QUIZ #7

① (7 pts) Given is the initial-value problem:

$$y'' - 3y' + 2y = te^{3t} \qquad y(0) = 1 \;, \; y'(0) = 2$$

Solve the problem using both methods: undetermined coefficients and variation of parameters. (You should get the same answers at the end).

Characteristic equation: $r^2 - 3r + 2 = 0 \;\rightarrow r = 1, 2$

$\rightarrow \boxed{y_h(t) = c_1 e^t + c_2 e^{2t}}$ | 2pts

<u>To find y_p:</u>

i) <u>Undetermined coefficients:</u> $y_p(t) = (At + B)e^{3t}$

$$y_p'(t) = Ae^{3t} + 3(At+B)e^{3t}$$

$$y_p''(t) = 3Ae^{3t} + 9(At+B)e^{3t} + 3Ae^{3t}$$

$$y_p''(t) - 3y_p'(t) + 2y_p(t) = 2Ate^{3t} + (3A + 2B)e^{3t} = te^{3t} + 0 \cdot e^{3t}$$

$$\rightarrow \begin{cases} 2A = 1 \\ 3A + 2B = 0 \end{cases} \longrightarrow \begin{matrix} A = \frac{1}{2} \\ B = -\frac{3}{4} \end{matrix}$$

So $\boxed{y_p(t) = \left(\frac{1}{2}t - \frac{3}{4}\right)e^{3t}}$

2pts

ii) <u>Variation of parameters:</u> $y_p(t) = v_1(t)e^t + v_2(t)e^{2t}$ $\quad W(e^t, e^{2t}) = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix} = e^{3t}$

$$v_1 = \int -\frac{te^{3t}e^{2t}}{e^{3t}}\,dt = \int -te^{2t}\,dt$$

$$\begin{matrix} u = -t \\ dv = e^{2t}dt \end{matrix} \longrightarrow \begin{matrix} du = -dt \\ v = \frac{1}{2}e^{2t} \end{matrix} \rightsquigarrow v_1 = -\frac{1}{2}te^{2t} + \int \frac{1}{2}e^{2t}dt = -\frac{1}{2}te^{2t} + \frac{1}{4}e^{2t}$$

2pts

$$v_2 = \int \frac{te^{3t}e^t}{e^{3t}}\,dt = \int te^t\,dt \qquad \begin{matrix} u = t \\ dv = e^t dt \end{matrix} \quad \begin{matrix} du = dt \\ v = e^t \end{matrix}$$

$$\rightarrow v_2 = te^t - \int e^t dt = te^t - e^t$$

So $y_p = \left(-\frac{1}{2}te^{2t} + \frac{1}{4}e^{2t}\right)e^t + (te^t - e^t)e^{2t} = \frac{1}{2}te^{3t} - \frac{3}{4}e^{3t} = \boxed{\left(\frac{1}{2}t - \frac{3}{4}\right)e^{3t}}$

So the general solution is : $y(t) = y_h(t) + y_p(t) = c_1 e^t + c_2 e^{2t} + (\frac{1}{2}t - \frac{3}{4})e^{3t}$

$y(0) = 1 \rightarrow c_1 + c_2 - \frac{3}{4} = 0$

$y'(t) = c_1 e^t + 2c_2 e^{2t} + 3(\frac{1}{2}t - \frac{3}{4})e^{3t} + \frac{1}{2}e^{3t}$

$\rightarrow y'(0) = 2 \rightarrow c_1 + 2c_2 - \frac{7}{4} = 0$

So we have: $\begin{cases} c_1 + c_2 = \frac{3}{4} \\ c_1 + 2c_2 = \frac{7}{4} \end{cases} \rightarrow \begin{matrix} c_1 = -\frac{1}{4} \\ c_2 = 1 \end{matrix}$

1 pt

$\therefore$ The solution to the initial-value problem is:

$y(t) = -\frac{1}{4}e^t + e^{2t} + (\frac{1}{2}t - \frac{3}{4})e^{3t}$.

② (3 pts) Given is the ODE: $y''' + 3y'' - 4y' - 6y = 0$.

Find 3 linearly independent solutions to this ODE. Use the Wronskian to check your answer.

Characteristic equation: $r^3 + 3r^2 - 4r - 6 = 0$

$\rightarrow r^3 + r^2 + 2r^2 + 2r - 6r - 6 = 0$

pts $\rightarrow r^2(r+1) + 2r(r+1) - 6(r+1) = 0 \rightarrow (r+1)(r^2 + 2r - 6) = 0$

$\rightarrow r = -1, -1 \pm \sqrt{7}$

$\rightarrow \{e^{-t}, e^{(-1+\sqrt{7})t}, e^{(-1-\sqrt{7})t}\}$ are linearly independent solutions to the ODE

Check using Wronskian: $W(e^{-t}, e^{(-1+\sqrt{7})t}, e^{(-1-\sqrt{7})t})(t) =$

$$\begin{vmatrix} e^{-t} & e^{(-1+\sqrt{7})t} & e^{(-1-\sqrt{7})t} \\ -e^{-t} & (-1+\sqrt{7})e^{(-1+\sqrt{7})t} & (-1-\sqrt{7})e^{(-1-\sqrt{7})t} \\ e^{-t} & (-1+\sqrt{7})^2 e^{(-1+\sqrt{7})t} & (-1-\sqrt{7})^2 e^{(-1-\sqrt{7})t} \end{vmatrix}$$

1 pt

$$\rightarrow W(e^{-t}, e^{(-1+\sqrt{7})t}, e^{(-1-\sqrt{7})t})(0) = \begin{vmatrix} 1 & 1 & 1 \\ -1 & -1+\sqrt{7} & -1-\sqrt{7} \\ 1 & (-1+\sqrt{7})^2 & (-1-\sqrt{7})^2 \end{vmatrix} \neq 0$$