

with the first and second relations given by left and right trivialization, respectively. Then $C \in \Gamma(SO(TG))$ takes T^*G, TG to E, F respectively.

The lift $\tilde{C} \in \Gamma(\text{Spin}(TG)) \subseteq \Gamma(Cl(\mathcal{T}))$ takes defining spinors $\mu, 1$ for T^*G, TG , to defining spinors φ, ψ for E, F .

A calculation yields our formula for $\psi \in \Omega(G)$.

4 Florent Schaffhauser, *Introduction to quasi-Hamiltonian spaces*

4.1 From Hamiltonian to quasi-Hamiltonian spaces

Start with a Lie group acting on a manifold, $G \curvearrowright M$, with $x \in \mathfrak{g} \mapsto x^\sharp \in \Gamma(TM)$. So now assume that the manifold has a symplectic structure, (M, ω) . For every function on the manifold there is a vector field called the Hamiltonian vector field:

$$(f : M \rightarrow \mathbb{R}) \mapsto X_f \in \Gamma(TM) \text{ such that } \underbrace{L_{X_f} \omega}_{\omega(X_f, \cdot)} = df$$

Definition 4.1 (Hamiltonian action). $\forall X \in \mathfrak{g}, \exists \mu^X : M \rightarrow \mathbb{R}$ such that $d\mu^X = L_{X^\sharp} \omega$.

$$\mu : M \rightarrow \mathfrak{g}^*, \quad x \mapsto (X \in \mathfrak{g} \mapsto \mu^X(x)) = \langle \mu(x), \cdot \rangle$$

$$\langle \cdot, X \rangle : \mathfrak{g}^* \rightarrow \mathbb{R}, \quad \mu^* \langle \cdot, X \rangle : M \rightarrow \mathbb{R}$$

For the Hamiltonian action we have $G \curvearrowright (M, \omega)$ with:

- $d\omega = 0$
- $\ker \omega = \{0\}$
- $g^* \omega = \omega$
- $\exists \mu : M \rightarrow \mathfrak{g}^*$ where $\forall X \in \mathfrak{g}, \exists \alpha^X \in \Omega^1(\mathfrak{g}^*)$ such that $i_{X^\sharp} \omega = \mu^* \alpha^X$

For the quasi-Hamiltonian action we have:

- $g^* \omega = \omega$
- $\exists \mu : M \rightarrow G, \forall X \in \mathfrak{g}, \exists \alpha^X \in \Omega^1(G)$ such that $i_{X^\sharp} \omega = \mu^* \alpha^X$ with μ equivariant.

Now $\alpha^X = \frac{1}{2}(\theta^L + \theta^R|X)$ where $(\cdot|\cdot)$ is an Ad-invariant non-degenerate product on \mathfrak{g} and θ^L, θ^R are Maurer-Cartan 1-forms. $g \in G, v \in T_g G, \alpha_g^X(v) = \frac{1}{2}(g^{-1} \cdot v + v \cdot g^{-1}|X)$.

Consequences are:

- $d\omega = \mu^* X$, with $X \in \Omega^3(G)$ a Cartan 3-form, $X_1(X, Y, Z) = ([X, Y]|Z)$
- $\ker \omega_x = \{X_x^\sharp : X \in \ker(\text{Ad}g + \text{id})\}$

4.2 Fundamental examples

So now we have $\mu : O \hookrightarrow \mathfrak{g}^*$, with

1. $\mu : \mathcal{C} \hookrightarrow G$
2. $G \times G \xrightarrow{\mu} G \times G$ with $(a, b) \mapsto (ab, a^{-1}b^{-1})$
3. $G \times G \xrightarrow{\mu} G$ with $(a, b) \mapsto aba^{-1}b^{-1}$
4. Products: $M_1 \times M_2 \rightarrow G$ with $(x_1, x_2) \mapsto \mu_1(x_1)\mu_2(x_2)$

4.3 Reduction

$\Sigma_{0,3} = S^2 \setminus \{s_1, s_2, s_3\}$, $\pi_1(\Sigma_{0,3}) = \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1\gamma_2\gamma_3 = 1 \rangle$. Then

$$\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma_{0,3}), G) = \{(u_1, u_2, u_3) \in \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3 \mid u_1u_2u_3 = 1\} = \mu^{-1}(\{1\})$$

and

$$\mathrm{Hom}_{\mathcal{C}}(\pi_1(\Sigma_{0,3}), G)/G = \mu^{-1}(\{1\})/G$$

Theorem 4.2. $\mu^{-1}(\{1\})/G$ is a symplectic manifold.

1. $i^*\omega$ is basic: $\mu^{-1}(\{1\}) \xrightarrow{i} (M, \omega) \exists \omega^{\mathrm{red}} \mid p^*\omega^{\mathrm{red}} = i^*\omega$
2. ω^{red} is symplectic.

5 Alexander Cardona, *Quantization of generalized complex manifolds*

5.1 Introduction

Let (M, ω) be a symplectic (compact) manifold and suppose that $[\omega] \in H_{\mathrm{dR}}^2(M, \mathbb{Z}) \implies \exists (\mathcal{L}, \nabla) \rightarrow M$ with $\Omega^{\nabla} = \omega$ is the prequantization line bundle.

The Poisson algebra $(C^\infty(M), \{\cdot, \cdot\})$ with $\{f, g\} = \omega(X_f, X_g)$ gives $C^\infty(M) \rightarrow O(\mathcal{H} = L^2(\mathcal{L})) : f \mapsto \hat{f} = f - \nabla_{X_f}$.

Vergne's definition of the quantization of M : (M, ω) a prequantizable symplectic manifold, J an almost-complex structure on M , with $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$. Then $\Omega^{ij}(M) = \Gamma(M, \Lambda^{ij}T^*M)$ with the Spin^c-Dirac operator:

$$\partial : \Omega_{\mathcal{L}}^{0,\mathrm{even}}(M) \rightarrow \Omega_{\mathcal{L}}^{0,\mathrm{odd}}(M)$$

Then $Q(M) = \mathrm{Ind}(\partial) = \ker(\partial) - \mathrm{coker}(\partial)$.

$G \curvearrowright (M, \mathcal{L})$ Hamiltonian equivariant action with moment map $\mu : M \rightarrow \mathfrak{g}^*$. $M_G = \mu^{-1}(0)/G$ is the Marsden-Weinstein reduction of M .

$$\mathcal{L} \rightarrow (M, \omega) \xrightarrow{i} \mu^{-1}(0) \xrightarrow{\pi_G} (M_G, \omega_G) \leftarrow \mathbb{L}_G$$

with $i^*\omega = \pi_G^*\omega_G$.

We have a representation of G on $\Gamma(\mathcal{L})$.

$$D_\xi s = \nabla_\xi s - \langle \mu, \xi \rangle s, \quad \xi \in \mathfrak{g}$$

More importantly, $\dim(Q(M_G)) = \dim(Q(M))^G$.

5.2 Basics on G.C.M.'s

Now we have $\mathcal{J} : \mathcal{T} \rightarrow \mathcal{T}$ an almost generalized complex structure with respect to the inner product on \mathcal{T} . Given \mathcal{J} an almost GCS, L the $+i$ -eigenbundle is maximal isotropic and $L \cap \bar{L} = \{0\}$. We define \mathcal{J} to be a GCS if the sections are closed under the Courant bracket: $[\Gamma(L), \Gamma(L)] \subset \Gamma(L)$. The lecture material for this section is not included here and can be found in Courant or Gualtieri's theses.

5.3 Definition of geometric quantization of G.C.M.

For prequantum line bundles: take (M, \mathcal{J}) to be a GCM and let us consider the anti-symmetric inner product and call Ω_0 its restriction to L .

$$\mathfrak{a} : \Gamma(L) \rightarrow \Gamma(TM), \text{ Weinstein-Zaubon}$$

$$\mathfrak{a}^* : H_{\text{dR}}^*(M) \rightarrow H_L^*(M)$$

Then (M, \mathcal{J}) is prequantizable if $i : \mathbb{Z} \rightarrow \mathbb{R}$ and $[\Omega_0] \in \mathfrak{a}^*(i^*H_{\text{dR}}^*(M))$

(M, \mathcal{J}) is prequantizable $\implies \mathcal{L} \rightarrow M$ with ∇ . \mathcal{J} a GCS $\implies L'$ maximal isotropic subbundle of $\mathcal{T}_{\mathbb{C}}$. Let ∇ be a Hermitian connection on \mathcal{L} , coupling it to d_L ,

$$\partial_L : \Omega_{L, \mathcal{L}}^{0, \text{even}}(M) \rightarrow \Omega_{L', \mathcal{L}}^{0, \text{odd}}(M)$$

gives that $Q(M) = \text{Ind}(\partial_L)$.

What conditions should be put on an action to have a generalized "quantization versus reduction" relation?

6 Chenchang Zhu, *A gerbe for the elliptic gamma function*

$E_\tau = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ is an elliptic curve with a line bundle of chern class 1. Specifically, this line bundle is $\mathcal{L} = \mathbb{C} \times \mathbb{C}/\mathbb{Z}^2$ and the action is defined as

$$(z, u)(n, m) = (z + m + n\tau, e^{2\pi i n x} (-1)^m u)$$

Elliptic Gamma functions:

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt$$

with

- $\Gamma(z + 1) = \Gamma(z)$
- $\Gamma(z + 1) = \frac{\sin wz}{w} \Gamma(z) \rightsquigarrow \Gamma(z + \sigma) = \sin(\pi z) \Gamma(z)$, a change of variables
- $\Gamma(z + \sigma) = \theta_0(z, \tau) \Gamma(z)$

So the idea is we present differentiable stacks as a groupoid and \mathbb{C}^* -gerbes as groupoids as well. We can actually find a gerbe over the universal triptic curves. I don't understand anything enough to take any more notes...

7 Alan Weinstein, *Group(oid)-like objects in Poisson geometry and algebra II*

From Chinchang's talk yesterday, she mentioned a triptic curve, $\mathbb{C}/\mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}$. You have to view this as a differentiable stack. Instead, let's consider something one might call a *diptic curve*, $\mathbb{R}/\mathbb{Z} + \lambda\mathbb{Z}$, where $\mathbb{R}/\mathbb{Z} \simeq S^1 = U(1)$ with $t \rightarrow e^{2\pi it}$. This is a very interesting example that we'd like to think of as a manifold dividing by a discrete group which should give us a 1-dimensional manifold, and differential stacks let us do this.

Another example that can be thought of as a symplectic orbifold is $\mathbb{R}^{2n}/\mathbb{Z}_2$. There is a singular point corresponding to the origin and it looks like a conic. The quotient of the origin by \mathbb{Z}_2 is not a free action. Typically, one can take any k , $\mathbb{R}^2/\mathbb{Z}_k$, and the k dictates the angle of the cone. If we look at the "Christmas tree ornament", with 3 and 4 at the poles, it actually belongs more to the Poisson manifold set, since the Hamiltonian vector fields would vanish at the poles. As a symplectic manifold, all the points are locally the same, so the poles aren't viewed any different. If you take two harmonic oscillators with frequency ratio of 3 to 4, then you actually get this ornament as the orbit space, so this is a natural symplectic object.

Instead we must consider groupoid, as a small category, and for us a very small category as just Lie groupoids with $G_1 \rightrightarrows G_0$ and $\{(g, h) | r(g) = l(h)\} = G_2 \rightarrow G_1 \circlearrowleft$. We can look at a subgroupoid closed under inversion (symmetric), and if we require a "wide" subgroupoid (containing the units) then we get equivalence relations.

As an example of how we can use the groupoid, take the Christmas ornament. First, consider the simpler examples, of a circle in \mathbb{R}^2 . For the first map $\mathbb{R}^2 \rightrightarrows \mathbb{R}^2$, $(0, z) \cdot (0, z) = (0, z)$, take $l = r = \text{id}$ and the second, $\mathbb{R}^2 \rightrightarrows \mathbb{R}^2$, $(1, -z) \cdot (1, z) = (0, z)$, $(1, z) = (0, -z)(1, z)$, with $l = -1, r = \text{id}$. So we're looking at the first component mod 2. So if I have a group H acting on a space K , we can think of a group element as an arrow with one end the l and the other the r , so in K , we have $k \xrightarrow{h} h \cdot k$.

There's also the notion of an equivalence relation. First take the objects to be discs, and have the annulus with arrows pointing out, and the l maps reverses the arrows with the r fixing the arrows. The atlas is these two charts, take the disjoint union of the two charts and the morphisms tell you how to glue one chart to the other, which is actually an equivalence relation. We get something like a sphere with a band in the middle.

In a similar vain, we can describe this ornament. So take the stack of 4 discs (the transformation groupoid \mathbb{Z}_4 acting on a disc), and the stack of 3 discs over another disc, and glue them together by an annulus, which resembles a clock. The map to one disc is a 3-fold covering and the other a 4-fold covering. We then similarly get a band in the middle of the ornament. Looking at the pole, it looks like a \mathbb{Z}_3 action on a point, usually a trivial action, but this doesn't give a trivial stack!

We need the notion of equivalence between groupoids. Since groupoids are categories, you might just think that the maps should be functors, but it turns out that's too restrictive of a notion. The appropriate idea is the notion of a *Bibundle*. This is the following: a space X with maps to G'_0 and G_0 and actions of the following $G'_1 \rightrightarrows G'_0$ and $G_1 \rightrightarrows G_0$. Roughly speaking, consider the moment map J from X to G_0 . The axioms are such that the whole thing becomes a groupoid. So if we have $J'(x) = J'(y)$, then there would be a unique groupoid element in G_0 from x to y . Equivalence of groupoids establishes an isotropy between orbits. It's more than just saying a 1-1 correspondence.

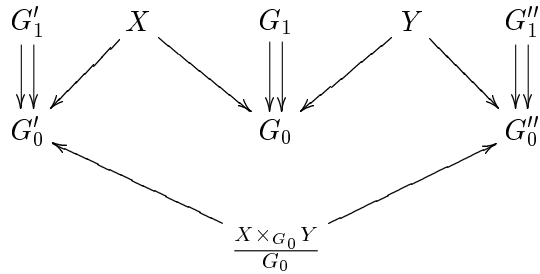
$$\begin{array}{ccc}
 G'_1 & & G_1 \\
 \Downarrow & \swarrow J' & \searrow J \\
 X & & X \\
 \Downarrow & & \Downarrow \\
 G'_0 & & G_0
 \end{array}$$

Definition 7.1. A *differential stack* is a Lie groupoid up to equivalence.

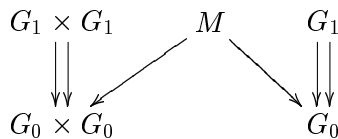
There's a more complicated definition which identifies a stack with a category. For example, take an ordinary manifold Y . Then the objects are maps(charts) f, f' from $X, X' \rightarrow Y$ and the morphisms are the maps $g : X' \rightarrow X$ between the charts. So the general notion is that given a Lie groupoid, the category that we attach to it is the category of all principle G_1 bundles. In general, a principle bundle will be a map from X to \underline{X} , with $J : X \rightarrow G_0$. When the groupoid is a group, it's just an ordinary principle bundle.

Again, the long awaited paper on this can be found at math.DG/0605694.

Now what would it mean to say a stack is a group. Mainly it should have a multiplication operation which is a morphism of stacks. Consider the morphism X with $G_1 \rightrightarrows G_0$ and $G'_1 \rightrightarrows G'_0$, then the Bibundle X will be principle over both G_0 and G'_0 . Why is this a good definition of a morphism? Because if I have another one Y between G and G'' , we can compose them over G_0 as $X \times_{G_0} Y$ over G_0 . It all looks pretty complicated, but one can nonetheless calculate these compositions.

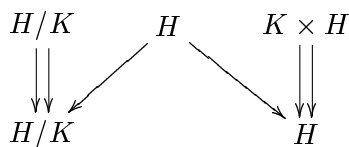


Now suppose we have a stack $G_1 \rightrightarrows G_0$, and take the product stack $G_1 \times G_1 \rightrightarrows G_0 \times G_0$, and a morphism M between them. Lifting to M from the product and then projecting to G_0 , we get a notion of multiplication, but it's multiple valued. This is all kept under control since these are actually actions on the left and right. It passes to an actual map of points $G_0/G_1 \times G_0/G_1 \rightarrow G_0/G_1$, which we can think of as a multiplication. Then we can add inverses and units, etc. and require commutativity.



So what's the example of a group in the category of stacks that's not just an ordinary group. Let $G \rightrightarrows \text{pt}$. The category is the principle G -bundle. So the stack $\text{pt} // G = BG$ is a principle G -bundle over the domain of the map. Sometimes people call this a stacking quotient. Again, consider $G \times G \rightrightarrows \text{pt}$ and $G \rightrightarrows \text{pt}$, with M between them, this corresponds to a *group homomorphism* $G \times G \rightarrow G$ which can only be thought of as group multiplication. This only works when G is abelian, and BG is an SLie group (*Stackey Lie group*).

Now take H a Lie group and K a normal subgroup. Look at the quotient $H // K$ and the stack $K \times H \rightrightarrows H$.



E.g. $\mathbb{Z} \subseteq S^1$ is a dense normal subgroup. Are there any non-abelian examples of dense normal subgroups? Some examples proposed turn out to be essentially like $\varphi^{-1}(K) \subseteq H' \xrightarrow{\varphi} H^{\text{abelian}} \supseteq K$. All of this is described in more detail in the paper with Blohmann. Consider the morphism H between the stacks $H/K \rightrightarrows H/K$ and $K \times H \rightrightarrows H$. Then $K = \mathbb{Z}, H = S^1$. For $H \times H$ over $K \times K \times H \times H \rightrightarrows H \times H$ and $H/K \times H/K \rightrightarrows H/K \times H/K$. Let M_0

connect them and then M is the bibundle over the entire thing equal to $(H \times H) \times_{H/K} H$.

$$\begin{array}{ccccc}
 K \times K \times H \times H & & H \times H & & H/K \times H/K \\
 \Downarrow & \swarrow & & \searrow & \Downarrow \\
 H \times H & & & & H/K \times H/K
 \end{array}$$

hence $M_0 = (H \times H) \times_{H/K} H$ is a bibundle

$$H \times H/H \times H \times K \times K \simeq H/H \times K$$

Consider the example given by the following diagram:

$$\begin{array}{ccccc}
 (S^1 \times S^1) \times_{S^1/\mathbb{Z}} S^1 & \xrightarrow{J_{S^1}} & S^1 & & \\
 \downarrow J_{S^1 \times S^1} & \searrow \exists! & \downarrow \pi & & \\
 & (S^1/\mathbb{Z} \times S^1/\mathbb{Z}) \times_{S^1/\mathbb{Z}} S^1/\mathbb{Z} & \longrightarrow & S^1/\mathbb{Z} & \\
 & \downarrow & & \downarrow \text{id} & \\
 S^1 \times S^1 & \xrightarrow{\pi \times \pi} & S^1/\mathbb{Z} \times S^1/\mathbb{Z} & \xrightarrow{m} & S^1/\mathbb{Z}
 \end{array}$$

We need to take equivalence before operation is associative and commutative (like loop spaces).

Next time we'll look at the space of functions on this quotient which turns out to be a Hopf algebra. For groupoids, we get the idea of a Hopfish algebra which will also be explained next time.

8 Eckhard Meinrenken, *Pure spinors and moment maps II*

8.1 Review

$\rho : \Gamma(\mathcal{T}) \rightarrow \text{End}_{C^\infty(M)}(\Omega(M))$, $v \oplus \alpha \mapsto 2(v) + \alpha$. $\varphi \in \Omega(M)$ is a *pure spinor* defining $E \iff \rho(x)\varphi = 0 \forall x \in \Gamma(E)$ and $\varphi \neq 0$ everywhere.

Example 8.1. 1. $\omega \in \Omega^2(M) \rightsquigarrow \varphi = e^{-\omega}$ defines $E = Gr_\omega$. 2. $\pi \in X^2(M) \rightsquigarrow \varphi = e^{-2\pi} \mu$ defines ... 3. G a Lie group, $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ invariant inner product on \mathfrak{g} . $E = \text{span}\{e(\xi) | \xi \in \mathfrak{g}\}$ where $e(\xi) = (\xi^L - \xi^R) \oplus B\left(\frac{\theta^L + \theta^R}{2}, \xi\right)$. $F = \text{span}$

These are obtained from T^*G , resp. TG , by a certain rotation $A \in \Gamma(O(\mathcal{T}G))$. Applying the lifted rotation $\tilde{A} \in \Gamma(\varphi_i \cap (\mathcal{T}G))$ to $\mu \in \Omega^{\text{op}}(G)$, resp. $1 \in \Omega(G)$.

Fix closed $y \in \Omega^3(M)$, $dy = 0$, $\implies d + y$ is a differential.

Let

$$Y(x_1, x_2, x_3) = [\rho(x_1), [\rho(x_2), [\rho(x_3), d + \eta]]] \in C^\infty(M)$$

Lemma 8.2. *The restriction of Y to $\Gamma(E)$ defines a tensor $Y_E \in \Gamma(\Lambda^3 E^*)$ “Courant tensor”.*

Proof. Let $\varphi \in \Omega(M)$ be a pure spinor defining E . Then $\forall x_i \in \Gamma(E)$,

$$\begin{aligned} \rho(x_1)\rho(x_2)\rho(x_3)(d+y)\varphi &= [\rho(x_1), [\rho(x_2), [\rho(x_3), d+\eta]]]\varphi \\ &= Y_E(x_1, x_2, x_3)\varphi \end{aligned}$$

Now the claim is evident. □

$$E = N_\varphi \subseteq \mathcal{T}$$

$$\rho(x_1)\rho(x_2)\rho(x_3)(d+y)\varphi = Y_E(x_1, x_2, x_3)\varphi, \quad x_i \in \Gamma(E)$$

8.2 Integrability

Definition 8.3. An almost Dirac structure $E \subset \mathcal{T}$ is a *Dirac structure* if it is integrable, i.e. $Y_E = 0$.

Equivalently, E is integrable $\iff \Gamma(E)$ preserved under the Courant bracket.

Proof.

$$\begin{aligned} Y(x_1, x_2, x_3) &= [\rho(x_1), [\rho(x_2), [\rho(x_3), d+\eta]]] \\ &= [\rho(x_1), \rho([[x_2, x_3]])] \\ &= \langle x_1, [[x_2, x_3]] \rangle \end{aligned}$$

Claim follows since E is maximal isotropic. □

Proposition 8.4 (Alexseev-Xu, Gualtieri). *$E \subset \mathcal{T}$ is a Dirac structure if and only if*

$$(d+y)\varphi = \rho(s)\varphi$$

for some $s \in \Gamma(\mathcal{T})$.

Given Dirac structure $E \subset \mathcal{T}$, let

$$[x, y]_E := [[x, y]], \quad (x, y \in \Gamma(E))$$

with $\rho([x, y]_E) = [\rho(x), [\rho(y), d+\eta]]$ and $p = pr_{TM} : E \rightarrow TM$.

Proposition 8.5. $(E, [,])$

This implies that $p(E) \subset TM$ defines a foliation of M .

Example 8.6. 1. $\varphi = e^{-\omega} E = Gr_\omega$, integrable $\iff d\omega = \eta$ (trivial foliation $p(E) = TM$).

2. $\varphi = e^{i(\pi)}\mu$ $E = Gr_\pi$, integrable $\iff \frac{1}{2}$
3. G a Lie group, $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$, $E = \text{span}\{e(\xi), \xi \in \mathfrak{g}\}$, $e(\xi) = (\xi^L - \xi^R) \oplus B\left(\frac{\theta^L + \theta^R}{2}, \xi\right)$.

Let

$$\eta = \frac{1}{12}B(\theta^L, [\theta^L, \theta^L]) \in \Omega^3(G)$$

Cartan 3-form.

Theorem 8.7. *The defining spinor $\varphi \in \Omega(G)$ for E satisfies*

$$(d + \eta)\varphi = 0$$

In particular, E is a Dirac structure. “Cartan-Dirac structure.”

Proof. For $x \in \Gamma(E)$,

$$\begin{aligned} \rho(x)(d + \eta)\varphi &= [\rho(x), d + \eta]\varphi \\ &= [i(\xi^L - \xi^R) + B\dots] \end{aligned}$$

□

$E = \text{span}\{e(\xi), \xi \in \mathfrak{g}\}$ Cartan-Dirac structure. $e(\xi) = (\xi^L - \xi^R)\dots$

Remark 8.8. 1. If $E \subset \mathcal{T}$ is a Dirac structure with respect to a closed $\eta \in \Omega^3(M)$ then the 2-forms on the leaves $S \subset M$ satisfy $d\omega = \iota^*\eta$.

2. The almost Dirac structure $F \subset \mathcal{T}$, $F = \text{span}\{f(\xi), \xi \in \mathfrak{g}\}$,

$$f(\xi) = \left(\frac{\xi^L + \xi^R}{2}\right) \oplus B\left(\frac{\theta^L - \theta^R}{4}, \xi\right)$$

is not integrable. Instead the defining pure spinor $\psi \in \Omega(G)$ satisfies

$$(d + \eta)\psi = \frac{1}{4} = \rho(e(\Xi)).\psi$$

where

8.3 Group-valued moment maps

Definition 8.9 (Alekseev-Malkin-M). A quasi-Hamiltonian \mathfrak{g} -space is a manifold M with a 2-form $\omega \in \Omega^2(M)$ and *moment map* $\Phi : M \rightarrow G$ such that

1. $d\omega = \Phi^*\eta$
2. M is a \mathfrak{g} -manifold, and Φ is \mathfrak{g} -equivariant.

$$3. \iota(\xi^\sharp)\omega = \Phi^*B\left(\frac{\theta^L + \theta^R}{2}, \xi\right)$$

$$4. \ker(\omega_m) = \{\xi_m^\sharp \mid \text{Ad}_{\Phi(m)}\xi = -\xi\}$$

Remark 8.10 (Xu, Burstyn-Crainic). Condition 4 can be replaced by equivalent condition

$$\ker(\omega_m) \cap \ker()$$

Theorem 8.11 (Burstyn-Crainic). *Conditions 2-4 above may be replaced by:*

$$\Phi \text{ is a strong Dirac map from } (M, Gr_\omega) \rightarrow (G, E_G)$$

IN particular, the \mathfrak{g} -action

The new viewpoint is extremely useful! E.g.

Theorem 8.12. *For any quasi-Hamiltonian G -space, (M, ω, Φ) , there exists an invariant volume form*

$$(\exp(\omega)\Phi^*\psi)^{\text{top}} \neq 0$$

$$\text{where } \psi_g = \det^{\frac{1}{2}}\left(\frac{\text{Ad}_g + 1}{2}\right) \exp\left(\frac{1}{2}B\left(\frac{\text{Ad}_g - 1}{\text{Ad}_g + 1}\theta^L, \theta^L\right)\right)$$

Immediate from our theory: volume form $\hat{=}$ pairing of pure spinors.

Remark 8.13. Get bi-vector field $\pi \in X^2(M)$, with

$$\exp(\omega)\Phi^*\psi = e^{-\iota(\pi)}(\exp(\omega)\Phi^*\psi)^{\text{top}}$$

From equation for $(d + \eta)\psi$ on deduces

$$\frac{1}{2}[\pi, \pi] = \frac{1}{4}\Xi^\sharp \in X^3(M)$$

(tri-vector field defined by $\Xi \in \Lambda^3\mathfrak{g}$)

Let $m : G \times G \rightarrow G$ group multiplication. Then

$$m^*\eta = pr_1^*\eta + pr_2^*\eta + d\tau,$$

$$\tau = \frac{1}{2}B(pr_1^*\theta^L, pr_2^*\theta^R) \in \Omega^2(G \times G)$$

Put

$$\tilde{\phi} = e^{-\tau}(pr_1^*\varphi \wedge pr_2^*\varphi) \in \Omega(G \times G)$$

Theorem 8.14 (Alekseev-Burstyn-M).

Theorem 8.15. Suppose (M_i, ω_i, Φ_i) are quasi-Hamiltonian \mathfrak{g} -spaces. Let

$$M = M_1 \times M_2 \text{ diagonal } \mathfrak{g}\text{-action}$$

Then (M, ω, Φ) is a quasi-Hamiltonian \mathfrak{g} -space.

Proof. $\Phi_1 \times \Phi_2 : (M_1 \times M_2, e^{-\omega_1} e^{-\omega_2}) \rightarrow (G \times G, \varphi^1 \wedge \varphi^2)$ is strongly Dirac.

$$m : (G \times G) \rightarrow$$

□

Example 8.16. Suppose $\sigma : G \rightarrow G$ is an involution preseving B . Let $H = G^\sigma$ its fixed point set. Then $M = G/H$ with $\omega = 0$ is a quasi-Hamiltonian $\mathbb{Z}_2 \times G$ -space. (It's the conjugacy class of (σ, e) .) Hence $M \times M$ with diagonal action is a quasi-Hamiltonian G -space.

Special Case: $G = (G \times G)^\sigma$, $\sigma(g_1, g_2) = (g_2, g_1)$, $M = G \times G/G \cong G$. So $M^2 = G^2$ is a quasi-Hamiltonian $G \times G$ -space. This called the *double*. One computes

$$\Phi(a, b) = (ab, a^{-1}b^{-1})$$

Fuse one more time, get

$$D(G) := G^2, \quad \Phi(a, b) = aba^{-1}b^{-1} \equiv [a, b]$$

Fuse several copies of $D(G)$

$$M = \underbrace{D(G) \times \cdots \times D(G)}_r = G^{2r}$$

$$\Phi(a_1, b_1, \dots, a_r, b_r) = \prod_{i=1}^r [a_i, b_i]$$

$$\omega = \dots$$

Theorem 8.17 (AMM). *The 2-form ω descends to the standard (Atiyah-Bott) symplectic form on*

$$M//G = \Phi^{-1}(e)/G = \text{Hom}(\pi_1(\Sigma), G)/G$$

Where Σ is a genus 3 surface.

Note: construction of ω doesn't require compactness of G and also works in complex category.

9 Yannick Voglaire, *Some aspects of non-formal deformation quantization II*

Now we will see how when we have a quantization of a Lie group, we can quantize the manifold itself. Recall the notion of WKB invariant quantization of (M, ω, ∇) .

$$u *_\nu v = \int_{M \times M} \underbrace{a_\nu e^{\frac{i}{\nu} \rho}}_{3\text{-pointfunction}} u \otimes v$$

- associative
- G -invariant, $G \subseteq \text{Aut}(M, \omega, \nabla)$
- stabilizes a fixed function space \mathcal{E}_ν

$$[u, v]_{*_\nu} := u *_\nu v - v *_\nu u = \frac{\nu}{i} \{u, v\} + o(\nu^2)$$

(G, ω) WKB quantization, $G \curvearrowright M \implies$ deformation of M .

Method to obtain invariant quantization of Lie groups:

$G \curvearrowright (M, \omega, \nabla)$ strongly hamiltonian action:

$$\exists \mathfrak{g} \rightarrow C^\infty(M) : X \mapsto \lambda_X, \quad \lambda_{[X, Y]} = \{\lambda_X, \lambda_Y\}$$

Definition 9.1. $*_\nu$ is covariant if

$$[\lambda_X, \lambda_Y]_{*_\nu} = \frac{\nu}{i} \{\lambda_X, \lambda_Y\}$$

From covariant to invariant:

Recall: X^\sharp such that $X_x^\sharp v = \frac{d}{dt} v(\exp(-tX)x)|_0$. Suppose $*_\nu$ is G -invariant. This means that representation

$$\begin{aligned} \mathfrak{g} \xrightarrow{\rho_\nu} C^\infty(M)[[\nu]] : \rho_\nu(X) &= \frac{i}{\nu} [\lambda_X, v]_{*_\nu} = \{\lambda_X, v\} + o(\nu^2) \\ &= X^\sharp v + o(\nu^2) \end{aligned}$$

Proposition 9.2. $\rho_\nu(X)$ is a derivation of $*_\nu$.

If we find $T_\nu : C^\infty(M)[[\nu]] \circlearrowleft$,

$$T_\nu^{-1} \circ \rho_\nu(X) \circ T_\nu = X^\sharp$$

then $*'_\nu$ defined by

$$u *_'_\nu v = T_\nu^{-1}(T_\nu u *_\nu T_\nu v)$$

and then X^\sharp will be derivations of this product.

Solvable Lie groups:

(R, ω) . We can hope to find $\varphi : R \rightarrow R$

$$\varphi^* \omega = \Omega$$

and such that Moyal product $*_\nu$ is $R0$ covariant. Suppose that we have T_ν as before. This implies we have an invariant associative product.

We can apply T_ν to Weyl product to find explicit formula for the kernel. We can take the image $\mathcal{E}_\nu := T_\nu^{-1}(S(R))$ as the function space.

So this is the general method and we shall now see an example.

Example 9.3. $G = SU(1, n)$. Take an Iwasawa decomposition ANK , $R = AN$.

Hermitian symmetric space non-compact type: G/K , G semisimple non-compact Lie group and K is maximal compact subgroup. Then G/K has a complex structure, action of G by holomorphic maps, and real rank of $G/K = \dim A$.

As G/K has an invariant symplectic form ω ,

$$R \xrightarrow{\cong} G/K$$

This gives a left-invariant symplectic form ω on R , $\omega = \varphi^* \omega$.

Proposition 9.4. $I : A \oplus N \rightarrow AN$ with $(a, n) \mapsto \exp(a) \exp(n)$ is a global Darboux chart.

Proposition 9.5. $\dim A = 1$

$$N = V \oplus RE \text{ Heisenberg algebra}$$

(V, Ω) symplectic vector space. We denote $A = RA$, $(a, v, z) = aA + v + zE$.

Here $T_\nu = F \circ \phi_\nu^* \circ F^{-1}$ and

$$(Fu)(a, v, \xi) = \int_{RE} e^{i\xi z} u(a, v, z) dz$$

$$\phi_\nu(a, \nu, \xi) = \left(a, \frac{1}{\cosh\left(\frac{\nu\xi}{2}\right)} \nu, \frac{1}{\nu} \sinh(\nu\xi) \right)$$

Then $K(p_0, p_1, p_2) = \cosh(2(a_1 - a_2)) \cosh(a_0 - a_2) \cosh(a_0 - a_1) \exp\left(\frac{i}{\nu} S_0(\cosh(a_1 - a_2)v_0, \cosh(a_2 - a_0)v_1, \cosh(a_0 - a_1)v_2) - \frac{i}{\nu} \bigoplus_{\text{Perm}(0,1,2)} \sinh(a_0 - a_1)z_2\right)$

Symplectic symmetric solvable Lie groups:

(M, ω, S) . $G =$ transvection group generated by $s_x \circ s_y \forall x, y \in M$. $\tilde{\sigma} = s_0 g s_0$ and for $(\mathfrak{g}, \sigma, \Omega)$ we have the following:

Theorem 9.6.

$$\{(M, \omega, s)\} \simeq \{(\mathfrak{g}, \sigma, \Omega)\}$$

By taking a central extension of \mathfrak{g} if necessary, we can assume that $(\mathfrak{g}, \sigma, \Omega)$ is an exact triple, i.e. $\Omega = \delta\xi$, $\mathfrak{g} = K \oplus P$ with K, P the ± 1 eigenspaces of σ , resp. and $[P, P] = K$ the adjoint action K on P is...

Suppose that (M, ω, s) , then $\mathfrak{g} = A \times_{\rho} B$, semidirect product of abelian Lie algebras with $\rho(A)B =: [A, B]$.

10 Zhuo Chen, *On transitive Lie bialgebroids and Poisson groupoids*

Lie bialgebroids are Lie algebroids $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho)$ whose dual \mathcal{A}^* also have algebroid structures, $(\mathcal{A}^*, [\cdot, \cdot]_*, \rho_*)$, such that

$$d_* : \Gamma(\Lambda^{\bullet} \mathcal{A}) \rightarrow \Gamma(\Lambda^{\bullet+1} \mathcal{A})$$

satisfies

$$d_*[X, Y]_{\mathcal{A}} = [d_*X, Y]_{\mathcal{A}} + (-1)^{1 \times 1+1}[X, d_*Y]_{\mathcal{A}}, \quad X, Y \in \Gamma(\Lambda^{\bullet} \mathcal{A})$$

We have that d_* corresponds to the Lie algebroid structure on \mathcal{A}^* with $\langle d_*f, \xi \rangle = \langle \rho_*(\xi), df \rangle$ and

$$\langle d_*X, \xi \wedge \eta \rangle = \rho_*(\xi)\langle X, \eta \rangle - \rho_*(\eta)\langle X, \xi \rangle - \langle X, [\xi, \eta]_* \rangle$$

So we denote a Lie bialgebroid by $(\mathcal{A}, \mathcal{A}^*)$ by:

$$(\mathcal{A}^*, d) \text{ or } (\mathcal{A}, d_*)$$

When \mathcal{A} is transitive, we say that (\mathcal{A}, d_*) is a transitive Lie bialgebroid.

The conclusion: If (\mathcal{A}, d_*) is a transitive Lie bialgebroid, then

$$d_* = [\Lambda, \cdot]_{\mathcal{A}} + \Omega$$

where $\Lambda \in \Gamma(\Lambda^2 \mathcal{A})$,

$$\Omega : \mathcal{A} \rightarrow L \wedge L \quad (L = \ker \rho)$$

is a 1-cocycle; i.e. $\Omega[X, Y]_{\mathcal{A}} = [\Omega(X), Y]_{\mathcal{A}} + [X, \Omega(Y)]_{\mathcal{A}}$

Ω should be regarded as $\Lambda^{\bullet} \mathcal{A} \rightarrow \Lambda^{\bullet+1} \mathcal{A}$. with

$$\Omega(f) = 0, \quad \Omega(X \wedge Y) = \Omega(X) \wedge Y - X \wedge \Omega(Y)$$

and (Λ, Ω) is compatible in the sense that:

$$\left[\frac{1}{2}[\Lambda, \Lambda]_{\mathcal{A}} + \Omega(\Lambda), \bullet \right]_{\mathcal{A}} + \Omega^2 = 0$$

as a map: $\Gamma\mathcal{A} \rightarrow \Gamma(\Lambda^3\mathcal{A})$.

The Converse: any pair (Λ, Ω) subject to this equation determines d_* and (\mathcal{A}, d_*) is a transitive Lie bialgebroid.

Uniqueness: if

$$d_* = [\Lambda, \cdot]_{\mathcal{A}} + \Omega = [\Lambda', \cdot]_{\mathcal{A}} + \Omega'$$

then $\Lambda' = \Lambda + \tau$, $\Omega' = \Omega - [\tau, \cdot]_L$ for some $\tau \in \Gamma(\Lambda^2 L)$.

Corollary 10.1. 1. $\dim(\mathcal{A}) = 1 \implies d_* = 0$

2. $\dim(L) = 1 \implies d_* = [\Lambda, \cdot]_{\mathcal{A}}$ (Λ is unique!)

3. $\rho_*(\xi) = \rho(\xi \lrcorner \Lambda)$ and $[\xi, \eta]_* = [\xi, \eta]_{\Lambda} + \Omega^*(\xi \wedge \eta)$.

4. If \mathcal{A} is not transitive, but $\text{im}(\rho_*) \subseteq \text{im}(\rho)$, then the conclusion is also true.

Coboundary (or exact) Lie bialgebroids are such $(\mathcal{A}, d_* = [\Lambda, \cdot]_{\mathcal{A}})$ for some $\Lambda \in \Gamma(\Lambda^2\mathcal{A})$.

Accordingly, (\mathcal{A}, d_*) is a coboundary if its corresponding (Λ, Ω) has

$$\Omega = [\tau, \cdot]_L \text{ for some } \tau \in \Gamma(L \wedge L)$$

(i.e., Ω is a coboundary)

10.1 Localization Theories

Let $F = L \wedge L$. \mathcal{A} is represented adjointly on F :

$$\mathcal{R} : \mathcal{A} \rightarrow CDO(F), \quad X \mapsto [X, \cdot]_{\mathcal{A}}, \quad \cdot \in \Gamma(L \wedge L)$$

The cohomology is such that

$$\partial^\bullet : \Gamma\text{Hom}(\Lambda^\bullet\mathcal{A}, F) \rightarrow \Gamma\text{Hom}(\Lambda^{\bullet+1}\mathcal{A}, F)$$

11 Kyosuke Uchino, *Courant brackets and omni-Lie algebras*

11.1 Courant Brackets

Definition 11.1. Let M be a smooth manifold. For any $(X, \alpha), (Y, \beta) \in \Gamma(\mathcal{T})$:

$$[(X, \alpha), (Y, \beta)] := ([X, Y], \mathcal{L}_X\beta - \iota_Y d\alpha)$$

The Courant bracket is used to define Dirac structures. Roughly speaking, Dirac structures are reductions of Poisson brackets. In other words, those are Poisson brackets on leaf spaces.

11.2 Omni-Lia algebras

Omni-Lie algebras are introduced by A. Wienstein.

Let V be a vector space. Set the following bracket on the space $gl(V) \oplus V$:

$$[(f, u), (g, v)] := ([f, g], f(v))$$

and set a nondegenerate symmetric...

Thus all Lie algebra structures on V are included in the omni-Lie algebra as substructures.

11.3 Main results

Idea: Let \mathcal{A} be an associative algebra and let $H^1(\mathcal{A}, \mathcal{A})$ (resp. $H_1(\mathcal{A}, \mathcal{A})$) is the 1st Hochschild cohomology (resp. homology) groups of the algebra.

If $\mathcal{A} = C^\infty(M)$, then

$$\Gamma(TM) = H^1(\mathcal{A}, \mathcal{A})$$

Basic Properties: Hochschild cochains are

$$C^n(\mathcal{A}, \mathcal{A}) := \text{Hom}_k(\mathcal{A}^{\otimes n}, \mathcal{A})$$

and the chains are

$$C_n(\mathcal{A}, \mathcal{A}) := \mathcal{A}^{\otimes n} \otimes \mathcal{A} = \mathcal{A}^{\otimes n+1}$$

- $H^1(\mathcal{A}, \mathcal{A}) = \text{Der}(\mathcal{A}) / \{ \text{inner-derivation} \}$

Algebraic derivations:

- $H^1(\mathcal{A}, \mathcal{A})$ is a Lie algebra
- For any $X \in H^1(\mathcal{A}, \mathcal{A}), \alpha \in H_n(\mathcal{A}, \mathcal{A})$: an algebraic Lie derivation $\mathcal{L}_X \alpha$ is well-defined, for example

Lemma 11.2 (Cartan's formulas). *Let $X, Y \in H^1(\mathcal{A}, \mathcal{A})$.*

Courant bracket on \mathcal{A} :

Definition 11.3 (Courant bracket on \mathcal{A}). For any $(X, \alpha), (Y, \beta) \in H^1(\mathcal{A}, \mathcal{A}) \oplus H_1(\mathcal{A}, \mathcal{A})$:

$$[(X, \alpha), (Y, \beta)] := ([X, Y], \mathcal{L}_X \beta - \iota_Y B \alpha)$$

Induced Courant bracket on \mathcal{A} :

The kernel of the bilinear form on $E(\mathcal{A})$:

$$J := \{e \in E(\mathcal{A}) \mid (e, \cdot) = 0\}$$

Lemma 11.4. *The kernel J is an ideal for the Courant bracket on \mathcal{A} .*

Proof. By an equivalence condition:

$$(e, [e', e''] + [e'', e']) =$$

□

Omin-Lie algebras vs. $\varepsilon(\mathcal{A})$:

Let V be a vector space over the field \mathbb{R} such that $\dim V \neq \infty$.

Theorem 11.5. *The omni-Lie algebra $gl(V) \oplus V$ is isomorphic to $\varepsilon(V[1])$, i.e.*

$$gl(V) \oplus V \cong \varepsilon(V[1])$$

Thus this relation induces an isomorphism:

$$vdv' \cong v \wedge v'$$

So we obtain

$$H_1(V[1], V[]) = \Omega_{V[1]\mathbb{R}}^1 \cong V \oplus \Lambda^2 V$$

Here we used an identification $dV \cong V$, $dv \cong v$. Thus

$$E(V[1]) \cong gl(V) \oplus V \oplus \Lambda^2 V$$

One can check that the kernel of the bilinear

12 Nan-Kuo Ho, *Yang-Mills connections on Nonorientable surfaces*

12.1 Introduction

$\Sigma \neq S^2$ compact closed surface. G compact connected Lie group. P principle G -bundle over Σ .

Consider:

- $\mathcal{A}(P)$ the space of all connections on $P \cong \Omega^1(\Sigma, P \times_G \mathfrak{g})$
- $\mathcal{G}(P)$ the group of gauge transformations

12.1.1 Symplectic [AB]

Indeed, it also inherits a complex structure from the Riemann surface:

12.1.2 Representation variety

For Σ orientable with genus $l > 0$,

$$\mathrm{Hom}(\pi_1(\Sigma), G)/G = \{(a_1, b_1, \dots, a_l, b_l) \in G^{2l} \mid \prod_{i=1}^l [a_i, b_i] = \mathrm{id}_G\}/G$$

For Σ nonorientable which is homeomorphic to the connected sum of m copies of $\mathbb{R}P^2$,

$$\mathrm{Hom}(\pi_1(\Sigma), G)/G = \{(c_1, \dots, c_m) \in G^m \mid \prod_{i=1}^m c_i^2 = \mathrm{id}_G\}/G$$

There is a one to one correspondence...

In other words,

$$\mathrm{Hom}(\pi_1(\Sigma), G)/G \cong \coprod_{[P]} \mathcal{M}_G(\Sigma)_P$$

The dimension of the moduli space \mathcal{M} :

$$\dim(\mathcal{M}_G(\Sigma_l)) = (2l - 2) \dim G + 2 \dim Z(G)$$

$$\dim(\mathcal{M}_G(\Sigma_m)) = (m - 2) \dim G + \dim Z(G)$$

Definition 12.1 (AMM). A quasi-Hamiltonian system is a triple (M, ω, μ) , consisting of a G -manifold M , an invariant 2-form ω ...

12.1.3 Yang-Mills flow (Morse theory) [AB][Da]

Fix P over Σ . The Yang-Mills functional on $\mathcal{A}(P)$ is defined by

$$L_P(\theta) =: \int_{\Sigma} \mathrm{tr}(F_{\theta} \wedge *F_{\theta})$$

Let

$$\mathcal{N}(P)_{\mu} =: \text{set of critical points of type } \mu$$

$$\mathcal{A}(P)_{\mu} =: \text{set of connections which flow to } \mathcal{N}(P)_{\mu} \text{ under the gradient flow of } L_P$$

Even though L_P is not a Morse-Bott function, $\{\mathcal{A}(P)_{\mu}\}$ still gives a stratification for $\mathcal{A}(P)$. Daskalopoulos proved ([AB]'s conjecture) that...

12.2 connectedness of the moduli space

Theorem 12.2 (AB). Σ closed, compact, orientable surface with $\mathcal{X}(\Sigma) < 0$. G is a compact connected Lie group. Then there is a bijection between $\pi_0(\mathcal{M}_G(\Sigma))$ (consider all P) and $\pi_1(G_{SS})$, where G_{SS} is the maximal connected semisimple subgroup of G .

Theorem 12.3 (Ho-Liu). Σ closed, compact, non-orientable surface with $\mathcal{X}(\Sigma) < 0$. G is a compact connected Lie group. Then there is a bijection between $\pi_0(\mathcal{M}_G(\Sigma))$ (consider all P) and $\pi_1(G)/2\pi_1(G)$.

Example 12.4.

$$G = SU(n), SP(n), \text{Spin}(n), SO(n), U(n)$$

$$\pi_1(G) = 0, 0, 0, \mathbb{Z}_2, \mathbb{Z}$$

Remark 12.5. Geometric interpretations, surfaces without boundaries.

12.3 Yang-Mills connections

Σ orientable:

Yang-Mills functional $L_P : \mathcal{A}(P) \rightarrow \mathbb{R}$:

$$L_P(\theta) = \int_{\Sigma} F_{\theta} \wedge *F_{\theta}$$

Yang-Mills connections: critical points of L_P .

Σ non-orientable:

consider the orientable double cover $\pi : \tilde{\Sigma} \rightarrow \Sigma$.

Proposition 12.6. The pull back principal G -bundle $\tilde{P} = \pi^*P \rightarrow \tilde{\Sigma}$ is topologically trivial

- $\pi^* : \mathcal{A}(P) \rightarrow \mathcal{A}(\tilde{P})$, $A \mapsto \pi^*A$ defines an inclusion, and

$$\pi^*(\mathcal{A}(P)) = \mathcal{A}(\tilde{P})^*$$

- $\mathcal{A}(\tilde{P})$ Kähler $\implies \mathcal{A}(P)$ totally geodesic, totally real, Lagrangian submanifold of $\mathcal{A}(\tilde{P})$.

Representation variety:

Fix a P on Σ . L_O is $\mathcal{G}(P)$ invariant.

Theorem 12.7 (AB). Σ orientable, $\mathcal{X}(\Sigma) < 0$,

$$\begin{aligned} \mathcal{M}_G^{YM}(\Sigma) &\cong \text{Hom}(\Gamma_{\mathbb{R}}(\Sigma_0^l), G)/G \\ &\cong \{(a_1, b_1, \dots, a_l, b_l, X) \in G^{2l} \times \mathfrak{g} \mid a_i, b_i \in G_i\} \end{aligned}$$

Idea: Holonomy along contractible loops on the double cover.

See the paper at math.SG/0605587

13 Laurent Classens, *Black Holes and group action deformations of Anti-de Sitter spaces*

M a manifold, and $(C^\infty, \cdot) \rightsquigarrow$ commutative and associative, $(A^M, *_M) \rightsquigarrow$ just associative. Consider

$$*_M : A^M \times A^M \rightarrow A^m$$

Then take a Lie group G , $(A^G, *_G)$ with $\tau : G \times M \rightarrow M$ and $\mu \in \text{Fun}(M)$. Take $x \in M$, then

$$(\alpha^x \mu)(g) = \mu(\tau_{g^{-1}} x)$$

Define

$$A^M = \{\mu \in \text{Fun}(M) \mid \forall x \in M, \alpha^x \mu \in A^G\}$$

and the product

$$(\mu *_M \nu)(x) = ((\alpha^x \mu) *_G (\alpha^x \nu))(e)$$

Now we will see how to define a black hole on an Anti-de Sitter space. Consider

$$\text{AdS}_l = \frac{SO(2, l-1)}{SO(1, l-1)}$$

which is $t^2 + \mu^2 - x^2 - y^2 - z^2 = 1$ in 3-dimensions. $S^2 = \frac{SO(3)}{SO(2)}$ and $SO(2, l-1) = ANK$ (abelian, nilpotent, compact). AN - maximal solvable group of $SO(2, l-1)$.

Now I will explain what a black hole structure is: $ls = \text{singularity} = \{\text{closed orbits of } AN \text{ and } \overline{AN}\}$. Then $\overline{AN} = \theta(AN)$. Then

$$BH = \{x \in \text{AdS} \mid l^k(x) \cap ls \neq \emptyset \forall \underbrace{\text{time-like } k}_{\|k\|^2 > 0} \in T_x \text{AdS}\}$$

One then shows that $S \subsetneq BH \subsetneq \text{AdS}$ (strict). The hope is that by just forgetting the definition and quantizing the de Sitter space, we can recover the structure in some way.

Now we'll talk a bit about the motivation of this definition. Actually, it was all pictures which won't be included here.

14 Alan Weinstein, *Group(oid)-like objects in Poisson geometry and algebra III*

Last time we talked about the quotient of a Lie group by any group can be treated with the notion of a stack. You can integrate any Lie algebroid to get a group-like object. The composition of paths is only associative after dividing out by an equivalence relation. This is a somewhat more general situation than our case here and the only reason we need stacks

is because it's a bad quotient space. See the preprint with Blohmann to really understand what's going on here.

Instead, let me present this as last time. We have the groupoid given by the action $\mathbb{Z} \times S^1 \rightrightarrows S^1$. Now this is our stack. Now take, $\mathbb{Z} \times \mathbb{Z} \times \mathbb{S}^1 \times S^1 \rightrightarrows S^1$. Call the bibundle between them

$$\tilde{\Delta} = (S^1 \times S^1) \times_{S^1//\mathbb{Z}} S^1 = \{(\theta_1, \theta_2, \theta_3) | \theta_1 + \theta_2 \equiv \theta_3 \pmod{2\pi\lambda\mathbb{Z}}\}$$

This is a multiple valued operation because you push it back down with all the twists by 2π .

What we want to look at today is the functions on this stacking group. When you have a group, there is an algebra of functions on the group, there are two ways of encoding them with the group structure. One is to use the group multiplication as a pull-back. If we're dealing with a finite group, we can identify the functions on \mathcal{G} tensored with itself. Consider them as a Banach or C^* -algebra. This is called a co-product. We can then go from $\mathcal{A} = \text{Fun}(\mathcal{G})$ to $\text{Fun}(\mathcal{G} \times \mathcal{G}) \approx \text{Fun}(\mathcal{G}) \otimes \text{Fun}(\mathcal{G})$ via Δ , the co-product. We then get a bi-algebra and have the following diagram:

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} & \xleftarrow{\Delta} & \mathcal{A} \\ \Delta \otimes \text{id} \downarrow & & \downarrow \Delta \\ \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} & \xleftarrow{\text{id} \otimes \Delta} & \mathcal{A} \otimes \mathcal{A} \end{array}$$

For the purpose of this talk, all the function will be \mathbb{C} -valued, though any ring works. Then the map $\varepsilon : \mathcal{A} \rightarrow \mathbb{C}$ is called the co-unit and $S : \mathcal{A} \rightarrow \mathcal{A}$ the antipode. Reversing all the diagrams and arrows, we get the axioms for a *Hopf* algebra. For the antipode, in a group, we replicate the group element together with the inverse and multiply it, i.e.

$$\begin{array}{ccccc} g & \longmapsto & (g, g) & \longmapsto & (g, g^{-1}) \\ \downarrow & & & & \downarrow \\ \text{pt} & \xrightarrow{\text{id}} & & \xrightarrow{\text{id}} & e \end{array}$$

Then S goes from the opposite algebra \mathcal{A}^{op} to \mathcal{A} . Then only way to make this an algebra homomorphism is to make this a set of vector spaces over \mathbb{C} and then we get a commutative diagram.

$$\begin{array}{ccccc} \mathcal{A} & \xleftarrow{m} & \mathcal{A} \otimes \mathcal{A} & \xleftarrow{\text{id} \otimes S} & \mathcal{A} \otimes \mathcal{A} \\ \uparrow \text{id} & & & & \uparrow \Delta \\ \mathbb{C} & \xleftarrow{\varepsilon} & & \xleftarrow{\varepsilon} & \mathcal{A} \end{array}$$

This problem will come back and haunt us later.

The second way of encoding the functions is by group convolution, i.e. take two functions on the group and getting $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ by

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1}g) dh$$

This operation is a little more problematic because we might not have a measure on the group. However, you can take densities instead of functions. This concept can be extended to groupoids and then you get a convolution on a groupoid. This operation is actually dual to the co-product under the circumstances you have a duality between \mathcal{A} and itself, for example in finite groups.

Now non-commutative geometry says that when you form a quotient space, instead of this algebra, $\text{Fun}(S^1/\mathbb{Z}) = \text{Fun}(S^1)^{\mathbb{Z}}$, one should take $\text{Fun}(S^1//\mathbb{Z}) = \text{Fun}(S^1) \rtimes \mathbb{Z}$ and this reflects the way \mathbb{Z} acts on S^1 , and thus the functions on S^1 . So what should the functions on this space look like? We have the spanning functions $e_1 = \exp(i\theta)$, $e_2 \mathcal{X}_1$. Then any function can be written as $\sum a_{pq} e_1^p e_2^q$. Then $e_2 e_1 = e^{2\pi i \lambda} e_1 e_2$ and we have a dependence on λ . We can think of these as functions on a 2-torus, $\mathcal{A}_0 \simeq \text{Fun}(\mathbb{T}^2)$, and $\mathcal{A}_\lambda = \text{Fun}(\mathbb{T}_\lambda^2)$, i.e. the non-commutative torus. One wonders whether \mathcal{A}_λ is a Hopf algebra itself, but the answer quickly is seen to be no. So there's no way to deform the coproduct from \mathcal{A}_0 to the one on \mathcal{A}_λ . However, this represents the functions on a stacking group, so one might be able to use this structure to put something like a Hopf algebra on $\text{Fun}(S^1) \rtimes \mathbb{Z}$. Now we get to the idea of a *Hopfish* algebra.

Then the idea of a bibundle turns out to be a bimodule in this scenario, i.e. ${}_A X_B$. We can compose bimodules as one would think

$$\text{Bim}(A, B) \times \text{Bim}(B, C) \rightarrow \text{Bim}(A, C), \quad {}_A X_B \cdot {}_B Y_C \stackrel{\text{def}}{=} {}_A (X \otimes_B Y)_C$$

In groupoids, we use the fact that the bibundle is principle on one side, but we don't seem to need that here. It's not clear what this corresponds to in the algebraic setting. Before we had $S^1//\mathbb{Z} \times S^1//\mathbb{Z} \xrightarrow{\tilde{\Delta}} S^1//\mathbb{Z}$, now we have $\mathcal{A}_\lambda \otimes \mathcal{A}_\lambda \xleftarrow{\tilde{\Delta}} \mathcal{A}_\lambda$. We get what we call a *sesquialgebra*.

Now we arrive at the idea of deciding what the antipode looks like. It took a while to figure out what the axiom should actually be. In general, look at a *sesquiunital* bialgebra:

So we look at the right $\mathcal{A} \otimes \mathcal{A}$ module $\text{Hom}_{\mathcal{A}}(\tilde{\varepsilon}, \tilde{\Delta})$. A preantipode is a left $\mathcal{A} \otimes \mathcal{A}$ module \tilde{S} with a nondegenerate pairing

$$\text{Hom}_{\mathcal{A}}(\varepsilon, \Delta) \times \tilde{S} \rightarrow \mathbb{C}$$

which is $\mathcal{A} \otimes \mathcal{A}$ linear. This is strong if $\tilde{S}^* \simeq \text{Hom}_{\mathcal{A}}(\tilde{\varepsilon}, \tilde{\Delta})$. Then \tilde{S} is an $(\mathcal{A}, \mathcal{A}_{\text{op}})$ bimodule.

Definition 14.1. \tilde{S} is an *antipode* if $\tilde{S} \sim \mathcal{A}$ as a left \mathcal{A} module.

This is equivalent to saying $\overset{S}{\sim} = S_{\mathcal{A}}$. Rather than saying S is a homomorphism, we say there is an object that is defined by Δ , etc. then this antipode should have this nice property. We call a sesquiunital algebra with an antipode (which is strong) a Hopfish algebra.

Corollary 14.2. *A Hopf algebra is a hopfish algebra.*

This is a good definition because it includes quasi-Hopf algebras. What was before a weaker form of co-associativity becomes exactly co-associativity. We also looked at weakening the strong condition on the antipode.

Now I'll quickly summarize the rest of the paper. One can take the idea of a Hopfish algebra and make into a tensor category. Somehow, hidden in the tensor product, one can recover the structure of the quotient S^1/\mathbb{Z} , which parallel what one does in Hopf algebras. The last remark is that we think that some of the problems involving non-degeneracy may be resolved in passing from these finite series to something like a C^* -algebra, which involves developing a Hopfish C^* -algebra, which is something we're about to start doing. We hope this is the beginning of the study of a new kind of object.

15 Nguyen Tien Zung, *Normal forms of Poisson structures I*

Questions: How do Poisson structures look like (locally)? How can we construct/classify them?

Recall that $(\pi\text{-Poisson structures on a manifold } P) \rightsquigarrow (\pi\text{-2-vector field such that } [\pi, \pi] = 0) \rightsquigarrow (\text{Poisson bracket } \{f, g\}_{\pi} := \langle df \wedge dg, \pi \rangle \text{ satisfies the Jacobi identity}) \rightsquigarrow (\exists \text{ Poisson submersion } (M, \omega) \rightarrow (P, \pi), \text{ quotient of symplectic manifold}) \rightsquigarrow (T^*M \xrightarrow{\#} TM, \langle \# \alpha, \beta \rangle := \langle \pi, \alpha \wedge \beta \rangle, \text{ a Lie algebroid, (singular) foliation on } M \text{ is a symplectic foliation } \omega = \pi^{-1}.)$

(All these points of view help understand the local structure).

15.1 Darboux theorem and generalization to Poisson case (Weinstein's Splitting theorem)

- (M^{2n}, ω) -symplectic manifold, $p \in M^{2n}$. Then \exists local coordinates $x_1, y_1, \dots, x_n, y_n$ in which

$$\omega = \sum dx_i \wedge dy_i$$

$\pi = \omega^{-1}$ nondegenerate Poisson.

- π_1 depends only on z_1, \dots, z_{n-2} and is a Poisson structure in a manifold of dimension $n - 2$.

$$\text{rank} \pi_1(p) = \text{rank} \pi(p) - 2$$

If $\text{rank}\pi_1(p) > 0$ then repeat the above process to find x_2, y_2 and so on. Then $\exists x_1, y_1, \dots, x_k, y_k, z_1, \dots, z_{n-2k}$ such that

$$\pi = \sum_{i=1}^k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \tilde{\pi}$$

Remark 15.1. – $\tilde{\pi}$ is unique (up to local isomorphisms) and is

- Equivariant version of splitting theorem?
- Equivariant Darboux theorem: $G \curvearrowright (M, \omega)$, the action fixes p . Then \exists Darboux coordinates $(x_1, y_1, \dots, x_m, y_m)$, in which the action of G is linear.

Proof. Coordinate by coordinate does not work, instead use Moser’s path method \square

- Poisson case: $G \curvearrowright (P, \pi)$, fixes p , does there exist coordinates (x_1, \dots, z_{n-2k}) in which

$$\pi = \sum \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \tilde{\pi}$$

and the action is linear?

Yes, if at least 1 of the following is satisfied:

- \exists equivariant momentum map (the action of G is Hamiltonian), (work with Monnier and Miranda, C^∞ , used Naoh-Moser)
- π satisfies an additional condition of cohomological type called “tameness” condition. (see Miranda and Z, `math.SG/0510523`, uses Moser’s path method)

Transverse Poisson structure:

N cuts symplectic leaves transversally. Intersections are symplectic submanifolds, where $N = \{x_i = y_i = 0\}$. On N we have the transverse poisson structure $\tilde{\pi}$.

Uniqueness (up to local isomorphism):

$H_q \subset T_q O(q)$ is the symplectic complement to $T_q N \cap T_q O(q)$.

We have a local fibration that intersects the symplectic leaves transversally. H_q is the symplectic complement to this leaf at q . Doing this for every point q you get a regular horizontal distribution giving you an Ehresmann connection ∇ for $(q \mapsto H_q)$. Then the Poisson structure is just

$$\pi = \pi_{\text{vert}} + \pi_{\text{hor}}$$

Then π_{hor} is non-degenerate in the horizontal direction giving a dual 2-form \mathbb{F} under the inverse where $\mathbb{F} \in \Omega^2(O(p)) \otimes C^\infty(P)$. The values depend on each point q however.

$$\pi \text{ is Poisson} \iff [\pi_{\text{vert}} + \pi_{\text{hor}}, \pi_{\text{vert}} + \pi_{\text{hor}}] = 0$$

\iff

1. v-v-v part =0, $\iff [\pi_{\text{vert}}, \pi_{\text{vert}}] = 0$, (π_{vert} is Poisson)
2. v-v-h part =0 $\iff \mathcal{L}_{\nabla} \pi_{\text{vert}} = 0$ (the connection preserves π_{vert}), in particular, π_{vert} is unique up to local isomorphism
3. v-h-h part =0 $\iff \partial_{\nabla} \mathbb{F} = 0$
4. h-h-h part =0 $\iff \text{Curvature}_{\nabla}(u, v) = \sharp_{\text{vert}}(d(\mathbb{F}(u, v)))$

Then we have the triple $(\pi_{\text{vert}}, \nabla, \mathbb{F})$ which allows us to construct a Poisson structure

Theorem 15.2 (Vorobjen). *This triple gives a Poisson structure*

Remark 15.3. Similar result holds for Dirac manifolds and we have

Poisson str. \leftrightarrow foliation by symplectic leaves

Dirac str. \leftrightarrow foliation by pre-symplectic leaves

Theorem 15.4 (Dreyour-Wade).

15.2 Normal Forms of Poisson structures which vanish at a point

π on P , $0 \in P$, $\pi(0) = 0$. Taylor expansion (in a local coordinate system):

$$\pi = \pi^k + \pi^{k+1} + \dots, \quad \pi^k \neq 0 \text{-homogeneous part of } \pi$$

(π^* does not depend on the choice of local coordinates)

$$0 = [\pi, \pi] = \underbrace{[\pi^k, \pi^k]}_{=0} + 2[\pi^k, \pi^{k+1}] + 2[\pi^k, \pi^{k+1}] + 2[\pi^k, \pi^{k+2}] \dots$$

i.e. π^k is a homogeneous polynomial Poisson structure of degree k .
 π may be viewed as a small deformation of π^k .

Questions:

1. Is π isomorphic to π^k ? (homogenization problem)
2. ...

15.3 (Formal) deformation and Poisson cohomology

π -Poisson tensor on P

$$\dots \rightarrow \gamma^{i-1}(P) \xrightarrow{d_\pi} \gamma^i(P) \rightarrow \gamma^{i+1}(P) \rightarrow \dots$$

$d_\pi \Lambda := [\pi, \Lambda]$, with $\gamma^i = \{i\text{-vector fields}\}$.

$$[\pi, \pi] = 0 \implies d_\pi \cdot d_\pi = 0 \text{ (differential complex)}$$

The cohomology of this complex is called Poisson cohomology (Lichnerowicz 1977?)

Interpretation of H^1, H^2, H^3 :

- $H_\pi^0(P) = \{F \in \text{Fun}(P) | X_F = 0\} = \{\text{Casimir function}\}$
- ...

$H_\pi^2(P)$ and deformations of π : $\pi + \varepsilon\pi'$ -Poisson deformation of π . Then

$$0 = [\pi + \varepsilon\pi', \pi + \varepsilon\pi'] = \underbrace{[\pi, \pi]}_{=0} + 2\varepsilon \underbrace{[\pi, \pi']}_{=0} + o(\varepsilon^2)$$

$\implies [\pi, \pi'] = 0$, i.e. π' is a 2-cocycle in the Lichnerowicz-Poisson complex.

If π' is a 2-cocycle, i.e. $\pi' = [\pi, Y]$, Y -vector field, then $\pi \dots$

15.4 Linearization Problem

$$\pi = \pi^1 + \pi^2 + \pi^3 + \dots, \quad \pi^1 \neq 0$$

Question: Does there exist a diffeomorphism Φ (formal, smooth, analytic,...) such that

$$\Phi_* \pi = \pi_1$$

General idea is that the relevant cohomology group is $H_{\pi_1}^2$.

Recall that Linear Poisson structures correspond to Lie algebras (finite-dimensional).

Fact: Lichnerowicz-Poisson complex for a linear Poisson structure π_1 on V can be restricted to homogeneous multi-vector fields of a given degree on V :

$$\dots \rightarrow \gamma_k^{i-1} \rightarrow \gamma_k^i \xrightarrow{d_{\pi_1}} \gamma_k^{i+1} \rightarrow \dots$$

with

$$\gamma_k^i = \left\{ \sum f_{i_1, \dots, i_k} \right\}$$

Theorem 15.5 (Crainic-Fernandez). *Assume $\pi = \pi_1 + \dots$ with $H^2(\mathfrak{g}, \mathbb{R}) = 0$. Then $\forall C^1$ -small deformation π' of π admits a point p' near 0, in which $\pi'(p') = 0$.*

16 Nguyen Tien Zung, *Normal forms of Poisson structures II*

Lemma 16.1. *Assume $\pi = \pi_1 + \pi_k + \pi_{k+1} + \dots$ ($k \geq 2$). If $H^2(\mathfrak{g}, S^k \mathfrak{g}) = 0$, then \exists diffeomorphism Φ such that $\Phi_* \pi = \pi_1 + \tilde{\pi}_{k+1} + \dots$ (kills π_k).*

Theorem 16.2 (Weinstein). *If $H^2(\mathfrak{g}, S^k \mathfrak{g}) = 0, \forall k \geq 2$, then \forall Poisson structures $\pi = \pi_1 + \dots$ is formally isomorphic to π_1 . In other words, \forall (formal) Poisson structures whose linear part corresponds to \mathfrak{g} is formally linearizable.*

(\mathfrak{g} is called *formally nondegenerate* in this case).

Example 16.3. \mathfrak{g} semisimple, then $H^2(\mathfrak{g}, W) = 0$, for any finite-dimensional module $W \implies \forall$ Poisson structures with a semisimple linear part is formally linearizable.

Remark 16.4 (Dufour). \exists many non-semisimple Lie algebras which are non-degenerate.

Example 16.5. $\mathfrak{g} = \mathbb{R} \ltimes_A \mathbb{R}^2$ a solvable Lie algebra, A - 2×2 matrix. For a generic A , this algebra is non-degenerate.

Idea: Once formal linearizability is OK, use K.A.M. theory, Naoh-Moser,... to achieve analytic/smooth linearizability (or more generally, analytic/smooth normalization).

Theorem 16.6 (Conn). *If \mathfrak{g} is semisimple, then any analytic Poisson structure whose linear part corresponds to \mathfrak{g} is analytically linearizable.*

Theorem 16.7 (Conn). *If \mathfrak{g} is semisimple, \mathfrak{g} is of compact type, then \forall smooth (C^∞) Poisson structures whose linear part corresponds to \mathfrak{g} is smoothly linearizable.*

Remark 16.8. The key word here is “compact”. If non-compact then not true in general (Weinstein: it is false if real rank of $\mathfrak{g} \geq 2$)

Question: What if \mathfrak{g} is not semisimple?

Low-dimensional case:

- $\dim \mathfrak{g} = 2, \mathfrak{g} = \mathbb{K} \ltimes \mathbb{K} = \text{Aff}(1)$, affine transformation of a line
Arnold: $\text{Aff}(1)$ is non-degenerate.
- $\dim \mathfrak{g} = 3$

16.1 Levi Decomposition and Linearization

Idea: Extend the Levi-Malcev decomposition

$$\mathfrak{g} = S \ltimes r$$

to ∞ -dimensional case. (Poisson algebras). Note S is semisimple and r radical.

Theorem 16.9. Assume $\mathcal{L} = \mathcal{L}_0 \supset \mathcal{L}_1 \supset \mathcal{L}_2 \supset \dots$ pro-finite Lie algebra. \mathcal{L}_i -Lie subalgebra, $\dim \mathcal{L}_i/\mathcal{L}_{i+1} < \infty$.

$$[\mathcal{L}_i, \mathcal{L}_j] \subset \mathcal{L}_{i+j}$$

Then \mathcal{L} admits a formal Levi decomposition.

$$\hat{\mathcal{L}} = \lim_{\leftarrow} \mathcal{L}/\mathcal{L}_i = S \ltimes \hat{R}$$

($\mathcal{L}/\mathcal{L}_2 = S \ltimes r, \hat{R} \dots$). Also, uniqueness of the Levi factor up to conjugations.

Let's say (x_1, \dots, x_s) -a basis of S . They are formal functions on V . Suppose $\mathfrak{g} = \mathcal{L}/\mathcal{L}_1 =$ the Lie algebra of the linear part of π .

- Case $s = \dim V$:
- General case: $\mathfrak{g} \cong \mathcal{L}/\mathcal{L}_1 = S \ltimes r$. $\hat{\mathcal{L}}$ admits a semisimple subalgebra....

$$p \mapsto (x_1(p), \dots, x_s(p))$$

momentum map of a formal action of S on (V, π) . (gives a semi-simple group S of formal ...)

Can complete (x_1, \dots, x_s) into a formal coordinate system (x_i, y_i) such that

$$\{x_i, x_j\} = \sum c_{ij}^k x_k, \{x_i, y_j\} = \sum b_{ij}^j y_j, \{y_i, y_j\} = ?, \text{ don't know}$$

- Smooth/analytic version: Extension of Conn results works for Levi decomposition of Poisson structures
- Application to algebroids: Similar results. In particular, Levi decomposition yields the *linearization theorem* for Lie algebras with a semisimple isotopy algebra.
- Application to non-semisimple Lie algebras:

Theorem 16.10 (Dufour-Z). *The algebras $\text{Aff}(u) = \mathfrak{gl}(u) \ltimes \mathbb{K}^n$ are non-degenerate.*

Proof. Levi decomposition plus some more work... □

Remark 16.11 (Bordemann, Petit, Makhlouf). $\text{Aff}(u)$ is strongly rigid (the enveloping algebra is rigid).

Remark 16.12. $\text{Aff}(u)$ is Frobenius (admits open symplectic leaf)

Question: Are all Frobenius algebras non-degenerate?

Answer: No, \exists degenerate Frobenius algebras.

Conjecture: The space of deformations is finite-dimensional for every Frobenius Lie algebra.

16.2 Various Problems

- Finite determinacy

$$\pi = \pi_1 + \cdots + \pi_k + \cdots$$

enough to know $\mathcal{J}^k(\pi)$ in order to know π up to isomorphism?

- Quadratic Poisson/quadratization? higher degree?
- Stability of higher-order singular points?
- Polynomial Poisson structures? We don't know much about this.
- Global properties of transverse Poisson structures?
- Normal Forms for compatible Poisson structures?
- Normal Forms for similar structures (e.g. Dirac, Jacobi, etc.)? From the point of Dynamical systems, after some reduction on a symplectic manifold, one ends up on a Poisson manifold and here Normal forms could be interesting.
- Normal Forms of vector fields on Poisson manifolds?

17 Kyonori Gomi, *Twisted K-theory and finite-dimensional approximation*

Problem in twisted K-theory: Realize twisted K-theory generally by means of finite dimensional geometric objects.

Theorem 17.1 (Main Result).

17.1 Twisted K-theory

Origins: P. Donovan and M. Karoubi

Applications: D-brane charges

17.1.1 Fredholm operators

\mathcal{H} : separable Hilbert space ($\dim \mathcal{H} = \infty$)

Definition 17.2. A *Fredholm operator* $f : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded operator

Example 17.3. $H^3(X; \mathbb{Z}) \cong \mathbb{Z}$,

- $X = S^3$

$$K(S^3; k) \cong 0$$

- $X = S^1 \times S^2$

$$K(S^1 \times S^2; k) \cong \mathbb{Z}$$

- $X = SU(3)$

17.1.2 Twisted vector bundle

- $\mathcal{U} = \{U_\alpha\}$: open cover of X

Remark 17.4. $(E_\alpha, \phi_{\alpha\beta} : \text{rank } r \implies r \cdot \delta(P) = 0.$

$$(\det \phi_{\alpha\beta})(\det \phi_{\beta\gamma}) = \dots$$

17.2 Hermitian general vector bundle

M.Furata, “Index Theory”.

Theorem 17.5 (Furata). *X compact. We can define a group by means of \mathbb{Z}_2 -graded Hermitian general vector bundles, which is isomorphic to $K(X)$.*

1. “ $h_\alpha \phi_{\alpha\beta} = \phi_{\alpha\beta} h_\beta$ ”
2. “ $\phi_{\alpha\beta} \phi_{\beta\alpha} = 1$ ”
3. “ $\phi_{\alpha\beta} \phi_{\beta\gamma} = \phi_{\alpha\gamma}$ ”

Theorem 17.6 (Main result). *X compact, P a $4+PU(\mathcal{H})$ -bundle. We can define a group by means of twisted \mathbb{Z}_2 -graded Hermitian general vector bundles, into which there exists a monomorphism*

18 Maxim Kontsevich, *Quantization and positive characteristic*

Joint work with Alekseev Belov, math/0512169, 0512171. Second paper proved by Y.Tsuchimoto a few years ago.

First, fix an integer $n \geq 1$, and define two algebras, $A_{n,\mathbb{C}} = \mathbb{C}\langle \hat{x}_1, \dots, \hat{x}_{2n} \rangle / [\hat{x}_i, \hat{x}_j] = \omega_{ij} \cdot 1 \simeq D(\mathring{A}_{\mathbb{C}}^n)$ an associative algebra. We denote $\hat{x}_i = x_i$ and $\hat{x}_{n+i} = \frac{\partial}{\partial x_i}$. $P_{n,\mathbb{C}} = \mathbb{C}[x_1, \dots, x_n]$ with Poisson bracket $\{x_i, x_j\} = \omega_{ij} \cdot 1$.

Conjecture 18.1. $\text{Aut}(A_{n,\mathbb{C}}) \simeq \text{Aut}(P_{n,\mathbb{C}})$. $\text{Aut}_{\text{field}}(\mathbb{C}) \rightarrow \text{Aut}_{\text{group}}(\text{Aut}(A_{n,\mathbb{C}}))$. $\text{Aut}(\mathbb{C})$ -equivalent means we can replace \mathbb{C} with an field k of characteristic 0.

Both groups are “ ∞ ”-dimensional algebraic groups.

$$\text{Lie}(\text{Aut}(A_{n,\mathbb{C}})) \not\cong \text{Lie}(\text{Aut}(P_{n,\mathbb{C}}))$$

We have that $\text{Sp}(2n; \mathbb{C})$ is contained in the automorphism group of both algebras. For (g_{ij}) in $\text{Sp}(2n; \mathbb{C})$, we have the transformations $\hat{x}_i \rightarrow \sum g_{ij} \hat{x}_j$ and $x_i \rightarrow \sum g_{ij} x_j$. Transvections $F \in \mathbb{C}[x_1, \dots, x_n]/\mathbb{C}^1$ and $T_F^A \in \text{Aut}(A_{n,\mathbb{C}})$, $T_F^P \in \text{Aut}(P_{n,\mathbb{C}})$ with the maps $\hat{x}_i \rightarrow \hat{x}_i$ for $i = 1 \dots n$ and $\hat{x}_{i+n} \rightarrow \hat{x}_{i+n} + \partial_i F(\hat{x}_1, \dots, \hat{x}_n)$.

Theorem 18.2. $\ker \rho = \ker \phi$ where

$$\rho : \text{Sp}(2n, \mathbb{C}) *_{\text{freeproduct}} (\mathbb{C}[x_1, \dots, x_n]/\mathbb{C} \cdot 1) \rightarrow \text{Aut}(A_{n,\mathbb{C}})$$

and

$$\phi : \text{Sp}(2n, \mathbb{C}) *_{\text{freeproduct}} (\mathbb{C}[x_1, \dots, x_n]/\mathbb{C} \cdot 1) \rightarrow \text{Aut}(P_{n,\mathbb{C}})$$

The plan is as follows:

1. Finite characteristic method and indications
2. Trigonometric and elliptic
3. Arithmetic support of holonomic D-modules

So for the conjecture, we need morphisms in both directions:

(\rightarrow): $\varphi : A_{n,\mathbb{C}} \circlearrowleft$,

$$\varphi(\hat{x}_i) = \sum_{I=(i_1, \dots, i_{2n}) \geq 0, |I| \leq \text{const}} c_{i,I} \hat{x}_1^{i_1} \cdots \hat{x}_{2n}^{i_{2n}}$$

with $\varphi^{-1}(\hat{x}_i) = \cdots \tilde{c}_{i,I}$. Then $R'_{(\varphi)} :=$ subring of \mathbb{C} generated by $c_{i,I}, \tilde{c}_{i,I}$, finitely generated integral domain. Invert a finite number of non-zero elements to get $R_\varphi \subset \mathbb{C}$, with R_φ/\mathbb{Z} smooth. φ can be considered as R_φ -linear automorphisms of A_{n,R_φ} .

For $R = R_\varphi$, $A_{n,R} \otimes \mathbb{Z}/p\mathbb{Z} = A_{n,R/pR}$ has a big center, with p a prime.

$$\text{Center} = R/pR[\hat{x}_1^p, \dots, \hat{x}_{2n}^p]$$

$A_{n,R/pR}$ is free module over Center of rank $= p^{2n}$. Azumaya algebra, form of $M(p^n \times p^n)$. Add central variables $y_i^p = \hat{x}_i^p$.

$$A \otimes_{\text{center}} R/pR[y_i] = R/pR[y_i] \langle \hat{x}_i \rangle / \hat{x}_i^p = y_i^D$$

, with $[\hat{x}_i, \hat{x}_j] = \omega_{ij}$ Then $[A, B] = 0, p = 0$, gives $(A + B)^p = A^p + B^p$
 φ preserves the center mod p so we get a polynomial automorphism

$$\mathring{A}_{R/pR}^{2n} \circlearrowleft$$

Lemma 18.3. For $p \gg 1$, this map above preserves a Poisson bracket.

Looking at R/p^2R , we can choose $a, b \in Z(R/pR)$ and choose a lift for $a', b' \in R/p^2R$.

Example 18.4. 1. $\hat{x}_i \rightarrow \sum g_{ij} \hat{x}_j$, $(g_{ij}) \in \text{Sp}(2n, R)$, $\hat{x}_i^p \rightarrow \sum g_{ij}^p \hat{x}_j^p \pmod p$.

2. $\hat{x}_i \rightarrow \hat{x}_i$ for $i = 1 \dots n$, and $\hat{x}_{i+n} \rightarrow \hat{x}_{i+n} + \partial_i F(\hat{x}_1, \dots, \hat{x}_n)$.

Continuing on,

$$\varphi(\hat{x}_i^p) = \sum_{|I| \leq \text{same const}} c_{i,I}^{(p)} \hat{x}_1^{p_{i_1}} \dots \hat{x}_{2n}^{p_{i_{2n}}}, \pmod p$$

This indicates that $c_{i,I}^{(p)} \in (R/pR)^p$.

Lemma 18.5. The above inclusion is true for $p \gg 1$.

Proof. R/pR is smooth/ $\mathbb{Z}/p\mathbb{Z}$, $a \in R/pR$ belongs to $(R/pR)^p \iff da = 0 \iff \delta_i a = 0 \pmod p$, where $\delta_1, \dots, \delta_N \in \text{Der}(R/\mathbb{Z})$ span the tangent space at every point of $\text{Spec}(R)$. This means we should prove that $\delta_j(\varphi(\hat{x}_i^p)) = 0$.

Let's denote $\alpha := \varphi(\hat{x}_i)$ and $\beta := \delta_j(\alpha)$. In $A_{n,R}$, \hat{x}_i is locally ad-nilpotent, i.e. $\forall y \in A_{1,R}$, $(\text{ad}\hat{x}_i)^M(y) = 0$ for $M \gg 1$. This gives that $\varphi(\hat{x}_i)$ is locally ad-nilpotent.

$$(\text{ad}\alpha)^M(\beta) = 0 \rightarrow (\text{ad}\alpha)^{p-1}(\beta) = 0$$

But this turns out to be

$$= [\alpha, [\alpha, \dots, [\alpha, \beta]] \dots] = \alpha^{p-1} \beta - \binom{p-1}{1} \alpha^{p-2} \beta \alpha + \dots + \beta \alpha^{p-1}, \pmod p, \quad p \gg M$$

Then

$$\varphi^{(p)}(x_i) := \sum_{|I|} \sqrt[p]{c_{i,I}^{(p)}} x_1^{i_1} \dots x_{2n}^{i_{2n}} \in R/pR$$

□

Conjecture 18.6. $\exists c_{i,I}^{\text{Poisson}} \in R \otimes \mathbb{Q}$ such that $\forall p \gg 1$,

$$\sqrt[p]{c_{i,I}^{(p)}} = c_{i,I}^{\text{Poisson}} \pmod pR$$

(\leftarrow): We know a priori how φ^{assoc} acts on the Centers.

In our case, the classification of the algebra in the Brouer group of $R/pR[y_1, \dots, y_{2n}]$ comes from the 1-form

$$\gamma = \sum_{i=1}^n y_{i+n} dy_i, \quad y_i = \hat{x}_i^p$$

with $d\gamma = \sum w_{ij} dy_i dy_j$.

Multiplicative version: $2n = 2$ variables. Then $\{x, y\} = xy$ and the 2-form is $\frac{dx}{x} \wedge \frac{dy}{y}$. Automorphisms of $(\mathbb{C}^\times)^2$ preserving $\{\cdot, \cdot\}$ give the following transformations:

$$S, T : (x, y) \rightarrow (y^{-1}, x), (xy, y)$$

and

$$M_{\lambda, \mu} : (x, y) \rightarrow (\lambda x, \mu y)$$

for $\lambda, \mu \in \mathbb{C}^\times$, and relations $S^4 = 1$, $S^2 = (ST)^3$, $M_{\lambda_1, \mu_1} M_{\lambda_2, \mu_2} = M_{\lambda_1 \lambda_2, \mu_1 \mu_2}$, $SM_{\lambda, \mu} S^{-1} = M_{\lambda^{-1}, \mu}$, $TM_{\lambda, \mu} T^{-1} = M_{\lambda, \mu}$.

If we consider the Birational automorphisms, then I should add an addition transformation

$$U : (x, y) \rightarrow (x(1+y)^{-1}, y)$$

with additional relations $UT = TU$, $US^2US^2 = 1$, $(S^{-1}U)^5 = 1$, $UM_{\lambda, 1}U^{-1} = M_{\lambda, 1}$, and $UM_{1, \mu}UM_{1, \mu^{-1}} = M_{1, \mu}UM_{1, \mu^{-1}}U$.

Conjecture 18.7. *This is a preservation of Birational Automorphisms of $(\mathbb{C}^\times)^2$ preserving $\{\cdot, \cdot\}$.*

Conjecture 18.8 (vague).

$$\omega_{ij} = -\omega_{ji} \in \mathbb{Z}, \quad \text{e.g. standard}$$

Then do Birational Automorphisms of $(\mathbb{C}^\times)^{2n}$ which preserve the bracket correspond to Non-commutative Birational Automorphisms of $\hat{x}_i \hat{x}_j = q^{\omega_{ij}} \hat{x}_j \hat{x}_i$ field of rational functions $\mathbb{C}(q)$. q is some primitive root of unity.

Now to move to a more general situation. Suppose X/\mathbb{C} smooth affine algebraic variety. Then we have the algebra over \mathbb{C} of differential operators on X , D_X , and M a D_X -module which are holonomic. This corresponds to a characteristic co-cycle $\text{Ch}(M) = \sum_\alpha n_\alpha [L_\alpha]$, $n_\alpha \in \mathbb{Z}$. Then $L_\alpha \subset T^*X$ is irreducible. L_α is the conormal bundle.

Now take $R \subset \mathbb{C}$ a finitely generated ring. X, M are defined over R .

$$\text{Spec}(R/p) \xrightarrow{\text{Fr}} \text{Spec}(R/p)$$

Then $Z(D_X \text{ mod } p) = O((T^*X)')$ and the support of M contained in $(T^*X)'_p$ is the support of $M \text{ mod } p$ as a module over the center, Z .

Question: Is it true that for $p \gg 1$ $\text{supp}_p M$ is a Lagrangian cycle?

Example 18.9. $X = \mathbb{A}^1$, $M = D_X/D_X \cdot P$, $P = \sum_{i+j \leq N} a_{ij} x^i (\frac{d}{dx})^j \neq 0$. This gives a polynomial $P_{(p)}$ in 2 variables with

$$\text{supp}_p M = \{P_{(p)} = 0\}$$

Consider,

$$\sum a_{ij} \left(\left(\begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \\ & & & & 1 \end{pmatrix} + \xi \cdot 1_{p \times p} \right)^i \left(\left(\begin{pmatrix} 0 & 1 & & \\ & \ddots & p-1 & \\ & & & 0 \end{pmatrix} + \eta \cdot 1_{p \times p} \right)^j \right) \in R/pR[\xi^p, \eta^p]$$

This is simply $P_{(p)}(\xi^p, \eta^p)$ with $\deg P_{(p)} = N$. If $P = x \frac{d}{dx} - \lambda$, this gives a curve for $\lambda \in \mathbb{Q}$ where $\xi^p \eta^p = \lambda^p - \lambda \pmod{p}$ so this is actually 0.

Definition 18.10. $\text{supp}_p M$ does not move with respect to p if $\exists L \subset T^*X \otimes \mathbb{Q}$ and $\text{supp}_p M = \text{Fr}(L \pmod{p})$ for $p \gg 0$.

Conjecture 18.11. *Support does not move* $\iff M$ is an extension of exponential motivic D -modules.

If $\varphi \in \text{Aut}(A_{n,\mathbb{C}})$, then M_φ a bimodule of $A_{n,\mathbb{C}}$ which in fact equals $A_{n,\mathbb{C}}$. The left action is standard. The right action is twisted by φ .

So how does the support of M move? the bounded degree coefficients of the equations belong to

$$\prod_{p=2,3} (R/pR) / \bigoplus_{p=2,3} (R/pR)$$

a finitely generated subring $R_M \subset R_\infty$. Then $\text{supp}_{\text{arith}} M$ is a Lagrangian subvariety in T^*X depending on $t \in \text{Spec}(R_M \otimes \mathbb{Q})$, call it L_t , where $xy = t$. Now $L_t \in \Gamma(L_t^{\text{smooth}}, \Omega_{\text{closed}}^1)$.

Theorem 18.12.

$$L_t \in \Omega_{\log}^1(L_t^{\text{smooth}})$$

If $L \subset T^*X$ is closed and smooth, then

$$H_1(L(\mathbb{C}); \mathbb{Z}) = 0 \rightarrow \exists! \text{ holonomic } D\text{-modules, with } \text{supp}_p M = L$$