QUIZ #5

1 a) (2pts) Is $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ diagonalizable? Why or why not?

b) (2 pts) Diagonalize $\begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$.

a) NO because:

$$\det \begin{bmatrix} -1-x & 1 & 0 \\ 0 & -1-x & 0 \\ 0 & 0 & 2-x \end{bmatrix} = (-1-x)(-1-x)(2-x) = 0 \rightarrow x = -1, 2$$

$$x = -1 \longrightarrow \text{Null}(A+I) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \rightsquigarrow x_2 = x_3 = 0 \quad x_1 \text{ free}$$

So $\dim(\mathcal{E}_{\lambda=-1}) = 1 < 2$

$x = 2$ is not a repeated root $\rightarrow \dim(\mathcal{E}_{\lambda=2}) = 1$

So $\dim(\mathcal{E}_{\lambda=-1}) + \dim(\mathcal{E}_{\lambda=2}) = 2 < 3 \rightarrow A$ is not diagonalizable.

b) $\det \begin{bmatrix} 5-x & -5 \\ 1 & 1-x \end{bmatrix} = (5-x)(1-x) + 5 = x^2 - 6x + 10 = 0$

$$\rightarrow x = \frac{6 \pm \sqrt{36-40}}{2} = 3 \pm i$$

$$\lambda = 3+i \rightarrow \text{Null}\begin{bmatrix} 5-3-i & -5 \\ 1 & 1-3-i \end{bmatrix} = \text{Null}\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} = \text{Null}\begin{bmatrix} 2-i & -5 \\ 2-i & -5 \end{bmatrix}$$

So x_2 is free $x_1 = \dfrac{5x_2}{2-i} \rightsquigarrow \mathcal{E}_{\lambda=3+i} = \text{span}\left\{\begin{bmatrix} \frac{5}{2-i} \\ 1 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} 2+i \\ 1 \end{bmatrix}\right\}$

$$\lambda = 3-i \rightarrow \text{Null}\begin{bmatrix} 5-3+i & -5 \\ 1 & 1-3+i \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 2-i \\ 1 \end{bmatrix}\right\}.$$

So $D = \begin{bmatrix} 3+i & 0 \\ 0 & 3-i \end{bmatrix} \quad P = \begin{bmatrix} 2+i & 2-i \\ 1 & 1 \end{bmatrix}.$

(2) (3 pts) Compute A^{15} where

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Diagonalize A to get $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ & some P (doesn't matter what P)

Such that $A = PDP^{-1}$

$\rightarrow A^{15} = PD^{15}P^{-1}$

but $D^{15} = \begin{bmatrix} 1^{15} & 0 & 0 \\ 0 & 1^{15} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = D$ so $A^{15} = PDP^{-1} = A$

i.e. $A^{15} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(Another way to do this is to realize that $A^2 = A \rightarrow A^{15} = A$).

(3) (3 pts) Let $T: \mathbb{P}_2 \rightarrow \mathbb{R}^3$ be a linear transformation &

$$T(p(t)) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix}$$

Let $\mathcal{B} = \{1, 2-t, 1+t+t^2\}$, $\mathcal{E} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ be 2 bases

for $\mathbb{P}_2, \mathbb{R}_3$ respectively. Compute $[T]_{\mathcal{E} \leftarrow \mathcal{B}}$.

$[T]_{\mathcal{E} \leftarrow \mathcal{B}} = \left[[T(1)]_{\mathcal{E}} \ [T(2-t)]_{\mathcal{E}} \ [T(1+t+t^2)]_{\mathcal{E}} \right]$

$T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ so $[T(1)]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T(\cancel{2}-t) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \rightarrow [T(2-t)]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$T(1+t+t^2) = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \rightarrow [T(1+t+t^2)]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$

$\rightarrow [T]_{\mathcal{E} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 1 & 0 & 7 \end{bmatrix}$