MIDTERM # 2

① If $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is orthogonal Then $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent.

Ⓐ TRUE　　　　Ⓑ FALSE

② Let $\vec{u}_1, \vec{u}_2$ be eigenvectors of a matrix A and let λ_1, λ_2 be the corresponding eigenvalues. If $\vec{u}_1$ and $\vec{u}_2$ are linearly independent then $\lambda_1 \neq \lambda_2$.

Ⓐ TRUE　　　　Ⓑ FALSE

③ Same hypothesis as ②. If $\lambda_1 \neq \lambda_2$ Then $\vec{u}_1$ and $\vec{u}_2$ are linearly independent.

Ⓐ TRUE　　　　Ⓑ FALSE

④ The matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ is diagonalizable.

Ⓐ TRUE　　　　Ⓑ FALSE.

⑤ Let A be a 5x5 matrix with a basis of eigenvectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$. Consider the matrix $Q = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4\ \vec{v}_5]$. Which of The following matrices is <u>always</u> diagonal?

Ⓐ QAQ^{-1}　　Ⓑ Q^TAQ　　Ⓒ $Q^{-1}AQ$　　Ⓓ None of These.

⑥ Is the set $\{1-t+t^2,\ 2-t^2,\ t+2t^2\}$ linearly independent ?

Ⓐ YES　　　Ⓑ NO

⑦ Let $\mathcal{E}$ denote the standard basis for $\mathbb{R}^2$ and $\mathcal{B} = \left\{ \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$.

The change of basis matrix $\underset{\mathcal{B} \leftarrow \mathcal{E}}{P}$ is

Ⓐ $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$　　Ⓑ $\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$　　Ⓒ Neither A nor B.

⑧ Is the following set : $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ orthonormal ?

Ⓐ YES　　　Ⓑ NO.

⑨ Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^3$ and $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

The coordinate vector $[\vec{x}]_{\mathcal{B}}$ is

Ⓐ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$　　Ⓑ $\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$　　Ⓒ $\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$　　Ⓓ None of these.

⑩ Let $\vec{x_1}, \vec{x_2}, \vec{x_3}$ be 3 linearly independent vectors in $\mathbb{R}^4$ and let $\vec{v_1}, \vec{v_2}, \vec{v_3}$ be orthogonal vectors coming from the Gram - Schmidt process.

Then : $\vec{v_1} = \vec{x_1}$　　$\vec{v_2} = \vec{x_2} - \dfrac{\vec{x_1}\cdot\vec{v_1}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1}$　　$\vec{v_3} = \vec{x_3} - \dfrac{\vec{x_2}\cdot\vec{x_1}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1} - \dfrac{\vec{x_2}\cdot\vec{v_2}}{\vec{v_2}\cdot\vec{v_2}}\vec{v_2}$

Ⓐ TRUE　　　Ⓑ FALSE

(11) Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$.

(11.1) One least squares solution $\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to the linear equation $A\vec{x} = \vec{b}$ is obtained by solving $A\vec{x} = \vec{c}$ where $\vec{c}$ is

(A) $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ (B) $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ (C) $\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ (D) $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

(11.2) The normal equation to $A\vec{x} = \vec{b}$ is:

$$\begin{aligned} 2x_1 + 12x_2 &= 4 \\ 12x_1 + 27x_2 &= 11 \end{aligned}$$

(A) TRUE (B) FALSE.

(12) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$: a basis for $\mathbb{R}^3$.

(12.1) The matrix $[A]_{\mathcal{B} \leftarrow \mathcal{B}}$ is:

(A) $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

(12.2) The characteristic polynomial for A is:

(A) $-\lambda(6-\lambda)^2$ (B) $\lambda^2(6-\lambda)$ (C) $(1-\lambda)(2-\lambda)(3-\lambda)$.

(13) Let A be an $m \times m$ matrix. Then $\text{Row}A = \text{Col}A$

(A) TRUE (B) FALSE.

(14) Let $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Is A diagonalizable? Why or why not?

$\det(A - xI) = (-1-x)^2(2-x) = 0$

So eigenvalues of A are $\lambda = -1$ (multiplicity 2), $\lambda = 2$ (multiplicity 1)

$\underline{\lambda = -1}$: $A + I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ has 2 pivots

So $\mathcal{E}_{\lambda=-1} = \text{Null}(A+I)$ has dimension $1 < 2 =$ multiplicity of $-$

$\rightarrow$ A is not ~~invertible~~ diagonalizable.

(15) Let $T: \mathbb{P}_2 \to \mathbb{P}_2$ be $T(a_0 + a_1 t + a_2 t^2) = (a_0 + a_1) + (3a_2 - a_0)t + a_1 t^2$

and $\mathcal{B} = \{1, t, t^2\}$ be a basis for dom(T), $\mathcal{E} = \{1, 1+t, 1+t^2\}$ be a basis for Codomain (T). Compute $\underset{\mathcal{E} \leftarrow \mathcal{B}}{[T]}$ and $[T(1+t+t^2)]_{\mathcal{E}}$.

$\underset{\mathcal{E} \leftarrow \mathcal{B}}{[T]} = \left[[T(1)]_{\mathcal{E}} \; [T(t)]_{\mathcal{E}} \; [T(t^2)]_{\mathcal{E}} \right]$

$T(1) = 1 - t \rightarrow [T(1)]_{\mathcal{E}} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ b/c $1 - t = 2.1 + (-1)(1+t) + 0.(1+t^2)$

$T(t) = 1 + t^2 \rightarrow [T(t)]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ b/c $1+t^2 = 0.1 + 0.(1+t) + 1(1+t^2)$

$T(t^2) = 3t \rightarrow [T(t^2)]_{\mathcal{E}} = \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix}$ b/c $3t = -3.1 + 3.(1+t) + 0.(1+t^2)$.

So $\underset{\mathcal{E} \leftarrow \mathcal{B}}{[T]} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$

$[T(1+t+t^2)]_{\mathcal{E}} = \underset{\mathcal{E} \leftarrow \mathcal{B}}{[T]} [1+t+t^2]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

(16) Let $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \in \mathbb{R}^3$ and H be the plane $2x + y - 2z = 0$.

Compute $\text{proj}_H \vec{v}$ and $\text{dist}(\vec{v}, \text{proj}_H \vec{v})$.

Solution: $2x + y - 2z = 0 \rightarrow x = \frac{-y}{2} + z$ $\quad y, z$: free

So a basis for H is $\left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

This basis turns out to be not orthogonal so we have to ~~$\text{proj}_H \vec{v}$ =~~ find an orthogonal basis for H.

Using Gram-Schmidt on $\{\vec{v_1}, \vec{v_2}\}$ where $\vec{v_1} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$:

$$\vec{w_1} = \vec{v_1} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{w_2} = \vec{v_2} - \frac{\vec{v_2} \cdot \vec{w_1}}{\vec{w_1} \cdot \vec{w_1}} \vec{w_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{-1}{5} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 2/5 \\ 1 \end{bmatrix}$$

Clearly $\{\vec{w_1}, \vec{w_2}\}$ is an orthogonal basis for H

So
$$\text{proj}_H \vec{v} = \frac{\vec{v} \cdot \vec{w_1}}{\vec{w_1} \cdot \vec{w_1}} \vec{w_1} + \frac{\vec{v} \cdot \vec{w_2}}{\vec{w_2} \cdot \vec{w_2}} \vec{w_2} = -\frac{1}{5} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \frac{14}{9} \begin{bmatrix} 4/5 \\ 2/5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 65/45 \\ 10/45 \\ 14/9 \end{bmatrix} \quad \left(= \begin{bmatrix} 13/9 \\ 2/9 \\ 14/9 \end{bmatrix} \right)$$

$$\text{dist}(\vec{v}, \text{proj}_H \vec{v}) = \| \vec{v} - \text{proj}_H \vec{v} \| = \left\| \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 65/45 \\ 10/45 \\ 14/9 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} -20/45 \\ -10/45 \\ 4/9 \end{bmatrix} \right\| = \sqrt{\frac{20^2}{45^2} + \frac{10^2}{45^2} + \frac{16}{81}}.$$