SOLUTIONS TO WORKSHEET # 3.

① Let $\mathcal{B} = \{1, 1+t, 1+t+t^2\}$ and $\mathcal{C} = \{2, 2-t, 1-t^2\}$ be two bases of $\mathbb{P}_2$. Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ & $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$.

<u>Solution:</u> $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left[[1]_{\mathcal{C}} \ [1+t]_{\mathcal{C}} \ [1+t+t^2]_{\mathcal{C}} \right]$

To find $[1]_{\mathcal{C}}$, first write $1 = a(2) + b(2-t) + c(1-t^2)$

$\rightsquigarrow \quad 1 = (2a+2b+c) + (-b)t + (-c)t^2$

$\rightarrow \quad \begin{cases} 2a+2b+c = 1 \\ -b = 0 \\ -c = 0 \end{cases}$ so $a = \frac{1}{2}$, $b = c = 0$

i.e. $[1]_{\mathcal{C}} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$

Similarly $[1+t]_{\mathcal{C}} = \begin{bmatrix} \frac{3}{2} \\ -1 \\ 0 \end{bmatrix}$ and $[1+t+t^2] = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

So $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$ & $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1}$.

② Let $\mathcal{B} = \{\vec{v_1} \cdots \vec{v_n}\}$ & $\mathcal{C} = \{\vec{w_1} \cdots \vec{w_n}\}$ be 2 bases of a vector space V. Show that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left[[\vec{v_1}]_{\mathcal{C}} \cdots [\vec{v_n}]_{\mathcal{C}} \right]$

<u>Solution:</u> Let $\vec{x} \in V$. We want to show: $\left[[\vec{v_1}]_{\mathcal{C}} \cdots [\vec{v_n}]_{\mathcal{C}} \right] [\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{C}}$

Since $\mathcal{B}, \mathcal{C}$ are bases, $\vec{x} = x_1 \vec{v_1} + \cdots + x_n \vec{v_n}$ for some $x_1, \dots, x_n$
$\vec{x} = y_1 \vec{w_1} + \cdots + y_n \vec{w_n}$ $\quad y_1 \cdots y_n$ in $\mathbb{R}$

i.e. $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $[\vec{x}]_{\mathcal{C}} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$.

We also have:
$$\begin{aligned} \vec{v_1} &= a_{11}\vec{w_1} + \cdots + a_{n1}\vec{w_n} \\ &\ \vdots \\ \vec{v_n} &= a_{1n}\vec{w_1} + \cdots + a_{nn}\vec{w_n} \end{aligned}$$
for some $a_{ij} \in \mathbb{R}$, $1 \le i,j \le n$

so then $\vec{x} = x_1\vec{v_1} + \cdots + x_n\vec{v_n}$

$$= x_1(a_{11}\vec{w_1} + \cdots + a_{n1}\vec{w_n}) + \cdots + x_n(a_{1n}\vec{w_1} + \cdots + a_{nn}\vec{w_n})$$

$$= (a_{11}x_1 + \cdots + a_{1n}x_n)\vec{w_n} + \cdots + (a_{n1}x_1 + \cdots + a_{nn}x_n)\vec{w_n}$$

So then $\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = y_1 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = y_n \end{cases}$ i.e. $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$.

but $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [\vec{x}]_{\mathcal{B}}$ $\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = [\vec{x}]_{\mathcal{E}}$ & $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \left[[\vec{v_1}]_{\mathcal{E}} \cdots [\vec{v_n}]_{\mathcal{E}} \right]$.

③ Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 1 & 6 \end{bmatrix}$ (This matrix may be different from the original version). Find a basis for Row A. Does this basis span Col A?

Solution:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 1 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} \right\}$ is a basis for Row (A)

This means Row(A) $= \mathbb{R}^3$ & since Col A $\leq \mathbb{R}^3$ & dim Col A = dim Row A = 3

Col A $= \mathbb{R}^3$. So $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} \right\}$ is actually a basis for $\mathbb{R}^3$.

④ Let $T: \mathbb{P}_2 \to \mathbb{R}^3$ be: $T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$. Find Ker T, range T.

Is T 1 to 1? Is T onto?

Solution: $\operatorname{Ker}(T) = \{p \mid T(p) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\} = \{p \mid p(0) = 0 \text{ \& } p(1) = 0\}$

If $p(t) = a + bt + ct^2$ & $p(0) = 0$ & $p(1) = 0$ then $\begin{cases} a = 0 \\ a + b + c = 0 \end{cases}$ so $\begin{cases} a = 0 \\ b = -c \end{cases}$

i.e. $\operatorname{Ker}(T) = \{rt - rt^2 \mid r \in \mathbb{R}\}$.

$\operatorname{Range}(T) = \{T(p) \mid p \in \mathbb{P}_2\} = \left\{ \begin{bmatrix} a \\ a+b+c \end{bmatrix} \mid \text{where } a, b, c \in \mathbb{R} \right\} = \mathbb{R}^2$

So T is not one to one because Ker T $\neq \{0\}$.

T is onto because range T $= \mathbb{R}^2$.