Math 1A Mock Final

1)Let f be a function such that $|f(x+h)-f(x)| \leq h^2$ for all x and h. Show that f is a constant.

We'll show: $f'(x) = 0$ for all x since this will imply that f is a constant.

By definition: $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

So $|f'(x)| = \lim_{h \to 0} \left| \frac{f(x+h)-f(x)}{h} \right|$

$= \lim_{h \to 0} \frac{|f(x+h)-f(x)|}{|h|}$

But $|f(x+h)-f(x)| \leq h^2$ so $|f'(x)| \leq \lim_{h \to 0} \frac{h^2}{|h|} = \lim_{h \to 0} |h| = 0$

$\Rightarrow |f'(x)| \leq 0$ and since $|f'(x)| \geq 0$, $|f'(x)| = 0$

Therefore, $f'(x) = 0$.

2)a)Prove that if $f'(x) = 0$ for all x in (a, b) then f is constant on $[a, b]$.
b)Prove that if $f'(x) = g'(x)$ for all x in (a, b) then there is a constant C such that $f(x) = g(x) + C$ for all x in $[a, b]$.

a) Take any $x_1 \neq x_2 \in [a,b]$, we'll show $f(x_1) = f(x_2)$

Mean Value Theorem shows that there is a c between x_1, x_2

such that $f'(c) = \dfrac{f(x_1) - f(x_2)}{x_1 - x_2}$

Since $c \in (a,b)$, $f'(c) = 0$ i.e. $\dfrac{f(x_1) - f(x_2)}{x_1 - x_2} = 0$

So $f(x_1) - f(x_2) = 0 \Rightarrow f(x_1) = f(x_2)$

Therefore, f is constant.

Alternatively, since any antiderivative of 0 is a constant and f is an antiderivative of $f' \equiv 0$, f is a constant.

b) This is an application of a). Let $h(x) = f(x) - g(x)$.

$\Rightarrow h'(x) = f'(x) - g'(x) = 0$ so h is a constant, say C, by a).

$\Rightarrow C = f(x) - g(x)$ hence $f(x) = g(x) + C$.

3) Sketch the graph of $f(x) = e^{-1/(x+1)}$.

Domain: $(-\infty, -1) \cup (-1, +\infty)$

Intercepts: $(0, \frac{1}{e}) \rightarrow$ y-intercept

No x-intercept.

Asymptotes: $\lim\limits_{x \to +\infty} e^{-\frac{1}{x+1}} = 1 = \lim\limits_{x \to -\infty} e^{-\frac{1}{x+1}} \rightarrow y = 1$: horizontal asymptote

$$\left.\begin{array}{l} \lim\limits_{x \to -1^+} e^{-\frac{1}{x+1}} = 0 \\ \lim\limits_{x \to -1^-} e^{-\frac{1}{x+1}} = +\infty \end{array}\right\} \quad x = -1 \text{ vertical asymptote.}$$

Intervals of increase, decrease, min, max:

$$f'(x) = e^{-\frac{1}{x+1}} \left(\frac{1}{x+1}\right)^2 = \frac{1}{(x+1)^2 e^{+\frac{1}{x+1}}} > 0 \rightarrow f \text{ always increases}$$

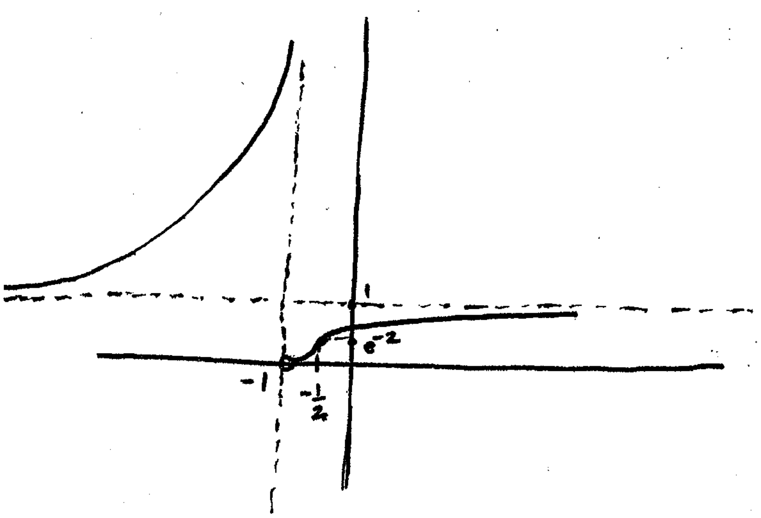
Concavity & points of inflection:

$$f''(x) = \frac{-1}{(x+1)^4 e^{\frac{2}{x+1}}} \left(2(x+1)e^{\frac{1}{x+1}} + (x+1)^2 e^{\frac{1}{x+1}} \left(\frac{-1}{(x+1)^2}\right)\right)$$

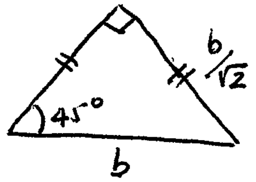
$$= \frac{-(2x+1)}{(x+1)^4 e^{\frac{1}{x+1}}}$$

$f''(x) = 0 \rightarrow x = -\frac{1}{2} \rightarrow$ inflection point $(-\frac{1}{2}, e^{-2})$

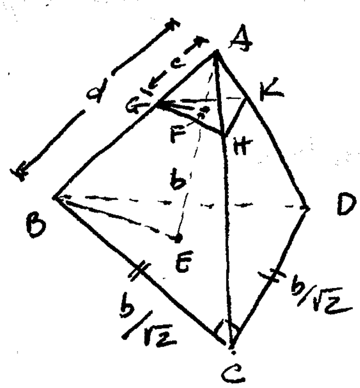
x		-1		$-\frac{1}{2}$	
f''	$+$	‖	$+$	0	$-$
f	CU	‖	CU		CD



4)Find the volume of a pyramid with height h and whose base is an isosceles right triangle with hypotenuse of length b.



The side of the right triangle has length $b\sin 45^\circ = \frac{b}{\sqrt{2}}$

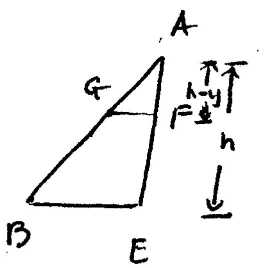


Each cross section is an isosceles right triangle similar to the base triangle.

Since ΔAGF is similar to ΔABE

$$\frac{AG}{AB} = \frac{AF}{AE} \quad \text{i.e.} \quad \frac{c}{d} = \frac{h-y}{h} \qquad (1)$$

Since ΔAGH is similar to ΔABC



$$\frac{AG}{AB} = \frac{GH}{DC} \Rightarrow \frac{c}{d} = \frac{GH}{b/\sqrt{2}} \qquad (2)$$

$$(1) \ \& \ (2) \Rightarrow \frac{h-y}{h} = \frac{GH}{b/\sqrt{2}}$$

$$\Rightarrow GH = \frac{b(h-y)}{\sqrt{2}\,h}$$

So area of ΔGHK (the cross section) is $\frac{1}{2}GH^2 = \frac{b^2(h-y)^2}{4h^2}$

So the volume of the pyramid is : $V = \int_0^h \frac{b^2(h-y)^2}{4h^2}\,dy$

$$\rightarrow V = \frac{b^2}{4h^2}\int_0^h (h^2 - 2hy + y^2)\,dy = \frac{b^2}{4h^2}\left(h^2y - hy^2 + \frac{y^3}{3}\Big|_0^h\right)$$

$$= \frac{b^2}{4h^2}\left(h^3 - h^3 + \frac{h^3}{3}\right) = \frac{b^2h}{3.4} = \frac{b^2h}{12}.$$

5)Show that the following two limits are equal.

$$\lim_{n\to\infty}\frac{1}{n}(e^{(1+1/n)^2}+e^{(1+2/n)^2}+\ldots+e^{(1+n/n)^2})$$

$$\lim_{n\to\infty}\frac{3}{n}\left(\frac{e^{(1)}}{2\sqrt{1}}+\frac{e^{(1+3/n)}}{2\sqrt{(1+3/n)}}+\frac{e^{(1+6/n)}}{2\sqrt{(1+6/n)}}+\cdots\frac{e^{(1+3(n-1)/n)}}{2\sqrt{1+3(n-1)/n}}\right).$$

The two limits are limits of Riemann sums:

+ The first is the integral $\int_1^2 e^{x^2}dx$ (the sum is computed using the right-hand point method).

+ The second is the integral $\int_1^4 \frac{e^u}{2\sqrt{u}}\,du$ (the sum is computed using the left-hand point method).

To show the limits are the same, it suffices to show

$$\int_1^2 e^{x^2}dx = \int_1^4 \frac{e^u}{2\sqrt{u}}\,du$$

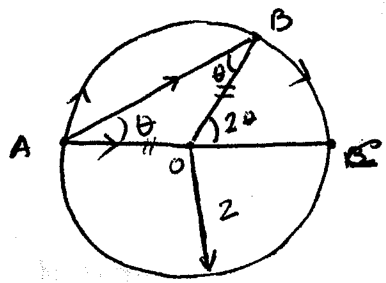
Here we use substitution: let $u = x^2 \Rightarrow du = 2x\,dx$

i.e. $dx = \frac{du}{2x} = \frac{du}{2\sqrt{u}}$

when $x = 1$, $u = 1$; $x = 2$, $u = 4$

So $\int_1^2 e^{x^2}dx = \int_1^4 e^u \cdot \frac{du}{2\sqrt{u}} = \int_1^4 \frac{e^u}{2\sqrt{u}}\,du$.

6)A woman at a point A on the shore of a circular lake of radius 2 mi wants to arrive at the point C diametrically opposite A on the other side of the lake in the shortest possible time. She can walk at the rate of 4 mi/h and row a boat at a rate of 2mi/h. How should she proceed?



The woman can row to B from A and walk to C from B or she can walk to B first and row to C.

Let's assume she rows from A to B first and walks to C from B.

To find the most efficient path, we need to find the angle $\theta = \widehat{BAC}$ that will minimize it.

We know $\widehat{BOC} = 2\theta$, $\widehat{AOB} = \pi - 2\theta$

So $AB^2 = AO^2 + OB^2 - 2\,AO.OB.\cos(\pi - 2\theta)$

$AB^2 = 8(1+\cos 2\theta) \longrightarrow AB = 2\sqrt{2}\sqrt{1+\cos 2\theta}$

The time it takes her to row from A to B is: $t_1(\theta) = \frac{AB}{2}$

$t_1(\theta) = \sqrt{2}\sqrt{1+\cos 2\theta}$

The arc $\overset{\frown}{BC}$ has length $2\theta.2 = 4\theta \Rightarrow$ the time it takes her to walk from B to C is $t_2(\theta) = \frac{\overset{\frown}{BC}}{4} = \frac{4\theta}{4} = \theta$.

So the total time in terms of θ is: $t(\theta) = t_1(\theta) + t_2(\theta)$

$\Rightarrow t(\theta) = \sqrt{2}\sqrt{1+\cos 2\theta} + \theta \qquad (0 \le \theta \le \pi/2)$

$t'(\theta) = \frac{-\sqrt{2}\sin 2\theta + \sqrt{1+\cos 2\theta}}{\sqrt{1+\cos 2\theta}} = 0 \Rightarrow \sqrt{2}\sin 2\theta = \sqrt{1+\cos 2\theta}$

So $2\sin^2 2\theta = 1+\cos 2\theta \Rightarrow 2(1-\cos^2 2\theta) = 1+\cos 2\theta$

$\Rightarrow 2\cos^2 2\theta + \cos 2\theta - 1 = 0$ so $\cos 2\theta = \frac{-1 \pm \sqrt{9}}{4} = \frac{1}{2}$ or -1

But $0 \le \theta \le \pi/2$ so $\cos 2\theta = \frac{1}{2}$ so $\boxed{\theta = \frac{\pi}{6}}$

7)Evaluate $\lim_{x\to 0^+} (\sin(x^2))^{(\sqrt{x^2+x}-x)}$.

First we find $\lim_{x\to 0^+} \ln(\sin x^2)^{\sqrt{x^2+x}-x} = \lim_{x\to 0^+} (\sqrt{x^2+x}-x)\ln \sin x^2$

$$= \lim_{x\to 0^+} \frac{x\ln x^2}{\sqrt{x^2+x}+x} \quad \left(= \frac{0}{0} \text{ since } \lim_{x\to 0^+} x\ln x^2 = 0 \text{ (check this) } \& \lim_{x\to 0^+} \sqrt{x^2+x}+x = 0\right)$$

L'Hospital

$$\overset{\text{L'Hospital}}{=} \lim_{x\to 0^+} \frac{\ln x^2 + x\frac{1}{x^2}(2x)}{\frac{2x+1}{2\sqrt{x^2+x}}+1} = \lim_{x\to 0^+} \frac{(\ln x^2+2)\,2\sqrt{x^2+x}}{2x+1+2\sqrt{x^2+x}}$$

We see that $\lim_{x\to 0^+} (2x+1+2\sqrt{x^2+x}) = 1$ and also

$$\lim_{x\to 0^+} (\ln x^2+2)\,2\sqrt{x^2+x} = \lim_{x\to 0^+} \frac{\ln x^2+2}{\frac{1}{2\sqrt{x^2+x}}} \overset{\text{L'Hospital}}{=} \lim_{x\to 0^+} \frac{\frac{2}{x}}{\frac{2x+1}{4(x^2+x)\sqrt{x^2+x}}}$$

$$= \lim_{x\to 0^+} \frac{8(x^2+x)\sqrt{x^2+x}}{x(2x+1)} = 0 \quad \text{(check this)}$$

So $\lim_{x\to 0^+} \ln(\sin x^2)^{\sqrt{x^2+x}-x} = 0$ so $\lim_{x\to 0^+} (\sin x^2)^{\sqrt{x^2+x}-x} = 1.$

8)Prove using ϵ-δ, that $\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$.

Given $\varepsilon > 0$.

First suppose $|x-1| < 1 \Rightarrow 1 < x < 3$

$\Rightarrow \frac{1}{|x|} < 1$

So $\left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|x-2|}{2|x|} < \frac{|x-2|}{2}$

Pick $\delta = \min(1, 2\varepsilon)$

$\Rightarrow$ If $|x-2| < \delta$ then $|x-2| < 1$ & $|x-2| < 2\varepsilon$.

We show: $\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$.

* If $\delta = 1$ then $|x-2| < 1$, $\frac{1}{|x|} < 1$, $\delta < 2\varepsilon$ i.e. $\frac{1}{2} < \varepsilon$.

So $\left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|x-2|}{2|x|} < \frac{\delta}{2} = \frac{1}{2} < \varepsilon$

* If $\delta = 2\varepsilon$ then $|x-2| < 2\varepsilon$

So $\frac{|x-2|}{2|x|} < \frac{|x-2|}{2} < \frac{2\varepsilon}{2} = \varepsilon$.

In either case, $\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$.

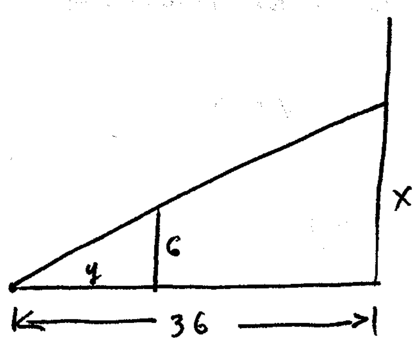
9)If f is a continuous function such that for all x, $\int_0^x f(t)dt = xe^{2x} + \int_0^x e^{-t}f(t)dt$, find an explicit formula for $f(x)$.

Differentiate both sides and use the Fundamental Theorem of Calculus : $f(x) = e^{2x} + 2x^2 e^{2x} + e^{-x} f(x)$

$$\rightarrow f(x)\left(1 - e^{-x}\right) = e^{2x}(1 + 2x^2)$$

$$\rightarrow f(x) = \frac{e^{2x}(1+2x^2)}{1 - e^{-x}}$$

10)A spotlight is shining on a building. The spotlight is 36 feet away from the building. A man that is 6 feet tall is walking toward the building. Calculate the rate of change in the length of his shadow if he is walking toward the building at a rate of 5 ft/s and he is currently 12 feet away from the building.



The spotlight is on the ground & 36 ft away from the building so the man's shadow is projected on the building

Let x be the length of the shadow

We want to calculate $\frac{dx}{dt}$ at the time the man is 12 ft away from the building

Let y be the distance b/w the man and the spotlight

$\rightarrow \frac{dy}{dt} = +5 \, ft/s$ (this is positive since the man walks away from the spotlight)

We have $\frac{y}{36} = \frac{6}{x} \Rightarrow x = \frac{6 \cdot 36}{y}$

$\rightarrow \frac{dx}{dt} = 6^3 \left(-\frac{1}{y^2}\right) \frac{dy}{dt} = 5 \cdot 6^3 \cdot \frac{-1}{y^2}$.

When the man is 12 ft away from the building, $y = 36 - 12 = 24$

$\rightarrow \frac{dx}{dt} = 5 \cdot 6^3 \cdot \frac{-1}{24^2} = \frac{-15}{8} \, ft/s$

This makes sense since the shadow shrinks as the man walks toward the building.