Poisson 2006: Conference Notes

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Contents

1 Marc Rieffel, *Strict deformation quantization: the example of toric manifolds*

Here I hope to present a class of examples to give a model of what one can hope to do for more complicated examples. First, from Quantum mechanics:
Given a Poisson manifold M, try to deform $A = C^\infty(M)$ into non-commutative algebras $A_\hbar$. Many of these ideas can be extended to non-compact manifolds with certain attention to details. One approach is to realize it as operators on a Hilbert space. So we want an actual product on the vector space of smooth functions with an involution, possibly depending on the Plank constant, $\hbar$, with a C^*-norm. This approach is strict deformation quantization. For $\hbar = 0$, we get the normal algebra A. Furthermore, you want the norms to be related to each other, and we call this map $\hbar \mapsto \widetilde{A}_\hbar$ a continuous field of C^*-algebras. we also need the condition:

$$\|(f \times_\hbar g - g \times_\hbar f) - i\{f, g\}\| \to 0$$

as $\hbar \to 0$. These express the "correspondence principle". One can also apply the above definition to a non-commutative C^*-algebra with a "Poisson bracket".

Now we move onto the simple class of examples: Let A be a unital C^*-algebra, e.g. $A = C(M)$. Let α be an action of $\mathbb{R}^d$ on A coming from a smooth action on M. Let A^∞ be the dense $*$-subalgebra of smooth elements for α. Let ∂_j be the derivation of A^∞ in the j-th direction for α, e.g. the vector field on M in the j-th direction for α. For any given $\theta \in M_d^{sk}(\mathbb{R})$ define a Poisson bracket on A^∞ by

$$\{a, b\} = \sum \theta_{jk}(\partial_j a)(\partial_k b)$$

We'll restrict our attention to actions of $\mathbb{T}^d$ where $d \geq 2$. We can clearly view this as $(\mathbb{R}^d/\mathbb{Z}^d)$ and define $e(r) = \exp(2\pi i r)$. For $n \in \mathbb{Z}^d$, set

$$A_n = \{a \in A : \alpha_l(a) = e(n \cdot t)a \forall t\}$$

Then $\bigoplus_n A_n$ is a dense $*$-subalgebra of A. These A_n are often called the isotopic components of A. Also,

$$A^\infty = \left\{\sum a_n : a_n \in A_n \text{ and } (\|a_n\|) \in S(\mathbb{Z}^d)\right\}$$

For θ as above and $a, b \in A^\infty$, set the twist by

$$a \times_\theta b = \sum_{m,n} a_m b_n e(m \cdot \theta n)$$

Denote the resulting algebra by A_θ^∞.

Take a covariant representation $(\pi, U, \mathcal{H})$, of $(A, \mathbb{T}^d, \alpha)$, (so $U_t\pi(\alpha)U_t^* = \pi(\alpha_t(a))$). We have $\mathcal{H} = \bigoplus \mathcal{H}_n$ for U. For $\xi \in \mathcal{H}$, with $\xi = (\xi_n)$, set

$$\pi^\theta(a)\xi = \sum_{m,n} \pi(a_m)\xi_n \cdot e(m \cdot \theta_n)$$

Then π^θ is a $*$-representation of A_θ^∞. If π is injective, then we have a C^*-norm, $\|a\|_\theta = \|\pi^\theta(a)\|$. We can show this norm is independent of $(\pi, U, \mathcal{H})$. The completion of A_θ^∞ for this norm is the C^*-algebra $\overline{A}_\theta$. Then $\hbar \to \overline{A}_{\hbar\theta}$ is a strict deformation quantization of A in the direction of the Poisson bracket from θ, Rieffel (1993). The algebras A_θ share some properties with A. There are many differences as well like the positive cones coming from actual "vector bundles", e.g. projective modules.

For the case of a manifold M, let $\Omega^*(M)$ be its deRham complex. The toric action on $C^\infty(M)$ lifts to an action on $\Omega^*(M)$ commuting with the exterior derivative d. So by the same techniques the exterior product on $\Omega^*(M)$ can be deformed, but the exterior derivative d left unchanged, to get a complex $\Omega^*(M_\theta)$ with $\Omega^0(M) = C^\infty(M_\theta)$. Note that it is not (graded) commutative in general.

Lie groups can also be deformed into quantum groups by our construction. Let G be a compact Lie group, and $H < G$ with $H \cong \mathbb{T}^d$. Let α be the action of $H \times H$ on $A = C(G)$ given by

$$(\alpha_{(s,t)}f)(x) = f(x \in xt)$$

For $\tilde{\theta}$ of size $2d \times 2d$ we can form $C(G_{\tilde{\theta}})$. This procedure will not respect the coproduct,

$$(\Delta f)(x, y) = f(xy)$$

in general, however, looking at $\tilde{\theta}$ of the form

$$\begin{pmatrix} \theta & 0 \\ 0 & -\theta \end{pmatrix}$$

and one gets a quantum group with unchanged coproduct, coidentity, and antipode, call it $C(G_\theta)$. This can be found in papers dating back to 1993.

Other developments that have received more attention by Connes and others include examples of non-commutative spaces A having a projective module given by a projection $p \in M_4(A)$ with A generated by the entries of p and with reduced Chern classes $ch_j(p) = 0$ for $j = 0, 1$ as happens for the 4-sphere S^4. They were led to $C(S_\theta^4$, using the action of a maximal torus in $SO(5)$. More generally, they show that if you have a compact Riemannian manifold, that is also a *spin manifold* with a Dirac operator, then if you have a toric action by *isometries* of M, then α lifts to a projective representation on $L^2(M, S)$ that commutes with D. The result is that one can twist the action of $C^\infty(M, S)$ so that it becomes a $C^\infty(M_\theta$ projective module, $C^\infty(M_\theta, S)$ with D still defined on it, and with completion still $L^2(M, S)$. We then get a Dirac operator for $C(M_\theta)$.

Connes keeps advocating "what is the non-commutative version of a Riemannian manifold"? He thinks that just knowing the Dirac operator gives you enough information to recover the structure. He proposes that we do this by "spectral geometry" $(A, \mathcal{H}, D)$ satisfying suitable axioms. They are satisfied for $A = C(M_\theta)$, $\mathcal{H} = L^2(M, S)$ and D. One recovers the "real" structure by the *charge conjugation operator* J on $L^2(M, S)$ twisted appropriately. You can also twist the Hodge star operator to get one $*_\theta$ on $\Omega(M_\theta)$.

Varilly showed in 2001 that if you look at the quantum group $C(G_\theta)$, then it acts on $C((G/K)_\theta)$ and this is a *homogenous subspace* of $C(G_\theta)$. For $G = SO(5)$, it acts on $C(S^4_\theta)$ and this is a homogeneous space. Furthermore, the action of $C((SO(5))_\theta)$ is by "isometries" for the "Riemannian structure" on $C(S^4_\theta)$ given by the Dirac operator $(D, L^2(M_\theta, S))$. Everything seems to fit together beautifully.

In the same year, Sitarz gave a variation of this construction that is also very useful. Instead, look at the Lie algebra and look at is universal enveloping algebra, $\mathcal{U}(\mathfrak{g})$ which is a Hopf algebra. Then the Hopf algebra has an action on the subalgebra $C^\infty(G/K)$. By a Drinfeld twist using $\mathfrak{h} = \mathrm{Lie}(H)$. one can deform $\mathcal{U}(\mathfrak{g})$ to a Hopf algebra $\mathcal{U}_\theta(\mathfrak{g})$. Only the coalgebra structure is deformed and the product remains unchanged. If G/K has a Dirac operator as above, so that $(G/K)_\theta$ does too, then the Dirac operator is also compatible with the action of $\mathcal{U}_\theta(\mathfrak{g})$. Thus $\mathcal{U}_\theta(\mathfrak{so}(5))$ acts on $\mathcal{U}(S^4_\theta)$ in a way compatible with the Dirac operator.

In a preprint, Landi (2006) study Yang-Mills for $C(S^4_\theta)$. Classically, there is an action of $SU(2)$ on S^7 with quotient S^4 (a Hopf fibration). This action commutes with an action of Spin(5) on S^7. Then any finite dimensional representation of $SU(2)$ "induces" a vector bundle on S^4 which is Spin(5)-equivariant. For the representation of $SU(2)$ on $\mathbb{C}^2$, one gets the classical "instanton" bundle on S^4 having the interesting Yang-Mills minima. Similarly, for a θ' determined by θ, there is an action of (ordinary) $SU(2)$ on $C(S^7_{\theta'})$ such that the fixed-point algebra is exactly $C(S^4_\theta)$. For each representation of $SU(2)$, you get a projective module over $C(S^4_\theta)$. For 2d representation, get the "quantized instanton bundle", denoted by $\mathcal{E}$.

Now then $\mathcal{U}_\theta(\mathfrak{so}(5))$ acts on $C^\infty(S^4_\theta)$ and $C^\infty(S^7_{\theta'})$, and $\mathcal{E}$ is equivariant for this infinitesimal action. One can look for connections on $\mathcal{E}$, that is maps

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^1(S^4_\theta)$$

satisfying the Leibniz rule:

$$\nabla(\xi, \omega) = (\nabla\xi)\omega + \xi \otimes d\omega$$

This extends to

$$\nabla : \mathcal{E} \otimes_A \Omega^p(S^4_\theta) \to \mathcal{E} \otimes_A \Omega^{p+1}(S^4_\theta)$$

and determines the curvature $F = \nabla^2$:

$$\nabla^2 : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^2(S^4_\theta)$$

On $\mathcal{E}$ one can have a natural Hermitian metric

$$\langle \xi, \eta \rangle_A \in A = C(S^4_\theta)$$

which extends to one on $\Omega^*(S^4_\theta)$. We say that a connection is compatible with this Hermitian metric if it satisfies the Leibniz rule:

$$d(\langle \xi, \eta \rangle_A) = \langle \nabla \xi, \eta \rangle_A + (-1)^{|\xi|} \langle \xi, \nabla \eta \rangle_A$$

We let $CC(\mathcal{E})$ denote the set of compatible connections on $\mathcal{E}$ for its Hermitian metric.

From the usual round Riemannian metric on S^4, for which $SO(5)$ acts by isometries, we get the deformed Hodge $*_\theta$ on $\Omega^*(S^4_\theta)$. For any $\nabla \in CC(\mathcal{E})$ with curvature F set:

$$YM(\nabla) = \int *_\theta \mathrm{tr}(F *_\theta F)$$

where tr is the A-valued trace on $\mathrm{End}_A(\mathcal{E})$ coming from the Hermitian metric on $\mathcal{E}$.

Let $\mathcal{G} = U\mathrm{End}_A(\mathcal{E})$, the group of unitary elements of $\mathrm{End}_A(\mathcal{E})$. It is the "gauge group" for $\mathcal{E}$, and acts on $CC(\mathcal{E})$ by conjugation. The functional YM on $CC(\mathcal{E})$ is invariant under the action of $\mathcal{G}$. The set of minima of YM is carried into itself by the action of $\mathcal{G}$, and the set of $\mathcal{G}$-orbits in the set of minima is the "moduli space" of minima.

The conformal Lie algebra $\mathfrak{so}(5,1)$ acts on $C^\infty(S^4)$ while $SO(5)$, and so $\mathbb{T}^2$, act by Ad on $\mathfrak{so}(5,1)$. Thus we can form $\mathcal{U}_\theta(\mathfrak{so}(5,1))$ and it acts on $C^\infty(S^4_\theta)$ and on $\Omega^*(S^4_\theta)$, leaving the Hodge structure invariant. It also acts on $C^\infty(S^7_{\theta'})$, and so infinitesimally on $CC(\mathcal{E})$. There is a canonical connection, ∇^0, coming from the connection of $\mathcal{E}$, which is minimal for YM. Landi uses the action of $\mathcal{U}_\theta(\mathfrak{so}(5,1))$ to show that the tangent space at ∇^0 in the moduli space of minima for YM is exactly 5-dimensional. That concludes this direction.

In another direction, Connes says that spectral geometry can be viewed as a "non-commutative metric space". The Lipschitz seminorm, L, on A is defined by

$$L(a) = \|[D, a]\|$$

Let $S(A)$ denote the state space of A, consisting of the positive linear functionals on A of norm 1. (the "non-commutative probability measures"). Define a metric ρ_L on $S(A)$ by

$$\rho_L(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : L(a) \leq 1\}$$

I want: (*) - The topology on $S(A)$ from ρ_L to agree with the weak-$*$ topology. Then we can define a quantum Gromov-Hausdorff distance. Hanfeng Li (2005) showed that for L on $C^\infty(M_\theta)$ coming from D the property (*) holds. Furthermore,

$$\mathrm{dist}_{qGH}(M_{\theta_1}, M_{\theta_2})$$

is continuous in θ_1 and θ_2.

2 Maxim Kontsevich, *Motivic quantization of integrable systems*

By a classical algebraic integrable system, we mean $X/\mathbb{C}$ with $\pi : \mathcal{T}^*X \to B$ and the fibers of π are Lagrangian. For a Toda lattice, we have

$$\frac{1}{2}\sum_i p_i^2 + \sum_i \exp(x_i - x_{i+1})$$

where $y_i = \exp(x_i)$ are algebraic coordinates. To quantize this, we take $\mathcal{D}_X$ which contains the large commutative subalgebra $\mathcal{X})(B)$. Motivic integrable systems were X over any field k. New: p-adic integrable systems.

Langlands Correspondence (funct. case):

Have a curve $C/\mathbb{F}_q$ with $(l,q)=1$, l prime. Then

$$\text{Irr.Rep } (\pi_1(C \times_{\text{Spec}(\mathbb{F}_q)} \text{Spec}(\overline{\mathbb{F}}_q) \to GL(N, \overline{\mathbb{Q}}_l))/\text{conj})^{\text{Fr}} = \text{ finite set}$$

$$= \text{ cuspidal autom. } \mathbb{C}\text{-rep of } GL(\text{ Adeles } C(\mathbb{F}_q))$$

Also,

$$\hat{\mathbb{Z}} = \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to \text{Aut}(\pi_1, ...)$$

A priori, one has that the GL group acts on $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We have a smooth projection X_t to $t \in C$. We can actually look at Fr^n and q^n to get an infinite tower.

Example 2.1. $C = \mathcal{P}^1 \setminus \{0,1,t,\infty\}$, $t \in \mathbb{F}_q$, $t \neq 0,1$. Representations go to $SL(2)$ with monodromy near each of the 4 singularities.

With this situation we'll associate a finite-dimensional commutative algebra over $\mathbb{Q}$, with a basis e_x, $x \in \mathbb{F}_q$, $e_x \cdot e_y = \sum_z c_{xyz} e_z$, for

$$c_{xyz} = \#\{w \in \mathbb{F}_q | w^2 = f_t(x,y,z)\} + \cdots$$

and

$$f_t(x,y,z) = (xy + yz + zx - t)^2 + 4xyz(1 + t - (x+y+z))$$

Theorem 2.2. *The set of representations is equal to* $\text{Hom}_{\text{alg}}(A, \overline{\mathbb{Q}})$.

Taking $\mathbb{F}_q \to \mathbb{F}_{q^n}$ we can get infinitely many. This leads to the following formalism: $\forall$ field k, there corresponds a tensor category C_k. The objects are the varieties over k and the morphisms are the following:

$$\text{Hom}_{C_k}(X_1, X_2) := K_0(\text{ schemes}/X_1 \times X_2)$$

an abelian group. The generators are $Y \to X_1 \times X_2$ modulo the relations $Y = Y' \cup Y''$. Composition is defined by

$$(Y \to X_1 \times X_2) \circ (Y' \to X_2 \times X_3) := (Y \times_{X_2} Y' \to X_1 \times X_3)$$

and tensor product $X_1 \otimes X_2 := X_1 \times X_2$. If $k = \mathbb{F}_q$, we have an infinite chain of tensor functors

$$\varphi_n : C_k \to \mathrm{Vect}/\mathbb{Q}, \quad X \to \mathbb{Q}^{X(\mathbb{F}_{q^n})}$$

Definition 2.3. Motivic integrable system is then a commutative associative algebra in C_k.

Now how is it related to usual integrable systems? Take $k = \mathbb{C}$ and replace $\sum n_\alpha Y_\alpha \to X^3$ by (l-adic) sheaf $\mathcal{F}$ with characteristic variety of $\mathcal{F}$ equal to a Lagrangian cycle $\mathcal{L}$ of $T^*(X^3)$. Then $T^*X \to B$ is an abelian fibration with

$$(T^*X)^3 \supset \{(v_1, v_2, v_3) | \pi(v_1) = \pi(v_2) = \pi(v_3), v_1 + v_2 = v_3 \text{ in fiber } \}$$

Then $\mathcal{L}$ is a Lagrangian subvariety in $T^*(X^3)$. In my example: $X = \mathbb{A}^1, B = \mathbb{A}^1$ then $(x, y) \mapsto y^2 x(x-1)(x-t)$ with $\mathbb{C}[x(x-1)(x-t)(\partial_t)^2] \subset \mathcal{D}_x$.

My conjecture is that the Langlands correspondence should work for any variety, not just curves. The arguments in favor of this conjecture are:

1. should work for curves with a motivic sheaf on $X^2 \times X = X^3$ by a diagonal embedding of GL_n where X is the modulus of stable bundles of rank N over C.
2. S surface, C a hyperplane with $\pi_1(S) \twoheadleftarrow \pi_1(C)$

We make the following observation: take k a local field and the Hilbert space $\mathcal{H} = L^2(k; \sqrt{|dx|})$, and $T_\rho : \mathcal{H} \to \mathcal{H}$. Then

$$(T_\rho \psi)(x) = \int_{y,z} K(x, y, z) \rho(z) \psi(y) |dy||dz|$$

is self-adjoint Hilbert-Schmidt operator, $[T_{\rho_1}, T_{\rho_2}] = 0$ with no correction terms!

For the 2-periodic Toda lattice: $X = \mathbb{G}_m \times \mathbb{G}_a$, a motivic integrable system, $\to \mathbb{G}_a$. On $\mathbb{A}^1$:

$$K_{\mathcal{X}}(x, y, z) = \cos^{+}_{\mathcal{X}} \left(\frac{x + y + z}{\sqrt{xyz}} \right) \left| \frac{dx}{x} \right|^{\frac{1}{2}} \left| \frac{dy}{y} \right|^{\frac{1}{2}} \left| \frac{dz}{z} \right|^{\frac{1}{2}}$$

3 Noriaki Ikeda *Deformation of graded Poisson (Batalin-Vilkovsky) structures*

3.1 Introduction

We are interested in these structures because, mathematically, many geometrical and algebraic structures described as a B-V structure. Trying to unify geometries, classify and define new ones. Physically, it's the structure of gauge theories.

3.2 Graded poisson structures on the Graded Vector Bundles

This is the AKSZ formulation. We define a super manifold as a cotangent bundle with reversed parity of the fiber. We can give this a grading with total degree of the fiber to be p. We denote a graded tangent bundle $T[p]M$. Now take a Poisson manifold N and shift the grading to get a supermanifold $\tilde{N}$ with an anti-bracket. This map is called the P-structure. The antibracket on $T^*[p]M$ satisfies some identities. It is defined as a bilinear form on a graded manifold with total degree $-p$ satisfying these identities.

Example 3.1. $E \oplus E^*$ has a natural pairing on the fiber We can define a Poisson bracket and consider a graded bundle

$$E[p] \oplus E^*[q]$$

. Then the Poisson bracket shifts to an anti-bracket.

Definition 3.2. S a function on a graded supermanifold is called a B-V action satisfying $(S, S) = 0$, the anti-bracket. It's called a Q-structure.

Definition 3.3. A B-V structure is a P-structure and Q-structure.

3.3 B-V structures of Abelian Topological Sigma Models

Consider two manifolds X and M with a map ϕ between them. A topological sigma model is independent of the metrics on the manifolds. Extend $\phi : X \to M$ to $\bar{\phi} : \Pi TX \to M$, and construct a B-V structure from $\bar{\phi}$. If $\dim X = n$, then natural geometry on $T^*[p]M$ is $p = n-1$. Then we put the antibracket on this graded cotangent bundle. In the case of our previous example, the total degree of the anti-bracket must be $-n+1$, with $q = n-p-1$ and consider

$$E[p] \oplus E^*[n-p-1]$$

. We actually want to consider $T^*[n-1]M$ with this example simultaneously. We then define the *form degree* and *ghost number* as a graded on the supermanifold related by

$$gh(F) = |F| - \deg(F)$$

3.4 Deformation Theory

We deform the B-V structures by fixing a P-structure on a graded bundle and considering the infinitesimal deformation of a Q-structure,

$$S = S_0 + \underbrace{gS_1 + \cdots}_{gS_1'}$$

such that $(S, S) = 0$ up to some cohomology, $\delta_0 = (S_0, *)$ with $\delta_0^2 = 0$. For any function G,

$$\int_X dG(\bar{\phi}, d\bar{\phi}) = 0 \iff H^2 = 0$$

i.e. no obstruction or D-brane, so the general solution is $S_1' = \int_X F(\bar{\phi}, d\bar{\phi})$ for any arbitrary function F.

Theorem 3.4. *IF a monomial $F(\bar{\phi}, d\bar{\phi})$ includes at least one $d\bar{\phi}$ and $(S_0, \int_X F) = 0$ then $\int_X F$ is δ_0-exact.*

3.5 Lie algebroid and observables

If we define

$$[e_1, e_2] = ((S, e_1), e_2)$$

with $\rho(e)F(\phi) = (e, (S, F(\phi)))$, where S is the Poisson Sigma model, then:

$$\mathcal{E} \text{ is a Lie algebroid} \iff (S, S) = 0$$

B-V structure is equivalent to the Lie algebroid structure on T^*M for 2d.

3.6 Structures in lower n

$n = 2$, the total bundle is $T^*[1]M = \Pi T^*M$. $S = S + gS_1'$ and $(S_1', S_1') = 0$ determines the algebraic and geometry structure so it corresponds to a Poisson bivector and the manifold is a Poisson manifold. This example is actually the 2d Poisson Sigma model.

Theorem 3.5. $(S_1', S_1') = 0$ *satisfying additional properties corresponds to a Courant algebroid on this bundle $\mathcal{E}$.*

If the above theorem holds, we call S the *Courant sigma model.* The form degree deformation defined an algebroid on the bundle on M defined from the master equation.

3.7 Quantum deformation theory

We define the B-V laplacian and add the gauge fixing term S_{GF},

$$S_q = S + S_{GF}$$

to restrict the theory on the Lagrangian submanifold of the total bundle and S_q satisfies the master equation. For an observable $\mathcal{O}$,

$$(S_q, \mathcal{O}) - i\hbar\mathcal{O} = 0$$

Substructures of B-V structures are for example, in $n = 2$, a symplectic Kähler structure. For $n = 3$, we get a generalized complex structure. Classically we have Courant, Dirac, generalized geometry, twisted Poisson, topological M, and G_2 models, but we don't understand the corresponding quantum geometry. In $n = 2$, we've already found corresponding quantum structures to Poisson and other classical models, such as the A and B models.

Future direction would be to look at what these quantum structure would be.

4 Alberto Cattaneo, *Deformation quantization and reduction*

Joint work with Giovanni Felder

By deformation quantization, recall this means to find an associative product on $C^\infty(M)[[\epsilon]]$.

Theorem 4.1 (Kontsevich)**.** *Every Poisson manifold has a deformation quantization.*

Remark 4.2. For other Poisson algebras, there is no existence result in general. We will have to consider such a a case. Kontsevish's theorem is a corollary of a larger theorem.

For reduction, recall our quotient space is a manifold and inherits a symplectic structure. A special case is the coisotropic case when $T^\perp C \subset TC$ for C a submanifold of a symplectic manifold M. Let N^*C be the *conormal bundle* of C:

$$0 \to N^*C \to T^*_C M \to T^*C \to 0$$

dual to

$$0 \to TC \to T_C M \to NC \to 0$$

where NC is the normal bundle.

Definition 4.3. C being coisotropic then means that $D := \pi^\sharp(N^*C) \subset TC$, and D is a singular integrable distribution on C.

If C/D is a manifold, it inherits a Poisson structure. We use this term because if $pi = \omega^{-1}$, then $\pi^\sharp(N^*C) = T^\perp C$.

Example 4.4. 1. M is coisotropic in M for every Poisson structure.

2. $\pi \equiv 0$. Any submanifold is coisotropic.

3. M, N Poisson manifolds, ϕ a map $M \to N$, then the graph of ϕ is coisotropic on $\overline{M} \times N \iff \phi$ is a Poisson map.

4. $\mathfrak{g}$ a Lie algebra, $M = \mathfrak{g}^*$ with Kirillov-Kostant Poisson structure. $\mathfrak{h}$ a subspace of $\mathfrak{g}$ and $\mathfrak{h}^0$ its annihilator. Then $C = \mathfrak{h}^0$ is coisotropic $\iff$ $\mathfrak{h}$ is a Lie subalgebra.

4.1 Algebraic description

Let A be a Poisson algebra. An ideal I in A as a commutative algebra is called a *coisotrope* if it also is a Lie subalgebra. Then A/I is an I-module.

Lemma 4.5. *If I is the vanishing ideal of a closed submanifold C of a Poisson manifold M and $A = C^\infty(M)$, then C is coisotropic $\iff$ I is a coisotrope. Moreover, $C^\infty(C/D) = (A/I)^I$. If C/D is not a manifold, take it as the definition.*

Let $N(I) := \{a \in A : \{a, I\} \subset I\}$ be the normalizer. Then $N(I)$ is a Poisson subalgebra, I is a Poisson ideal in $N(I)$, and $N(I)/I$ is isomorphic to $(A/I)^I$ as a commutative algebra! So one may induce a Poisson structure on $(A/I)^I$. Is there a deformation quantization of it?

4.2 Coordinate description

Let $\{x^I\}$ be adapted local coordinates of M. C locally is given by $x^I = 0$, $I > k = \dim C$. C coisotropic $\iff$ $\pi^{\mu\nu}|_C = 0$.

4.3 Strongly regular submanifolds

A good generalization of the notion of preseymplectic submanifold is a submanifold C of M such that $\operatorname{rank}(TC + \pi^\sharp N^*C)$ is constant.

Remark 4.6. If M is symplectic, then stronly regular = symplectic. It turns out that $K := (\pi^\sharp)^{-1}TC \cap N^*C$ is a subbundle (actaully a subalgebroid of T^*M) and $\pi^\sharp(K)$ integrable characteristic distribution.

Theorem 4.7 (Calvo-Falcet, C-Zambon)**.** *C strongly regular in M $\implies$ $\exists C' \subset M$ Poisson-Dirac submanifold which contains C as a coisotropic submanifold.*

C' Poisson-Dirac: for every symplectic leaf $\mathcal{O}$ of M, $C' \cap \mathcal{O}$ symplectic in $\mathcal{O}$. C' inherits a Poisson structure with these intersections as its symplectic leaves.

The Poisson sigma model is a 2d topological field theory with target a Poisson manifold whihc produces Kontsevich's star product.

$$S(X, \eta) := \int_\Sigma \langle \eta, dX\rangle + \frac{1}{2}\langle \eta, \pi^\sharp(X)\eta\rangle$$

where Σ is a 2-manifold. X a map $\Sigma \to M$.

Boundary conditions ("branes"): We have the variational principle where we split over the surface and the boundary:

$$\delta S = \int_\Sigma - \int_{\partial\Sigma}$$

We must impose the boundary term to vanish, otherwise it gets messy. Setting δX or η to zero accomplishes this. Precisely, fix $C \subset M$ and require $X(\partial\Sigma) \subset C$ and

$$\iota^*_{\partial\Sigma}\eta \in \Gamma(T^*(\partial\Sigma) \otimes X^*N^*C)$$

Instead, think of this as a Hamiltonian problem. Let $\Sigma = I \times \mathbb{R}$. Write $\eta = \lambda dt + \zeta$, with $I = [0, 1]$, $t \in \mathbb{R}$ with λ a path in $\Gamma(I, X^*T^*M)$ and ζ a path in $\Gamma(I, T^*I \otimes X^*T^*M)$. So $(X, \zeta) \in T^*PM$ has a canonical symplectic structure. λ is a Lagrange multiplier imposing: $dX + \pi^\sharp(X)\zeta = 0$. If $\mathcal{C}$ is the space of solutions, and $\mathcal{C}_C := \{(X, \zeta) \in \mathcal{C} : X(0) \in C\}$, then

1. $\mathcal{C}_C$ coisotropic in $T^*PM \iff C$ coisotropic in M.
2. $\mathcal{C}_C$ presymplectic in in $T^*PM \iff C$ strongly regular in M.

In the Lagrangian approach, compatibility with symmetries $\implies C$ is strongly regular. We can rescale π to $\pi_\epsilon := \epsilon\pi$. Then $\forall\epsilon$, C still satisfies the strong regularity condition. For $\epsilon = 0$, C is even coisotropic. Considering coisotropic submanifolds in no loss of generality.

In the perturbative expansion of the PSM, let Σ be the disc and C the coisotropic submanifold of M. $f \in C^\infty(C)$, $u \in \partial\Sigma$, $\mathcal{O}_{f,u} := f(X(u))$. Fix ∞ in $\partial\Sigma$, and define

$$(f * g)(x) := \int_{(X,\eta)\in\mathcal{M}_C : X(\infty)=x} e^{\frac{i}{\hbar}S}\mathcal{O}_{f,u}\mathcal{O}_{g,v}$$

For $C = M$: the result does not depend on u and v. If defines a star product for the given Poisson structure with $\epsilon = i\hbar$. This is actually Kontsevich's star product. For $C \neq M$, one expects to get a star product for the given Poisson structure on $C^\infty(C/D)$. This is actually not the case in general.

Problem 1: There exists a potential anomaly (in $H^2_\delta(N^*C)$- Lie algebroid cohomology) Problem 2: In the absence of anomaly, on e get an associative algebra on the quantum invariants.

4.4 Many branes

It is possible to cut the boundary of the disc into two pieces and associate to each of them a different coisotropic submanifold C_1, C_2. If the anomalies vanish, this produces a bimodule structure on $C^\infty(C_1 \cap C_2/D)[[\epsilon]]$ for the deformation quantizations of C_1/D and C_2/D. Putting three different coisotorpic submanifold on three different components of the boundary yields morphism of bimodules.

What happens with more than three boundary conditions? Not clear. Actually, the topology changes and some propagators can no longer be closed forms.

4.5 Comments

Quantization of coisotropic submanifolds can also be achieved by the BVF method. The present method seems to be more systematic. The construction of bimodule structure is something new. In particular if applied to graphs, it yields a quantization of Poisson maps. The construction of morphisms of bimodules may also be interesting. E.g. sometimes you might have a bunch of Poisson maps with relations, so you want to quantize the product and preserve the associativity. A dream is to use this procedure to get quantum groups by deformation quantization of Poisson-Lie groups. This method has also interesting applications to Lie theory. (work in progress with C.Torossian). You can recover lots of results from representation theory using this method.

Questions: In the symplectic case, does this anomaly show up? Locally it doesn't clearly. But can you relate it to this Atiyah-Bott case? Probably. Is there a rigorous version? No, well, there is for the tube expansion. The fact it's not rigorous is that we don't know how to do it exactly. In principle, you should get an associative product, but you don't. In you have a singular symplectic leaf in your space can you quantize these? Yes, it doesn't matter whether your reduced space is singular or not. Would this help if you wanted to quantize the non-commutative of a foliation? In principle, there should be version where you go to the symplectic groupoid and you find something.

5 J.H. Lu, *Some aspsects of a Poisson structure on a complex semi-simple Lie group*

Joint work with Sam Evens.

Let G be a complex semisimple simply connected Lie group. For example, $G = SL(n, \mathbb{C})$. We will discuss a Poisson structure π_0 on G. This π_0 is actually invariant under the conjugation action by a fixed maximal torus H. So if $\Sigma \subset G$ is a symplectic leaf of π_0,

$$H \cdot \Sigma = \bigcup_{h \in H} h\Sigma h^{-1}$$

an H-orbit of leaves. It's good to organize these H-orbits in a certain way. In particular, we will decompose $G = \bigcup_{t \in H/W,\, w \in W} F_{t,w}$ (disjoint), with W being the Weyl group, where

1. Each $F_{t,w}$ is a finite union of H-orbits of symplectic leaves.
2. $F_{t,w}$ is an affine variety and in general singular.

For each t, w we will have a Poisson map

$$\mu : X_{t,w} \to F_{t,w}$$

where $X_{t,w}$ is a smooth regular Poisson variety. In fact it is a single H-orbit of symplectic leaves in some X. μ is a resolution of singularities. So this fits into the study of desingularization of Poisson varieties. This leads into the study of "symplectic varieties" or symplectic singularities (Beauville, Fu, Kaledin, Ginsburg, Namikawa,...). An example is maybe nilpotent coadjoint orbits in a semisimple Lie group, or their closures. They are actually pretty rare.

Recall that a set of coordinates $\{x_i\}$ on a Poisson variety P is said to be *log-canonical* if

$$\{x_i, x_j\} = c_{ij} x_i x_j, \quad c_{ij} \in \mathbb{C},\ \forall i, j$$

This plays the role of double coordinates. For each t, w we will construct log-canonical coordinates on $X_{t,w}$ and $F_{t,w}$. This leads to things called "cluster varieties" or (cluster algebras).

First, let's recall some background in Lie theory ($G = SL(n, \mathbb{C})$):
Nilpotent variety in $\mathfrak{g}$:

$$\mathcal{N} = \{x \in \mathfrak{g} : (\mathrm{ad}_x)^n = 0 \text{ for some } n\} \subset \mathfrak{g} \cong \mathfrak{g}^* \text{ by Killing}$$

Finite union of adjoint orbits. Singular Poisson subvariety of $\mathfrak{g}$.
For $SL(2, \mathbb{C})$: 2 adjoint orbits in $\mathcal{N}$, through 0 and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $\begin{pmatrix} x & y \\ z & -x \end{pmatrix} \in \mathcal{N} \iff x^2 + yz = 0$.
For $SL(3, \mathbb{C})$: 3 adjoint orbits in $\mathcal{N}$ through 0, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ (subregular), $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = x_2$ (regular orbit), $\mathcal{N}$ is smooth exactly at the orbit through x_2.
Springer resolution: $B \subset G$ Borel.

$$\mu : T^*(G/B) \to \mathfrak{g}^* \cong \mathfrak{g}, \ \mathrm{im}\mu = \bigcup_{g \in G} \mathrm{Ad}_g \mathfrak{a} = \mathcal{N}$$

with $T^*(G/B) \cong G \times_B \mathfrak{a}$, where $\mathfrak{a} = [\mathfrak{b}, \mathfrak{b}]$ the nilradical of $\mathfrak{b} = \mathrm{Lie}(B)$. So $\mu : T^*(G/B) \to \mathcal{N}$.

Definition 5.1. Let V be a normal irreducible variety over $\mathbb{C}$ with V_s denoting its subset of smooth points. A resolution of singularities of V is a pair (X, μ) where X is smooth and irreducible, $\mu : X \to V$ is a proper, surjective morphism such that $\mu|_{\mu^{-1}(V_s)} : \mu^{-1}(V_s) \to V_s$ is an isomorphism.

Definition 5.2. If (V, π) is a Poisson variety, a symplectic resolution of (V, π) is a resolution of singularities (X, μ) such that X is symplectic and μ is Poisson.

Steinberg fibers in G: Let $\chi_1, ..., \chi_r$ be the characters of the r fundamental representations of G where $r = \dim H$. Define $\chi : G \to \mathbb{C}^r$, and the fibers of χ are the Steinberg fibers. Then $G = SL(n, \mathbb{C})$, so

$$\det(\lambda I - g) = \lambda^n - \chi_1(g)\lambda^{n-1} + \chi_2(g)\lambda^{n-2} + \cdots + (-1)^{n-1}\chi_{n-1}(g) + (-1)^n \cdot 1$$

For $t \in H$, let $F_t = \chi^{-1}(\chi(t))$. Then $G = \bigcup_{t \in H/W} F_t$ (disjoint union).

$F_x = \mathcal{U}$ unipotent variety. For $SL(n, \mathbb{C})$: $F_t =$ all matrices whose Jordan forms have t on the diagonal. In general, F_t is a finite union of conjugacy classes. It can be singular. In Lie theory, there is a well know resolution of this called Grothendieck simultaneous resolution:

$$\begin{tikzcd}
G \times_B B \arrow[r, "\mu"] \arrow[d, "\kappa"'] & Gm \arrow[d, "\chi"] \\
H \arrow[r, "\chi"] & \mathbb{C}^r
\end{tikzcd}$$

where the action is given by $[gb_1, {b_1}^{-1}bb_1] = [g, b]$ with $\mu : [g, b] \mapsto gbg^{-1}$, and $\chi : [g, tn] \mapsto t$. For $t \in H$, $\kappa^{-1}(t) = G \times_B (tN) = X_t$.

Theorem 5.3 (Grothendieck, Slodowy). *For any $t \in H$, $\mu : X_t \to F_t$ is a resolution of singularities.*

Bruhat cells: B_-: opposite Borel of B.

$$G = \bigcup_{w \in W} BwB_-$$

Then $SL(3, \mathbb{C}) = \{g = (a_{ij})_{3\times 3} : \det = 1\}$. $W = \{1, r_1, r_2, r_1r_2, r_2r_1, w_0 = r_1r_2r_1 = r_2r_1r_2\}$ and to list a few,

$$BB_- : a_{33} \neq 0, \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \neq 0$$

$$Br_1B_- : a_{33} \neq 0, \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

For $t \in H$, $w \in W$, define:

$$F_{t,w} = F_t \cap BwB_-, \; X_{t,w} = X_t \cap \mu^{-1}(BwB_-) \subset X = G \times_B B$$

Theorem 5.4 (Evens-L). *1. $F_{t,w} \neq 0 \; \forall t, w$*

2. $\mu : X_{t,w} \to F_{t,w}$ is a resolution of singularities.

Example 5.5. $SL(3, \mathbb{C})$,

$$\mathcal{U} \cap Bw_0B_- = \left\{ \begin{pmatrix} x & y & a \\ z & b & 0 \\ c & 0 & 0 \end{pmatrix} : abc = -1, x + b = 3, xb - yz - ac = 3 \right\}$$

$(b-1)^3 + yzb = 0$. Then $u = b - 1, v = y, w = zb$ and $u^3 + vw = 0$, a Kleinian singularity.

Poisson structures: π_0 on G and π on $X = G \times_B B$. Use the theory of Poisson Lie groups. Step 1: G has a standard Poisson Lie group structure π_G. It has Heisenberg double $(G \times G, \pi_+)$. Identify

$$p : (G \times G)/G_{\text{diag}} \to G : (g_1, g_2)G_{\text{diag}} \mapsto g_1{g_2}^{-1}$$

$\pi_0 \overset{\text{def}}{=} p_*(\pi_+)$ roughly the dual of (G, π_G). This was done by Alekseev-Malkin (1990). Step 2: $\forall t \in H$, $tN \subset (G, \pi_0)$ is coisotropic, $p^{-1}(tN) \subset (G \times G, \pi_+)$ is coisotropic, $p^{-1}(tN)/\sim \cong G \times_B (tN) = X_t$. $X = \bigcup_{t \in H} X_t, \pi)$.

6 H. Bursztyn, *Reduction of Courant algebroids and GCS's*

Joint work with Gualtieri and Calvacanti

So we'd like to understand what's going on in reduction of GCS's and Courant algebroids. One example would be $S^1 \curvearrowright \mathbb{C}^2 \setminus \{0\}$ with $e^{i\theta}(z_1, z_2) = (e^{i\theta}z_1, e^{i\theta}z_2)$. In the complex case, you have $\mathbb{C}^* \curvearrowright \mathbb{C}^2 \setminus \{0\}$ which reduces to $\mathbb{C}\mathcal{P}^1$. For the symplectic case, you have $S^3 \hookrightarrow \mathbb{C}^2 \setminus \{0\}$ which reduces to $S^3/S^1 = \mathbb{C}\mathcal{P}^1$. This example is actually a Kähler reduction and the two cases are compatible.

6.1 Courant algebroids

Model: $TM \oplus T^*M = \mathcal{T}$.

- $\langle X + \xi, Y + \eta \rangle = \eta(X) + \xi(Y)$
- $[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - i_Y d\xi$

Dirac structures: $L < \mathcal{T}$ such that

- $L = L^2$ (Lagrangian)
- $\Gamma(L)$ involutive (integrable)

In particular, we can have $L =$graph(π) or $=$graph(ω). So $L = +i$-eigenbundle $\subseteq \mathcal{T} \otimes \mathbb{C}$ of $\mathcal{J} : \mathcal{T} \to \mathcal{T}$ with $\mathcal{J}^2 = -\mathrm{id}$. The idea is now that if you can reduce the larger structure, then you can reduce what's inside, i.e. the GCS. You can also twist the integrability condition for $H \in \Omega^3_\varphi(M)$ and you have the extra term in the Courant bracket $+i_Y i_X H$.

Definition 6.1. A *Courant algebroid* is a vector bundle E over M with

- a nondegenerate symmetric bilinear form on the fibers
- $\pi : E \to TM$
- $[\cdot, \cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$

and satisfying 5 other properties:

1. $[e, e] = \frac{1}{2}\mathcal{D}\langle e, e \rangle$, $\mathcal{D} = \pi^* \circ d : C^\infty(M) \to \Gamma(E)$
2. $[e_1, f e_2] = f[e_1, e_2] + (\mathcal{L}_{\pi(e_1)} f) e_2$
3. $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$
4. $\pi([e_1, e_2]) = [\pi(e_1), \pi(e_2)]$
5. $\mathcal{L}_{\pi(e)} \langle e_1, e_2 \rangle = \langle [e, e_1], e_2 \rangle + \langle e_1, [e, e_2] \rangle$

We're more interested in *Exact* Courant algebroids:

$$0 \to T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \to 0$$

with $\nabla : TM \to E$ and $H(X,Y,Z) = \langle [\nabla X, \nabla Y], \nabla Z \rangle$. For $[H] \in H^3_X(M)$ is the *Severa class*. This is important because you might for example start with some 3 form that reduces to something with curvature.

Now given a Courant algebroid over M, with some symmetries we get to a reduced Courant algebroid E_{red} over M_{red} and we'd like to know what these symmetries are.

6.2 Extended actions, moment maps

Usually:

$$\mathfrak{g} \to \mathfrak{X}(M)$$

or

$$\begin{tikzcd}
\mathfrak{g} \arrow[rr] \arrow[dr] & & \mathrm{sym}(TM) & & {[X, \cdot]} \\
& \mathfrak{X}(M) \arrow[ur] & & X \arrow[ur, mapsto] &
\end{tikzcd}$$

We want:

$$\begin{tikzcd}
\mathfrak{a} \arrow[rr] \arrow[dr] & & \mathrm{sym}(E) & & {[e, \cdot]_c} \\
& \Gamma(E) \arrow[ur] & & e \arrow[ur, mapsto] &
\end{tikzcd}$$

But $\Gamma(E)$ is not a Lie algebra!

$$\Omega^1(M) \to \Gamma(M) \to \mathfrak{X}(M)$$

Notice that the last three axioms say precisely that this is an adjoint action. In order to make sense of what $\mathfrak{a}$ is, we wanted to know what the right object is for $\Gamma(E)$. In order to have Lie algebras, we have a Courant algebra over $\mathfrak{g}$:

$$\text{Leibniz algebra } (\mathfrak{a}, [\cdot,\cdot]) \text{ with}$$

$$\mathfrak{a} \xrightarrow{\pi} \mathfrak{g} \text{ (morphism)}$$

This is *Exact* if $\mathfrak{h} = \ker(\pi) \to \mathfrak{a} \to \mathfrak{g}$ is exact, and $[\mathfrak{h}, \mathfrak{h}] = 0$.

Now what is an extended action? A usual action is something from a Lie algebra to a vector field. This now tells us how to complete the diagram to get:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{h} \arrow[r] \arrow[d, dashed] & \mathfrak{a} \arrow[r] \arrow[d, "\rho"] & \mathfrak{g} \arrow[d, "\psi"] \\
& \Omega^1(M) \arrow[r] & \Gamma(E) \arrow[r] & \mathfrak{X}(M)
\end{tikzcd}$$

The top row is an exact Courant algebra and the bottom a morphism of Courant algebras.

An *Extension* of G-action on M, $\psi : \mathfrak{g} \to \mathfrak{X}(M)$ with the above diagram and two conditions:

- $\rho(\mathfrak{h}) \subseteq \Omega^1_{\mathfrak{a}}(M)$
- (*) $\mathfrak{g} = \mathfrak{a}/\mathfrak{h}$ action on E integrates to a G-action.

Then $E = \mathcal{T}$ with canonical G-equivariant structure. For (*) it is enough to find a $\mathfrak{g}$-equivariant splitting:

$$\begin{tikzcd}
E \arrow[r, "\pi"] & TM \arrow[l, bend left, dashed, "\nabla"]
\end{tikzcd}$$

So a *Moment map* for extended actions is the following diagram:

$$\begin{tikzcd}
& \mathfrak{h} \arrow[r] \arrow[dl, dashed, "\hat{\mu}"'] \arrow[dd, "\nu"] & \mathfrak{a} \arrow[r] \arrow[dd, "\rho"] & \mathfrak{g} \arrow[dd, "\psi"] \\
C^\infty(M) \arrow[dr, "D"'] & & & \\
& \Omega^1_{\mathrm{Cl}}(M) \arrow[r] & \Gamma(E) \arrow[r] & \mathfrak{X}(M)
\end{tikzcd}$$

Then you get $\mu : M \to \mathfrak{h}^*$ which is equivariant and the moment map condition is $\ker d\mu = \mathrm{Ann}(\nu(\mathfrak{h}))$.

Example 6.2.

1. $\mathfrak{g} \to \mathfrak{X}(M)$ ordinary action
2. Symplectic action $G \curvearrowright (M, \omega)$:

   $$\psi : \mathfrak{g} \to \mathfrak{X}(M)$$

   Want extension to $E = \mathcal{T}$ ($H = 0$).

   Get a Courant algebra:

   $$0 \to \mathfrak{g} \to \mathfrak{a} = \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g} \to 0$$

   with

   - $[(g_1, h_1), (g_2, h_2)] = ([g_1, g_2], [g_1, h_2])$ “hemi-semi”

- $\rho : \mathfrak{g} \oplus \mathfrak{g} \to \Gamma(E)$ with $(g, h) \mapsto \psi(g) + i_{\psi(h)}\omega$
- $\mu : M \to \mathfrak{g}^*$ quivariant with $i_X\omega = d\mu$ a moment map.

Then say we have an exact Courant algebroid with an extended action. If $K \subset E$, let S be the leaf of $\Delta_S = \pi(K^\perp = \mathrm{Ann}(\gamma(\mathfrak{h}))$ and P the leaf of $\Delta_P = \pi(K^\perp + \pi(K) = \mathrm{Ann}(\gamma(\mathfrak{h})) + \psi(g)$.

6.3 Reduction of Courant algebroids

If $\rho(\mathfrak{a}) = K \subseteq E$ is isotropic, then

$$E_{\text{red}} = \frac{K^\perp|_P}{K|_P}/G \to M_{\text{red}}$$

is an exact Courant algebra.

6.4 GCS, GKS,...

7 Viktor Ginzburg, *Coisotropic intersections*

We plan to look at Lagrangian intersections with $L \subset (W, \omega)$ closed, φ_t Hamiltonian, and

$$L_t = \varphi_t(L) \cap L \neq \emptyset$$

There is one question here: what do I mean by $\varphi_t \approx \mathrm{id}$? For C^1, this is a theorem by Weinstein from the early 70's. For C^0, this is due to Laudenbach-Sikorov.

For H_t $t \in [0, 1]$, we have

$$\|H_t\| = \int_0^1 (\max H_t - \min H_t) dt$$

and $\|\varphi\| = \inf_{\varphi = \varphi_H} \|H\|$ is the Hofer norm. This is also called the energy of φ. Now there exists a constant $\Delta_L > 0$, $\varphi(L) \cap L = \emptyset$ which implies $\|\varphi\| > \Delta_L$, which is due to Gromov, Floer, Chekavov.

Once a submanifold is not Lagrangian, the fact that the maps we consider are Hamiltonian makes no difference. About a year ago, generalization of this fact to submanifolds of smooth co-dimension was answered, which led me to as the question about coisotropic intersections. Let's say that $M^{2n-k} \subset (W^{2n}, \omega)$ and $\ker(\omega|_M) =$ integrable distribution. Let $\mathcal{F}$ be the characteristic foliation with $\dim = k$. Now let's ask the following question: suppose φ_t is hamiltonian and $\varphi_t \approx \mathrm{id}$, is it true that $\varphi(M) \cap M \neq \emptyset$? This turns out to be connected to many other things in symplectic geometry, in particular dynamics of Poisson maps.

The first case we'd like to explore is the *infinitesimal version*: Take H Hamiltonian, X_H not tangent to M. It's rather easy to see that tangency points of X_M and M correspond to $dH|_{\mathcal{F}} = 0$ and also to critical points of H along $\mathcal{F}$. In a way, this means these things must exist in abundance. Now the coisotropic intersections are governed by the foliated morse theory of $\mathcal{F}$.

So consider the graph of $\mathcal{F}$:

$$G_M = \{(x, y) \in \overline{W} \times W | x \text{ and } y \text{ on the same leaf of } \mathcal{F}\}$$

This is in fact Lagrangian in $\overline{W} \times W$. So take $\phi_t : \overline{W} \times W \circlearrowleft$ with $\phi_t = (\mathrm{id}, \varphi_t)$. Then $\phi_t(G_M) \cap G_M$ corresponds to $\{x \in M | x \text{ and } \varphi_t(x) \text{ on the same leaf }\} \subset \varphi_t(M) \cap M$. Now the Lagrangian intersection property gives that the first part is not empty. But the problem remains that this Lagrangian "thing" is not a Lagrangian submanifold. If all leaves of $\mathcal{F}$ are closed, then it works, i.e. symplectic reduction with respect to a group action.

Furthermore, the question of leaf-wise intersection has already been studied to some extent back to Moser (1979) and Hofer (80's) and more recently by Dragov (2005). But suppose W is exact, $\omega = d\lambda$, e.g. $W = \mathbb{R}^n$, $d\lambda|_{\mathcal{F}} = 0$, and λ is closed along $\mathcal{F}$, then $[\lambda|_{\mathcal{F}}] \in H^1(\mathcal{F})$ which is the Liouville class. Is this non-empty as well?

So let's look at Poisson maps:

$$\begin{tikzcd}
\mathfrak{g} \arrow[rr, "\varphi"] & & B \\
& \text{closed,sympletic} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$\Gamma(\varphi) \subset \overline{B} \times P$, φ Poisson $\iff$ Γ_φ coisotropic (due to Weinstein). Consider when φ is a submersion and fiber bundle. Then $\mathcal{F}$ =(fibers of $\varphi)^\omega$. Note that $\mathcal{F}$ is globally flat if $\pi_1(B) = 0$.

So take φ_t a family of Poisson maps and start deforming. For an exact deformation, we have α_t exact for all t, and

$$\alpha_t = \varphi_t^* i_{\frac{\partial \varphi_t}{\partial t}} \omega_B \text{ closed 1-form on } P$$

Then φ_t is exact $\iff$ $\exists \phi_t : \overline{B} \times P$ with $\phi_t(\Gamma_\varphi) = \Gamma_{\varphi_t}$. Now look at the intersection:

$$\Gamma_\varphi \cap \Gamma_{\varphi_t} = \{x \in P | \varphi_t(x) = \varphi(x)\}$$

These points are exactly the analogs of coisotropic intersection.

Example 7.1. $\varphi = \mathrm{id}$, $P \to P = B$. This is exact $\iff$ φ_t is Hamiltonian. Arnold's conjecture is

$$\{x \in P | \varphi_t(x) = x\} = \mathrm{Fix}(\varphi_t)$$

Example 7.2.

$$\varphi_t = \varphi_B^t \varphi \varphi_P^t$$

is an exact deformation with φ_B^t and φ_P^t being Hamiltonian.

Our results are the following:

1. Take any P, B, φ, $\varphi_t \approx \varphi(C^1)$, then $\exists\{x \in P | \varphi_t(x) = \varphi(x)\}$. (follows from Moser's theorem)
2. $\pi_1(B) = 0$, $\varphi_t \approx \varphi$ (C^0), then we have the same result. (follows from Lagrangian intersection property)
3. $V = \mathbb{T}^{2n}$, $\pi_2(P) = 0$, φ_t anything, then we also have the same result. This follows from a version of coisotropic intersections.

Theorem 7.3. *Say M is a* stable *coisotropic submanifold of W (not very restrictive condition). Then $\exists\Delta_M > 0$, $\varphi(M) \cap M = \emptyset \implies \|\varphi\| > \Delta_M$.*

8 Marco Gualtieri, *D-branes on Poisson varieties*

What I'm really talking about today are holomorphic Poisson structures. For (M, I) a complex manifold and $\{\cdot,\cdot\}$ a holomorphic Poisson bracket. In the case of $\mathbb{CP}^2$, $\pi \in H^0(\Lambda^2 T) = H^0(O(3))$ and $[\pi, \pi] = 0$ is satisfied simply from the dimension.

Now given a GCS, write $\pi = P + iQ$, with P, Q real, Poisson of type $2,0 + 0,2$. Then

$$\mathcal{J} = \begin{pmatrix} I & Q \\ & -I^* \end{pmatrix}, \; \mathcal{J}^2 = \begin{pmatrix} -1 & IQ - QI^* = 0 \\ & -1 \end{pmatrix}$$

Now $\mathrm{sym}[\cdot,\cdot]_C = \mathit{Diff}(M) \ltimes \Omega^2_{\mathrm{Cl}}$ has a bigger symmetry group. Then $\mathrm{sym}(\mathcal{J}_I) = (\varphi, B)$: $e^B \mathcal{J}_I^\varphi e^{-B} = \mathcal{J}_I \iff \varphi \in \mathrm{Hol}(I)$ and B of type $(1,1)$. Often we think of B as the curvature of some unitary connection on L. Naturally, then the symmetry group is bigger than one would think.

So what are the symmetries of $\mathcal{J}_{I,\pi} = (\varphi, B)$?

$$\begin{pmatrix} 1 & \\ B & 1 \end{pmatrix} \begin{pmatrix} I^\varphi & Q^\varphi \\ & -I^{*\varphi} \end{pmatrix} = \begin{pmatrix} I & Q \\ & -I^* \end{pmatrix} \begin{pmatrix} 1 & \\ B & 1 \end{pmatrix}$$

1. $Q^\varphi = Q$
2. $I^\varphi - I = QB$
3. $BI^\varphi = -I^* B$

So actually a Poisson structure can transform a B-field into a diffeomorphism! $Q : T^* \to T$.

So how can you actually produce a solution to these equations? Say, if you have a Q, how do you produce a B that controls the change in the structure.

Theorem 8.1 (Hamiltonian Symmetry). *Let $f \in C^\infty(M,\mathbb{R})$, φ_t be Hamiltonian flow by $X = Qdf$, then (φ_t, B_t) solves 1,2,3 if*

$$B_t = \int_0^t (dd_s^c f) ds$$

This is a time dependent 2-form corresponding to the time-dependent complex structure $I^{\varphi_t} = I_t$.

Subojects: Whenever you have a GCS, there's a notion of a G.C. submanifolds which correspond to D-branes.

$$(F \in \Omega^2(S), S) \xhookrightarrow{f} (M, \mathcal{J}),\ H \in \Omega^3_{\mathrm{Cl}}(M)$$

where

1. $f^*H = dF$
2. $f_* \mathrm{Gr}_F$ is complex in $\mathcal{T}$, with $\mathrm{Gr}_F \subset \mathcal{T}_S \subset \mathcal{T}_M = \mathcal{T}$, where Gr_F is the graph of F.

Example 8.2. $(S,F) \subset M$, $\mathcal{J} = \mathcal{J}_I$, and

- F closed
- S complex submanifold
- F of type (1,1)

So what do Branes for holomorphic Poisson structures look like?

$$\begin{pmatrix} I & Q \\ & -I^* \end{pmatrix} \begin{pmatrix} X \\ FX \end{pmatrix} = \begin{pmatrix} IX + QFX \\ -I^*FX \end{pmatrix}$$

Then we have the equation:

$$\underbrace{FI + I^*F}_{2,0+0,2} + FQF = 0$$

The properties are then:

1. $Q = 0$, F of type (1,1)
2. *nonlinear* in F, so there is no obvious tensor structure on branes
3. *non-perturbative*: suppose we take tQ with $t \to 0$, and F a solution for $t = 1$. How do I find solutions for other values of t? actually, $\frac{1}{t}F$ is a solution for all t.

An easy solution is $F = 0$. Then act by Hamiltonian transformation $(S^\varphi \hookrightarrow M, F + B)$, with (φ_t, B_t), then

$$F = 0 + \underbrace{\int_0^t (dd^c_S f) ds}_{B_t}$$

solves the equation. $I + QF = I^{\varphi_t}$ is another complex structure. You get a whole set of integrable complex structures.

Second method for examples: use a generalized Kähler structure. When you think of things in GKS terms, and you take the commutator $[J_+, J_-]g^{-1} = Q$, then you get a holomorphic Poisson structure with respect to $J_\pm$ (Hermitian, not Kähler). Using the identity, $[J_+, J_-] = (J_+ - J_-)(J_+ + J_-)$, you get

$$J_+ - J_- = \underbrace{[J_+, J_-]g^{-1}}_{Q} \underbrace{g(J_+ + J_-)^{-1}}_{F}$$

For $F = g(I_+ + I_-)^{-1}$, as long as $\mathcal{J}_A$ is type 0 (for $(\mathcal{J}_A, \mathcal{J}_B)$ a GKS), then this is invertible and gives a solution to our equation for I_+. Then (I_+, Q) is a holomorphic Poisson structure and F is a brane.

Furthermore, you can go backwards by taking a brane and produce a GKS, i.e. if $F(J_+ + J_-) = g$ is positive definite, then you get a GKS. You should think of this case as the brane giving you an *ample* line bundle, or calling this an “ample brane”.

Example 8.3. Suppose we look at the holomorphic symplectic case such that π is invertible, $\pi^{-1} = \Omega^{2,0}$.

- (I, Ω), $F_t = \int_0^t (dd^c_S f)ds$ a brane solution, with $I^{\varphi_t} = I + QF_t$ with this being a Hamiltonian deformation, so there will be a bunch of points where the complex structures agree. Like deforming a Lagrangian so it intersects itself.
- But if you have a GKS solution, then find Hyper-Kähler form J with $I - J = Q(\omega_I + \omega_J)$, $\omega_I + \omega_J$ a brane. This is like deforming a completely different Lagrangian.

9 B. Tsygan, ***Deformation quantization of stacks and gerbes***

Joint work with P. Bressler, A.Gorokhovsky, and R. Nest.

Stack (for US), or a groupoid: will be a topological space with an open cover and a sheaf of algebras on every open set. We also have an isomorphism of sheaves on the intersections $G_{ij} : A_i \xrightarrow{\sim} A_j$ with $G_{ij}G_{jk} = \mathrm{Ad}(c_{ijk})G_{ik}$, $c_{ijk} \in A_i$ (on $U_i \cap U_j \cap U_k$) and the relation $c_{ijk}c_{ikl} = G_{ij}(c_{jkl}) \cdot c_{ijl}$ on U_{ijkl}. We can refine the cover and get an isomorphism of two such stacks.

Variant: instead of $\coprod_i U_i \leftleftarrows \coprod_{ij} U_i \cap U_j$, any ètale groupoid $X_0 \leftleftarrows X_1$, e.g. $\Gamma \curvearrowright X$, A_X on X_j, with $G_\gamma : A_X \xrightarrow{\sim} A_X$.

This leads to a sheaf of categories on M.

$$U \subset M : \mathcal{C}(U) = \{ \text{ Collections } \mu_i \in \mathcal{A}_i\text{-modules on } U_i$$

with maps h_{ij} between them. Then if $\mathcal{A}$ is a stack, then Collections (μ_i, h_{ij}) are twisted A-modules.

Now a deformation of a stack A is another stack $A_i[[\hbar]], *_i$ with $a *_i b = ab + \hbar \cdots$, $\mathcal{G}_{ij}(a) = G_{ij}(a) + \hbar \cdots$, and $\mathcal{C}_{ijh} = c_{ijk} + \hbar \cdots$. If we quotient the deformations of A by isomorphisms, what do we get?

During the talk, I will ask the following questions:

1. How do we describe this quotient in terms of a differential graded Lie algebra $\mathcal{L}(A)$? (the Hochschild complex of A...)

2. Let A_0 be a *gerbe*: M a manifold, $A_i = \mathcal{O}_{U_i}$, $G_{ij} = 1$ and $(c_{ijk} \in Z^2(M, \mathcal{O}^*)$ (a Cech two co-cycle) Then the quotient space will be isomorphic to $\{$ Formal twisted Poisson structures $\}$ modulo some equivalences in the C^∞ case. Then $\pi = \hbar\pi_0 + \hbar^2\pi_1 + \cdots$ with $[\pi, \pi] = \langle H, \pi \wedge \pi \wedge \pi \rangle$. In this theory we're working with a limit where the twisted Poisson structures starts with a normal Poisson structure, i.e. $[\pi_0, \pi_0] = 0$. Now this DGLA, $\mathcal{L}_A$ is equivalent to what? it's a new kind of L^∞ structure given by H (the closed 3-form).

3. Stacks arise more naturally, or canonically, than sheaves of algebras. Moreover, their obstructions to being a sheaf are non-trivial and interesting.

4. $\mathcal{L}(A)$ computes (by definition) the Hochschild cohomology of A. What about Hochschild and cyclic homology?
   Answer: (Theorem) when $A = A_0$ a gerbe: $CC^-_\bullet(A_0) \simeq \Omega^\bullet_M[[u]]$, and the differential will be $ud_{\text{dR}} + u^2 H$.
   (Theorem) Also important is that in general: $CC^-_\bullet(A)$ is a recipient of *Chern character* of (perfect) twisted A-modules

Motivating examples:

1. (M, ω), canonical stack, namely, locally on $U_i : \mathring{A}^{\hbar}_{U_i} = \mathcal{O}_{U_i}[[\hbar]]$ coming from ∇_i. We get a stack immediately from the gauge equivalence $\nabla_i \sim \nabla_j$. If M is complex, and ω holomorphic, when is this stack isomorphic to a sheaf of algebras?

2. More substantial version: (Kashiwara-Scharpira-Polescello...) a stack of algebras not over $\mathbb{C}[[\hbar, \hbar^{-1}]]$ but over $k = \partial_t{}^{-1} \sim \hbar$.

3. $M = T^*X$, $\Sigma \subset T^*X$ coisotropic (C^∞ case), then you get a foliation $\mathcal{F}$ on Σ such that $\Sigma/\mathcal{F}$ is symplectic, ω_B. B is transverse to the leaves, and the holonomy goupoid of $\mathcal{F}$ acts on B preserving ω_B. The canonical stack deformation of that! Why is it

interesting? Well, one can start with $A_\Sigma = \{$ FTOs with $WF = \underbrace{\Sigma \times_B \Sigma}_{=L_\Sigma}\}$. Then $L_\Sigma \circ L_\Sigma = L_\Sigma$ and you get $A_\Sigma(a_\hbar)_{\hbar\to 0}$ which corresponds to a canonical stack on B.

For $D \in A_\Sigma \rightsquigarrow$ Projection $P_D \in A_\Sigma \rightsquigarrow$

$$\text{Index-like invariants of } D = \langle \text{ Cocycle of } A^\hbar_\Sigma, ch(P_D)\rangle$$

Can compute cyclic homology, etc. of A_0 and its deformations...

If $\sigma : U_\sigma := U_{i_0} \cap \cdots \cap U_{i_n}$ (simplex) contained in τ with $I_\sigma \subset I_\tau$, then I want to define the Hochschild cohomology. Then $\mathcal{L}(A) = \{$ collections $\omega_\sigma \in \Omega^\bullet(\Delta^\sigma) \cdot C^\bullet_{\text{local}}(M_\sigma)$ such that $\omega_\sigma|_{\Delta^\tau} = \omega_\tau$ in some sense natural sense $\}$.

10 R. Fernandes, *Stability of symplectic leaves*

Joint with M. Crainic.

So the subject today is a very classic subject. From foliation theory, we start by fixing a manifold M and $k \in \mathbb{N}$. Then $\mathrm{Fol}_k(M) \subset \mathrm{Gr}_k(TM)$ (C^2 topology).

Definition 10.1. A compact leaf L of $\mathcal{F} \in \mathrm{Fol}_k(M)$ is called *stable* if $\exists F \in U \subset \mathrm{Fol}_k(M)$ and tubular neighborhood $V \supset L$ such that every foliation in U has a leaf in V diffeomorphic to L.

Linear holonomy: $h : \pi_1(L, x_0) \to GL(\nu(L)_{x_0})$, ($\nu(L) = TM/TL$).

Theorem 10.2 (Reeb, Thurston, ...). *If L is a compact leaf and $H^1(\pi_1(L, x_0), \nu(L)_{x_0}) = \{0\}$, then L is stable.*

Actions: Fix a manifold M and connected Lie group G. $\mathrm{Act}(G; M) \subset \{G \to \mathit{Diff}(M)\}$ in C^2-topology.

Definition 10.3. An orbit $\mathcal{O}$ of $\alpha \in \mathrm{Act}(G; M)$ is stable if every nearby action has nearby diffeomorphism to $\mathcal{O}$.

Linear isotropy representation: $\rho : G_x \to GL(\nu(G)_x)$

Theorem 10.4 (Hirsh and Stowe). *If $\mathcal{O}$ is a compact orbit and $H^1(G_x, \nu(\mathcal{O})_x) = \{0\}$, then $\mathcal{O}$ is a stable orbit.*

If G_x is discrete, and G is 1-connected, then the first theorem implies the second.

Poisson Geometry: Fix manifold M,

$$\mathrm{Poiss}(M) = \{\pi : [\pi, \pi] = 0\} \subset \Gamma(\Lambda^2 TM) \ C^2\text{-topology}$$

Definition 10.5. A symplectic leaf S of $\pi \in \mathrm{Poiss}(M)$ is *stable* if every nearby Poisson structure has a nearby symplectic leaf diffeomorphic to S.

Theorem 10.6 (Crainic, RF). *Let S be a compact symplectic leaf of $\pi \in \mathrm{Poiss}(M)$. If $H^2_\pi(M;S) = \{0\}$, then S is stable.*

Poisson cohomology: $d_\pi : \mathfrak{X}^\bullet(M) \to \mathfrak{X}^{\bullet+1}(M)$, $d_\pi = [\pi_1]$.
Relative Poisson cohomology (Ginsburg and Lu): $d_\pi : \underbrace{\mathfrak{X}^\bullet(M,S)}_{\Gamma(\Lambda^\bullet T_S M)} \to X^{\bullet+1}(M,S)$, $d_\pi Q = [\pi, \tilde{Q}]|_S$.

- If $S = \{x_0\}$, then $H^\bullet(M, \{x_0\}) = H^\bullet(\mathfrak{g}_{x_0})$
- For any leaf S: $H^\bullet(M,S) = H^\bullet(T^*_S M)$.

Corollary 10.7. *For $S = \{x_0\}$ a 0-dimensional leaf, if $H^2(\mathfrak{g}_{x_0}) = \{0\}$, then $x_)$ is a stable zero of π.*

Corollary 10.8. *If $M = \mathfrak{g}^*$, with $\mathfrak{g}$ compact, simple, then every coadjoint is stable.*

Example 10.9. For $\mathfrak{so}^*(2)$ all the spheres about the origin are stable. For $\mathfrak{sl}^*(2,\mathbb{R})$, the leaves outside the hyperbolic cone are not stable. In the case of $\mathbb{R}^2 \setminus \{0\} \times \mathbb{R}^2 \setminus \{0\}$, then $\pi = \partial_{\theta_1} \wedge \partial_{\theta_2} + a\partial_{r_1} \wedge \partial_{r_2}$.

Now in the Lie algebroid world:

$$(A, [\cdot,\cdot], \rho) \leftrightarrow \text{ Fiber-wise linear Poisson structures on } A^* \to M$$

Crainic and Maerdijk: studied deformation cohomology of A

$H^\bullet_{\mathrm{def}}(A) :=$ Cohomology of Poisson complex with FL multivector fields on $A^* \to M$

On the other hand, one can define the relative deformation cohomology for a leaf L of A:

$$H^\bullet_{\mathrm{def}}(A; L) \equiv \text{“Relative Deformation cohomology”}$$

Theorem 10.10. *If L is a compact leaf of A and $H^2_{\mathrm{def}}(A,L) = \{0\}$, then L is stable.*

Here you may vary in a direction different from the Poisson structure, so it is very different from the previous result.

Going back to Poisson manifolds:

$$H^2_\pi(M;S) \hookrightarrow H^2_{\mathrm{def}}(T^*M, L)$$

For any Lie algebroid, $H^2_{\text{def}}(A; L) = H^1(A|_L; \nu(L))$. Now assume further that $A|_L$ is integrable. Then this is the same as $H^1(\mathcal{G}(A|_L); \nu(L)) = H^1(G_x, \nu(L)_x)$ with $G_x = \mathcal{G}(A|_L)_x$ (the isotropy group $\mathcal{G}$). Then one sees that $\mathcal{G}(\mathcal{F}) = \Pi_1(\mathcal{F})$ and $G_x = \pi_(L, x)$. So deformations in the sense of Lie algebroid theory is the same as normal deformations.

So the groupoid integrating my Lie algebroid is the action groupoid: $\mathcal{G}(A) = \tilde{G} \ltimes M$. Then $H^1(\tilde{G}_x, \nu(\mathcal{O})_x) = H^1(G_x, \nu(\mathcal{O}_x))$.

Proof. (Sketch)

Fix a tubular neighborhood $\nu(S) \to M$ over S. We then have the short exact sequence

$$\nu^*(S) \to T^*_S M \xrightarrow{\pi^\sharp} TS$$

with $G : TS \to T^*_S M$. Then $\nabla_x S = [G(x), S]$, $s \in \Gamma(\nu^*(S))$. $\nu^*(S) \to S$ is a bundle of Lie algebras and $\nu(S)_x$ has a linear Poisson structure $(d_s\pi)^\perp$.

Vorobjev: (linear approximation of $d_S\pi \in \text{Poiss}(\nu(S))$)

- S (0-section) is a symplectic leaf of $d_S\pi$
- symplectic leaves of $d_S\pi$ with $\dim S$ are teh image of flat sections through $\nu_0(S)$ (zeroes of $(d\pi)^\perp$)
- $d_S\pi$ induces $(d_S\pi)^\perp$ on $\nu(S)_x$

Idea of Proof: Construct a diffeomorphism $\phi_\theta : \nu(S) \to M$ such that $\{\phi_\theta(S(x)) : x \in S,$ with $s \in \Gamma_0(\nu_0(S))\}$ are the symplectic leaves of θ whenever $\theta \sim \pi$ and $S \sim 0$. This goes in two steps:

1. Construct ϕ_0 such that $\{\phi_\theta(S(x)) : s \in \Gamma_0(\nu_0(S))\}$ contains the points of θ with rank $= \dim S$.
2. Argue by contradiction: $S_n \to 0$, $S_n \in \Gamma_0(\nu_0(S))$ and $\theta_n \to \pi$, $\theta_n \in \text{Poiss}(M)$, such that $\|\theta_n^V(S_n)\| \neq 0$. Then

   $$\frac{\theta_n^V(S_n)}{\|\theta_n^V(S_n)\|} \to 0 \neq C \in \mathfrak{X}^2(M, S) \text{ (cocycle)}$$

   and $H^2(M, S) = \{0\} \implies \exists w \in W := \Gamma_0(\nu_0(S))^\perp \subset \Gamma_0(\nu(S))$,

   $$\lim_{n \to +\infty} \frac{d}{dt} \|\theta_n^V(S_n + tw)\||_{t=0} = \|C\| \neq 0$$

   $\implies S_n$ is not a local minimum of $w \in W \mapsto \|\theta_n^V(S_n + tw)\|$ but $w \in W \mapsto \|\pi^V(w)\|$ has a structure local minimum at zero.

□

11 T. Holm, *Orbifold cohomology of abelian symplectic reductions*

Joint work with Rebecca Goldin and Alan Knutson.

We'll actually be looking at how a normally hard computation is rather easy.

11.1 Symplectic background

Suppose we have a Hamiltonian torus action on a symplectic manifold. Three main examples we should keep in mind are: 1. $S^1 \curvearrowright S^2$ by rotations, 2. $\mathbb{T}^d \curvearrowright \mathbb{R}^{2d} \to \mathbb{R}^d$, 3. coadjoint orbit $\mathbb{T} \curvearrowright \mathcal{O}_\lambda \subseteq \mathfrak{g}^* \to \mathfrak{t}^*$ where $\mathcal{O}_\lambda = SO(5)/\mathbb{T}$.

Theorem 11.1 (Atiyah, Guillemin-Sternberg). *M compact, Hamiltonian $\mathbb{T}$-space, then $\phi(M)$ is a convex polytope.*

Theorem 11.2 (Marsden-Weinstein). *The* symplectic reduction *at $\alpha \in \mathfrak{t}^*$ is*

$$M//_\alpha\mathbb{T} := \phi^{-1}(\alpha)/\mathbb{T}$$

. Then α regular $\implies$ $M//_\alpha\mathbb{T}$ is a symplectic orbifold.

11.2 Orbifolds and their cohomology

Topological space modeled on $\mathbb{R}^n/\Gamma$, Γ a finite group. It's really better to think of these as groupoids or stacks: better for sub-objects, morphisms. For an example of an orbifold, recall the Christmas ornament for Alan Weinstein's lecture last week in the Workshop.

Cohomology:

1. singular cohomology of underlying topological space

$$H^\bullet(X; R)$$

2. cohomology of the "orbifold": If $X = M/G$, then

$$H^\bullet_{\text{orb}}(X; R) := H^\bullet_G(M; R)$$

3. Chen-Ruan cohomology of X is as a vector space

$$H_{\text{CR}}(X; R) = H_{\text{orb}}(I^1(X); R)$$

   There's a funny grading and a shift corresponding to how g acts on $\nu(Y \subseteq X)$ where $Y \subset X^g$.

Consider the easier example of a sphere with a $\mathbb{Z}_2$ singularity at the north pole. Then $H^\bullet(X;\mathbb{Z}) = \frac{\mathbb{Z}[x]}{\langle x^2\rangle}$, $|x| = 2$ and $H^\bullet_{\text{orb}}(X;\mathbb{Z}) = \frac{\mathbb{Z}[x]}{\langle 2x^2\rangle}$, $|x| = 2$.

For stringy Betti numbers and stringy Hodge numbers, Chen-Ruan introduced a ring with these invariants. For an orbifold X, its first inertia orbifold is

$$I^1(X) := \{(x,(g)_{\Gamma_x})|x \in X, (g)_{\Gamma_x} \text{ is a conjugcy class in } \Gamma_x\}$$

So there's also a funny product for the Chen-Ruan cohomology:

$$I^2(X) = \{(x,(g_1)_{\Gamma_x},(g_2)_{\Gamma_x})\}$$

mapping down to

$$I^1(X) \quad (x,(g_1)_{\Gamma_x}),\ (x,(g_2)_{\Gamma_x}),\ (x,((g_1g_2)^{-1})_{\Gamma_x})$$

with $\alpha,\beta \in H_{\text{orb}}(I^1(X)) \implies \alpha\cdot\beta := (e_3)_*\left((e_1^*(\alpha)\smile e_2^*(\beta)\smile \underbrace{e(V_{\text{ob}})}_{\text{mysteriousobstruction}}\right)$.

Hard to prove associativity.

11.3 Why symplectic computations are easier

Cohomology of reductions:

Theorem 11.3 (Kirwan). *M compact, Hamiltonian $\mathbb{T}$-space. The inclusion $\phi^{-1}(\alpha) \hookrightarrow M$ induces a* surjection

$$\ker(\not\alpha) \to H^*_{\mathbb{T}}(M;\mathbb{Q}) \xrightarrow{\not\alpha} H^*_{\mathbb{T}}(\phi^{-1}(\alpha);\mathbb{Q})$$

where the first two components are often computable. When α is regular,

$$H^*_{\mathbb{T}}(\phi^{-1}(\alpha)) \cong H^*_{\text{orb}}(M//_\alpha\mathbb{T})$$

Theorem 11.4 (G-H-K). *M compact, Hamiltonian $\mathbb{T}$-space. The inclusion $\phi^{-1}(\alpha) \hookrightarrow M$ induces a surjection*

$$\bigoplus_{g\in\mathbb{T}} H^*_{\mathbb{T}}(M^g;\mathbb{Q}) \to \bigoplus_{g\in\mathbb{T}} H^*_{\mathbb{T}}(\phi^{-1}(\alpha)^g;\mathbb{Q})$$

In fact, these are $(\mathbb{R}\times\mathbb{T})$-graded rings, $\mathcal{K}$ is a graded ring map and for α regular,

$$\bigoplus_{g\in\mathbb{T}} H^*_{\mathbb{T}}(\phi^{-1}(\alpha)^g;\mathbb{Q}) \cong H^*_{\text{CR}}(M//_\alpha\mathbb{T};\mathbb{Q})$$

Comments:

1. This last part works for any $X = M/\mathbb{T}$ where M is stably complex.
2. $\bigoplus_{g\in\mathbb{T}} H^*_{\mathbb{T}}(M^g;\mathbb{Q}) \hookrightarrow \bigoplus_{g\in\mathbb{T}} H^*_{\mathbb{T}}(M^{\mathbb{T}};\mathbb{Q})$ and we can define the ring structure directly.
3. There's always a finite $\Gamma \subseteq \mathbb{T}$ so that we only compute $\bigoplus_{g\in\Gamma}$.

12 J.P. Dufour, *Stability of higher order singularities for Poisson strucutres and algebroids.*

Joint work with A. Wade.

Theorem 12.1 (Crainic-Fernandes,2004). *Let x be a singular point of a Poisson structure π such that the Lie algebra $\mathfrak{g}$ corresponding to the linear part $\pi^{(1)}$ satisfies $H^2_{\mathrm{CE}}(\mathfrak{g},\mathbb{R}) = 0$. Then x is a stable singularity (every near enough Poisson structure π' has a singular point near x)*

"Conversely", if $\mathfrak{g}$ is a Lie algebra such that for every Poisson structure π with a singular point where $\mathfrak{g}$ *is the linear part*, then this singular point is stable. Then we have $H^2(\mathfrak{g},\mathbb{R}) = 0$. Do we have the following?

$$x \text{ stable } \iff H^2(\mathfrak{g},\mathbb{R}) = 0$$

There are many *stable* singular points with $\pi^{(1)} = 0$ (hence $\mathfrak{g}$ is commutative, and thus $H^2(\mathfrak{g},\mathbb{R}) \neq 0$)

So we start with a Poisson structure π and a singularity point x. We can suppose our manifold is $\mathbb{R}^n$ and the singular point is 0. Taylor expansion:

$$\pi = \underbrace{\pi^{(k)}}_{\neq 0} + \cdots + \underbrace{\pi^{(l)}}_{\text{lhomogeneousterm}} + \cdots$$

We have the chain:

$$\mathcal{C}^{(s)}(\pi^k) : \nu_1^{(s-k+1)} \to \nu_2^{(s)} \xrightarrow{\partial_2^{(s)}} \nu_3^{(s+k-1)} \xrightarrow{\partial_3^{(s+k-1)}} \cdots$$

with $\nu_q^{(r)} = \{r \text{ homogeneous } q\text{-vectors }\}$ and

$$\partial_q^{(r)} : A \mapsto [\pi^{(k)}, A]$$

Theorem 12.2. *If $H^2\mathcal{C}^{(s)}(\pi^{(k)}) = 0$, for $s = 0, 1, ..., k-1$, then the singular point* (0) *is* stable.

When $k = 1$, $H^2(\mathcal{C}^{(0)}(\pi^{(1)}) = 0 \iff H^2(\mathfrak{g},\mathbb{R}) = 0$

$$\mathcal{C}^{(0)} : 0 \to \nu_2^{(0)} \xrightarrow{\partial_2^{(0)}} \nu_3^{(k-1)}, \ \mathcal{C}^{(k-1)} : \nu_1^{(0)} \xrightarrow{\partial_1^{(0)}} \nu_2^{(k-1)} \xrightarrow{\partial_2^{(k-1)}} \cdots$$

Let $\Sigma_k^l(\pi) =$ "the set of points x where the condition

$$\partial_2^{(0)}, ..., \partial_2^{(k-2)} \text{ are 1-1 } \mathrm{im}\partial_1^{(0)} = \ker \partial_2^{(k-1)}, \ l = \dim \mathrm{im} \partial_1^{(0)}$$

works when we replace 0 by x"

Theorem 12.3. *Under conditions of the previous theorem,*

- $\Sigma_k^l(\pi)$ *is an* l*-codimensional submanifold near the origin*
- *For* π' *(a Poisson structure) near enough to* π*,* $\Sigma_k^l(\pi')$ *is also an* l*-codimensional submanifold which is near* $\Sigma_k^l(\pi)$

Proof. (Sketch). Let $\mathcal{T}$ be a topological space, A and B finite dimensional vector spaces and M a manifold. Suppose we have also a continuous map

$$\mathcal{T} \to C^\infty(M, A), \ \ \pi \mapsto F_\pi$$

and another continuous map

$$M \times \mathcal{T} \to C^\infty(A, B), \ \ (x, \pi) \mapsto G_{x\pi}$$

I suppose $G_{x\pi}(F_\pi(x)) = 0$, $G_{x\pi}(0) = 0\ \forall x, \pi$. Also suppose that there is a point x_0 in π and π_0 in $\mathcal{T}$ such that

$$\begin{cases} F_{\pi_0}(x_0) = 0 \\ \mathrm{im} d_{x_0} f_{\pi_0} = \ker d_0 G_{x_0 \pi_0} \end{cases}$$

- ${F_{\pi_0}}^{-1}(0)$ is a submanifold of M of codimension equal to the dimension of $\mathrm{Ind}_{x_0} F_{\pi_0}$
- for π near π_0 we have the same for ${F_\pi}^{-1}(0)$ (near x_0)

Choose N_0 any complementary sub-vector space of $T_0 = \mathrm{im} d_{x_0} F_{\pi_0}$ Then:

- $G_{x\pi}|_{N_0}$ is 1-1 near 0 for (x, π) near (x_0, π_0) and you get that $G_{x_0\pi_0}|_{N_0}$ is 1-1.
- Also ${F_\pi}^{-1}(0) = {F_\pi}^{-1}(N_0)$. If $F_\pi(x) \in N_0$, $G_{x\pi}(F_\pi(x)) = 0$, $G_{x\pi}(0) = 0$ and $F_\pi(x) = 0$

So how do I get to the theorem? Take $\mathcal{T} = \{$ Poisson structure $\pi\}$ of C^{2k} topology. Then

$$F_\pi(x) : \mathcal{J}^{k-1}(\pi)(x) \to A = \{\mathrm{J}^{k-1} \text{ jet } \}$$
$$G_{x\pi} : A \to B = \{\mathrm{J}^{2k-2} \text{ jets of 3-vectors } \}$$

and $\mathcal{J}^{2k-2}[\pi, \pi] = 0$. □

For examples, $k = 1$ see [C-F]. It gets interesting at $k = 2$. Then we get a quadratic Poisson structure where its modular vector field is a linear vector field.

Theorem 12.4 (Monnies). *When* $\lambda_i + \lambda_j \neq 0$*,* $(\forall\ i, j)$*, then we have* $H^2(\mathcal{C}^0(\pi^{(2)}) = 0$*,* $H^2(\mathcal{C}^{-1}(\pi^{(2)})) = 0$*.*

What about higher order terms, $k > 3$? Then

$$\pi^{(k)} = I \wedge X^{k-1}, \ \ I = \sum x_i \partial x_i, \ \ X^{k-1} = \sum x_i^{k-1} \partial x_i$$

13 M. Crainic, *The Weil complex of a Lie algebroid*

Joint work with Camilo Arias Abad and based on previous work with others. Related to work of Mehta, Roytenberg, Severa, Weinstein (the "supermanifold picture"), Blaom ("the basic curvature"), Behrnand (action algebroids).

When trying to work with supermanifolds, I found it hard to write down formulas, and I wanted global formulas, not in coordinates, so I feel that this direction works better.

13.1 To mention a few unrelated known results

- Invariant functions: If $\mathcal{G}$ has 0-connected s-fibers, then there's a 1-1 correspondence between $\mathcal{G}$-invariant functions on M and A-invariant functions on M
- 1-cocylcles: If $\mathcal{G}$ has 1-connected s-fibers, then there's a 1-1 correspondence between $\{c \in C^\infty(\mathcal{G})$ c-multiplicative $\}$ and $\{$ sections $l \in \Gamma(A^*) : l([\alpha, \beta]) = -L_\beta(l(\alpha)) + L_\alpha(l(\beta))\}$.
- Van Est map: it's a graded algebra map

  $$\phi : H^*_{\text{diff}}(\mathcal{G}) \to H^*(A)$$

  If G has k-connected fibers, then ϕ is an isomorphism in all degrees $\leq k$. The same holds in the presence of coefficients $E \in \text{Rep}(\mathcal{G})$.
- Closed multiplicative 2-forms: If $\mathcal{G}$ has 1-connected s-fibers: Then there's a 1-1 correspondence between multiplicative closed forms $\omega \in \Omega^2(\mathcal{G})$ and bundle maps $\sigma : A \to T^*M$ satisfying:
  1. $\sigma([\alpha, \beta]) = L_\alpha(\sigma(\beta)) - L_\beta(\sigma(\alpha)) + d(\sigma(\alpha), \rho(\beta))$
  2. $(\sigma(\alpha), \rho(\beta 0) = -(\sigma(\beta), \rho(\alpha))$
- Multiplicative 1-forms: ...

13.2 "Infinitesimal" models for the cohomology of $B\mathcal{G}$

Note: We have the universal principal $\mathcal{G}$-vundle $E\mathcal{G} \to B\mathcal{G}$ and $\Omega^*(B\mathcal{G})$ is the basic part of $\Omega^*(E\mathcal{G})$.

1. If $\mathcal{G} = \text{Lie}(G)$, G-compact: the model for $\Omega^*(E\mathcal{G})$: the Weil complex

   $$W(\mathfrak{g}) = (\Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*), d_{\mathfrak{g}} + i)$$

   and the basic part gives:

   $$H^*(BG) = S^{\text{inv}}_{\mathfrak{g}^*}$$

2. If $\mathcal{G} = \text{Lie}(G)$, not necessarily compact,
$$H^*(BG) = H^*(C^*_{\text{diff}}(G; S\mathfrak{g}^*), \delta + \cdots)$$

Cartan Model If $\mathcal{G}$-comes from an action on M of a compact Lie group G: then $B\mathcal{G} = EG \times_G M$ hence
$$H^*(B\mathcal{G}) = H^*_G(M)$$

The infinitesimal model for the "De Rham complex of $E\mathcal{G} = EG \times M$" is
$$W(\mathfrak{g}_M) := W(\mathfrak{g}) \otimes \Omega^*(M) = \Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*) \otimes \Omega^*(M).$$

The basic part of this is Cartan's complex of "equivariant forms" and this computes $H^*_G(M)$.

Question: $\mathcal{G}$-Lie groupoid, A-its Lie algebroid:

1. Is there a Weil complex $W(A)$?
2. Is there an "infinitesimal model f" computing $H^*(B\mathcal{G})$?

Such cohomologies will be relevant to the computation sf the cyclic homology of the convolution algebra of $\mathcal{G}$.

13.3 The Weil complex of A

What is it? It's a bigraded algebra:
$$W(A) = \bigoplus_{k,p} W^{k,p}(A)$$
with two differentials: d increasing k and δ increasing p.

Explicitly: An element in $W^{k,p}(A)$ is a sequence. The leading component is an anitsymmetric map while the second term D_1 controls the failre of the first to be $C^\infty(M)$-linear. The third controls the second and so on. The therefore have a relation.

Differentials: d: on D_0 it is the De Rham differential action on $\Omega^k(M)$ and δ: on D_0 is Koszul.

Linear description: In the presence of a (classical) connection ∇ on A, we have an isomorphism
$$W(A) \cong W_\nabla(A) = \Gamma(\Lambda A^* \otimes SA^* \otimes \Lambda T^*M)$$
where the differential on the right hand side depend on ∇ in a non-trivial way. For instance, when $A = \mathfrak{g}_M$-associated to a Lie algebra action of $\mathfrak{g}$ on M, choosing ∇-the obvious flat connection,
$$W_\nabla(\mathfrak{g}_M) = W(\mathfrak{g}) \otimes \Omega^*(M)$$
with Kalkman's differential!

Low dimensional cocycles:

- the infinitesimal data corresponding to closed multiplicative 2-forms are precisely the cocycles which are concentrated in $W^{2,1}(A)$.
- the infinitesimal data corresponding to multiplicative 1-forms are precisely the δ-cocycles which are concentrated in $W^{1,1}(A)$.
- the infinitesimal data corresponding to closed multiplicative 1-forms are precisely the coycles which are concentrated in $W^{1,1}(A)$.

The Van Est map: will relate the two complexes:

$$\phi : \Omega^*(\mathcal{G}_*) \to W^{*,*}(A)$$

(compatible with all the structures).

Theorem 13.1. *If $\mathcal{G}$ has p_0-connected s-fibers, then ϕ induces isomorphism in the δ-cohomology in all degrees $p \leq p_0$.*

This really tells us something about the map induced by ϕ at the level of the spectral sequences associated to the double complexes.

Corollary 13.2. *The 1-1 correspondences for closed multiplicative 2-forms (and its twisted version) as well as the one for 1-forms follow as a consequence of the Van Est isomorphism.*

A way to approach $H^*(B\mathcal{G})$: make sense of

$$W^{k,p}(A) = C^p(A; S^k(\mathrm{Ad}(A)^*)$$

and then integrate.

13.4 The adjoint representation of A: Intuition

Sloppy: it should be "the only non-trivial representation of A which comes from free". May think of it as "the tangent transversal structure".

Example 13.3. 1. IF $A = \mathfrak{g}$, the usual

2. If $A = T\mathcal{F}$ (foliation), the normal bundle with the action given by the usual "Bott connection"
3. In general, we have a Lie algebra (bundle)

   $$\mathfrak{g}_A = \ker(\rho)$$

   and a foliation

   $$\mathcal{F}_A = \mathrm{im}(\rho)$$

   and the adjoint should be some combination of the two. There are two representations of A, the only problem being that they are not really vector bundles. But even when they are (A is regular), this is not exactly the correct answer
4. if A is not regular these two representations are not even vector bundles.

13.5 ∞-representations

We use Quillen's superconnections (in the algebroid setting), which we require to be flat. More precisely, an ∞-representation of A is a graded vector bundle together with a degree one map

$$D : C^*(A; E) \to C^{*+1}(A; E)$$

satisfying the Leibniz identity, and such that $D^2 = 0$. Explicitly, D is built up in several levels.

13.6 The adjoint (∞-) representations

We contruct it with teh hlelp of a classical connection ∇ on A and we will call it

$$\underset{\cdot}{+}\nabla(A) \in \mathrm{Rep}_{\infty}(A).$$

The graded bundle is

$$\mathrm{Ad}(A) = A \oplus TM$$

with A in degree 0 and TM in degree 1. The components of ∞-action are (there are only 3):

- Level 0: $\rho : A \to TM$
- Level 1: The A-connection on $\mathrm{Ad}(A)$ which we consider is the basic A connection associated to ∇
- Level 2: $\overline{R}_{\nabla} \in C^2(A; \mathrm{Hom}(TM, A))$ is the basic curvature of ∇.

Note: If one chooses another connection then the adjoints are the same.

13.7 Back to the Weil complex

Choose a connection ∇ in A. Then define

$$W_{\nabla}(A) := \Gamma(\Lambda(A^*) \otimes S(A^*) \otimes \Lambda(T^*M))$$

with two differentials:

- First write this as
  $$W_{\nabla}(A) := C^*(A; S(\mathrm{Ad}^*_{\nabla}))$$
  and take the resulting "Koszul differential"
- Then write it as
  $$W_{\nabla}(A) = \Omega^*(M; E) = C^*(TM : S(E^*))$$
  where $E = A \oplus A$ (degree 0/1). E is actually a representation up to homotopy of TM (using ∇). Take the resulting differential.

These give $W_\nabla(A)$ the same structures that $W(A)$ has.

Theorem 13.4. *For any connection ∇, one has isomorphisms compatible with all the structure:*

$$W(A) \cong W_\nabla(A)$$

IN particular, the cohomology of $S(A)$ is isomorphic to $H^*(M)$.

For equivariant cohomology: use

$$C^*_{\mathrm{diff}}(\mathcal{G}; S(\mathrm{Ad}^*_\nabla))$$

Can be done. Can one replace it by smaller complexes when $\mathcal{G}$ is proper?

14 Ping Xu, *Twisted cohomology for differentiable stacks*

Joint work with Jean-Louis Tu.

14.1 Motivation

M compact, have the isomorphism:

$$K^*(M) \otimes \mathbb{C} \xrightarrow{\mathrm{ch}} H_{\mathrm{dR}}(M, \mathbb{C})$$

Mathai-Stevenson gave the isomorphism for $\alpha \in H^3(M, \mathbb{Z})$

$$K^*_\alpha(M) \otimes \mathbb{C} \xrightarrow{\mathrm{ch}} H(\Omega^\bullet(M), d\mathrm{tr})$$

So we have the question, what about M replaced by a stack?

Definition 14.1 (differentiable stack)**.** It's the orbit space of Lie groupoids keeping track of isotropy geometry, or also can be thought of as Lie groupoids modulo Morita equivalences. (For Orbifolds, this is a bad quotient space.)

Now I'll describe the program I'll try to finish: $\mathfrak{X}$ a differentiable stack,

$$\alpha \in H^3(\mathfrak{X}, \mathbb{Z}) \rightsquigarrow C^*(\mathfrak{X}, \alpha) \supset C^\infty_C(\mathfrak{X}, \alpha)$$

Then from Connes,

$$K^*_\alpha(\mathfrak{X}) = K_*(C^*(\mathfrak{X}, \alpha)) \xrightarrow{\mathrm{ch}} HP_*(C^\infty_C(\mathfrak{X}, \alpha))$$

For $(\alpha = 0)$,

1. $\mathfrak{X}$-manifold, Connes, Pflaum.
2. orbifold, Baum-Connes, Brylinski-Nistov, Crainic
3. Quotient stack, Brylinki (abelian case), Baum-Brylinski-Macpherson, Block-Getzler,

So even without twisting, it's really hard to compute this cohomology.

14.2 de Rham Cohomology

Have $\Gamma_1 \rightrightarrows \Gamma_0$ a Lie groupoid. Then $\forall p \in \mathbb{N}$,

$$\Gamma_p = \{(x_1 ... x_p) | x_1 ... x_p \text{ makes sense } \}$$

So then the de Rham cohomology of the groupoid is the cohomology of the double complex:

$$H^*_{\mathrm{dR}}(\Gamma) = H(\Omega(\Gamma_\bullet), \delta)$$

with $\delta = \pm\partial + d$.

So what do I mean by Morita equivalence? Suppose I have the groupoid $\Gamma_1 \rightrightarrows \Gamma_0$, and a surjective submersion $\varphi : X \to \Gamma_0$, then

$$\Gamma[x] = x \times \Gamma \times x = \{(x, \gamma, y) | \varphi x = s\gamma, \varphi y = t\gamma\}$$

Definition 14.2. So $\Gamma_1 \sim \Gamma_2$ Morita equivalent if $\exists \Gamma$ and Morita morphism $\Gamma_1 \leftarrow \Gamma \to \Gamma_2$.

Proposition 14.3. $G \sim_M H \implies H^*_{\mathrm{dR}}(G) \simeq H^*_{\mathrm{dR}}(H)$

Example 14.4.

1. M a manifold, $\varphi : X \to M$ a surjective submersion, then in particular given an open cover, then $\varphi : \coprod U_i \to M$ and you can get the Cech groupoid for $X \times_M X : \coprod U_{ij} \rightrightarrows X : \coprod U_i$.
2. $\mathfrak{X}$-orbifold: can be represented by an ètale proper groupoid $\Gamma_1 \rightrightarrows \Gamma_0$.

   Lemma 14.5. $H^*_{\mathrm{dR}}(\mathfrak{X}) = H^*_{\mathrm{dR}}(\Gamma_0) = H^*(\Omega(\Gamma_0), d)$
3. $\mathfrak{X} = (M/G)$: G acts on M smoothly from the right. Then $\Gamma : M \times G \rightrightarrows \Gamma_0 : M$ by $s(m, g) = m$, $t(m, g) = mg$, $(m, g)(n, h) = (m, gh)$.

14.3 S^1-gerbes over a differentiable stack

Recall that an S^1-bundle over $\mathfrak{X}$ modulo equivalences is equal to $H^2(\mathfrak{X}, \mathbb{Z})$, so then a gerbe is just a geometric realization of classes in $H^3(\mathfrak{X}, \mathbb{Z})$.

Then an S^1-gerbe over a differential stack is then the S^1-central extension of Lie groupoids modulo Morita equivalences.

Definition 14.6. is a morphism of groupoids such that $\ker \varphi = \Gamma_0 \times S^1$ and is in the center of $\widetilde{\Gamma}$.

Remark 14.7. $\varphi : \widetilde{\Gamma} \to \Gamma$ is an S^1-principle bundle.

One can define a connection and curving:

$$\widetilde{\Gamma} \to \Gamma$$

Connection: $\theta \in \Omega^1(\widetilde{\Gamma})$ connection 1-form for the S^1-bundle such that $\partial\theta = 0$.
Curving: given θ, a 2-form $B \in \Omega^2(\Gamma_0)$ such that $d\theta = \partial B$.
3-curvature: given $(\theta \cdot B)$,

$$\Omega = dB \in \Omega^3(\Gamma_0)^{\Gamma}, \;\; \partial\Omega = \partial dB = d\partial B = 0$$

and $[\Omega] \in H^3_{\mathrm{dR}}(\Gamma_\bullet)$.

Proposition 14.8 (Behrend-Xu). $[\Omega]$ *integer class is actually the Diximier-Douady class.*

Theorem 14.9 (Tu-Xu). *If $\mathfrak{X}$-orbifold, or $[M/G]$, with G acting on M properly, and any $\alpha \in H^3(\mathfrak{X}, \mathbb{Z})$, then $\exists$ an S^1-groupoid central extension $\widetilde{\Gamma} \to \Gamma$ admitting connections and curving, (θ, B) such that $[\Omega] = \alpha$.*

14.4 Transgression map and twisted cohomology

Real quickly, using an S^1-central extension with connection and curving on Γ, under the transgression map, you get a flat line bundle over the inertial groupoid Γ. From this you can arrive at the twisted cohomology. We don't have time to continue into more detail.

15 S. Waldmann, *Locally noncommutative space-times*

Joint work with D. Bahns (formal), Heller (C^*-algebra).

This topic really has Poisson geometry at the boundary, so it more so has Poisson geometry as an application.

15.1 Motivation

If you were interested in fundamental physics, there are 3 main interactions, and then gravity which you can't attack with normal quantum theory. At the moment, there's not field theory to use with gravity. We have to expect some interaction between

$$\text{gravity} \;\oplus\; \text{QFT}$$

at small distances, i.e. 10^{-33} cm, Planck's scale.

As an intermediate step, people have recently discussed non-commutative space times. So what are the hopes that this hopes to resolve:

- introduces some sort of non-commutativity, the singularities in normal general relativity become more smooth, i.e. somewhat solvable.

- QFT behaves better on such non-commutative space-times because of certain uncertainty relations, and maybe the problems will disappear

Example 15.1 (NC Minkowski space). $M = \mathbb{R}^4$, η flat metric, $\theta \in \Gamma^\infty(\Lambda^2 TM)$ constant.

$$f \star g = \mu \circ e^{\frac{i\lambda}{2}\theta^{kl}\partial_k \otimes \partial_l} f \otimes g$$

with $\mu(f \otimes g) = fg$ the normal Weyl-Moyal product and λ the Planck area.

Note: this is highly non-geometric and most likely, physicists use this because it's simple but I'm not advocating this as a model by any means. This is the main source of criticism.

The main idea doesn't depend so much on the star product you use, but rather to study field theory on M_θ, i.e. based on $(C^\infty(\mathbb{R}^4)[[\lambda]], \star)$. There are many problems: (Conceptual) $\theta, \star$ constant, not at small distances but they show up everywhere. This is somehow believed to be at the root of doing fourier transforms of these constant global objects which gives singularities.

I want to set up:

- geometric formulation
- generic space-times
- formal/C^*-algebraic deformation quantization
- only works at small distances

Then the main idea is to consider $M \rightsquigarrow M \times M$. We then need to talk about at least two points to have a notion of distance and also to look at some non-commutative support close to $\Delta_M \subset M \times M$.

So take M pseudo-riemannian (actually need only ∇) and choose: $U \subseteq TM$ an open neighborhood of M, $V \subset M \times M$ an open neighborhood of Δ_M. Then $\phi : U \xrightarrow{\sim} V$ with $\phi(u_p) = (\exp_p(-v_p), \exp_p(+v_p))$.

15.2 Noncommutativity at small distances

Consider $\widetilde{\theta} \in \Gamma^\infty(\Lambda T(M \times M))$, $[\widetilde{\theta}, \widetilde{\theta}] = 0$, $\mathrm{supp}\widetilde{\theta} \subseteq V$ (close to Δ), Then θ and $\phi^*\widetilde{\theta}$ is our Poisson structure on TM with supp $\subseteq U$. We require $\mathrm{supp}\theta \cap T_pM$ to be compact. and to be symmetric under flips

$$v_p \mapsto -v_p, \ (q, q') \leftrightarrow (q', q)$$

Then under the semi-classical version, choose the corresponding $\widetilde{\star}$ on $M \times M$ with $\star$ on TM and $\mathrm{supp}\star \subseteq \mathrm{supp}\theta$.

So this is not yet what we really want. This already achieves non-commutativity at small distances, but I'd like to strengthen the whole thing so that the non-commutativity is only at small distances. So we require further that if $f|_V$ is constant along the relative coordinates, ϕ, it should behave like a scalar, i.e.

$$\forall g, \quad f\widetilde{\star}g = fg = g\widetilde{\star}f$$

iff on TM: $u \in C^\infty(M)$

$$\pi^*u \star g = \pi^*ug = g \star \pi^*u$$

This means that $\star, \theta$ are *vertical*, i.e. multidifferential of D is called vertical if we can do the following:

$$D(f_1, ..., f_r\pi^*u, ..., f_k) = \pi^*uD(f_1, ..., f_k), \quad \forall r$$

More geometrically this means they differentiate only along the fibers. Then the question is can we arrange this?

Example 15.2.

1. $\alpha \in \Gamma^\infty(\Lambda^2 TM)$ a vertical lift to $\alpha^\vee \in \Gamma^\infty(\Lambda^2 T(TM))$ which is vertical. Poisson structure, but constant along the fibers
2. locally: choose $X_1, ..., X_d$ on $\mathbb{R}^d$ such that
   (a) $\mathrm{supp} X_i \subseteq B_\varepsilon(0)$
   (b) $X_i(0) = e_i$
   (c) $[X_i, X_j] = 0$

   on $TM = U \times \mathbb{R}^d$ put the X_i in the fiber direction and consider

$$\theta = \frac{1}{2}\theta^{ij} X_i \wedge X_j, \quad \theta^{ij} \text{ constant}$$

For the second example, we need the following theorem for an arbitrary vertical Poisson structure:

Theorem 15.3. *For any vertical θ there exists a vertical $\star$ with* $\mathrm{supp}\star = \mathrm{supp}\theta$. *In fact:*

$$\{ \textit{ formal vertical } \theta \ \}/\{ \textit{ formal vertical diffeomorphisms } \}$$

in 1-1 correspondence with

$$\{ \textit{ formal vertical } \star \ \}/\{ \textit{ formal vertical equivariance } \}$$

Proof. (Idea).
Step 1: Vertical HKR Theorem

$$U^{(1)}_{\text{ver}} : \Gamma^\infty_{\text{ver}}(\wedge^\bullet T(TM)) \xrightarrow{\sim} HH^\bullet_{\text{ver}}(C^\infty(TM))$$

Step 2: Extend $U^{(1)}$ to L_∞-morphism by Kontsevich's L_∞-morphism on $\mathbb{R}^d$. □

15.3 Vertical $*$-products

So we can use the inclusion maps:

$$i_p"T_pM \hookrightarrow TM$$

and the Poisson map for $\theta|_{T_pM}$, then $i_p^*\{f,g\}_\theta = \{i_p^*f, i_p^*g\}_{\theta_p}$ is the same for $\star \rightsquigarrow \star_p$. We then push forward $\theta_p, \star_p$, to M via $\exp_p$. So we use $(C^\infty(M)[[\lambda]], \widetilde{\star}_p)$ to relate the two points (q, q') under a geodesic midpoint that takes us to the neighborhood where exp is a diffeomorphism. We can measure $d^2 \in C^\infty(TM)$ by $d^2(u_p) = g_p(v_p, v_p)$ which is equal to $\delta_{v_p}(d^2)$ and replace $\delta_{v_p} \rightsquigarrow w_{u_p} = \delta_{v_p} + \cdots$ positive functions with respect to $\star$.

15.4 C^*-algebraic version

In general I don't know how to do it. But in this case of commuting vector fields, then there is a C^*-algebraic version because in the area of compact support, they have commuting flows. Then you're in the game of Rieffel of deforming the functions on the manifold.

15.5 Outlook

Some of the remaining questions are perhaps doubling the degrees of freedom by replacing M by $M \times M$. There are some interpretational issues here. Also, we can also deform the δ, but it's certainly not unique and are the bounds on this. Furthermore, we usually look for transitive actions, what about this? Lastly, what about non-commutativity when you're looking at 3 or 4 points coming together. So we can repeat the whole game for higher order cartesian products and maybe use them for regularization. It's not easy what verticality should be in that case.