MIDTERM #1

1. If $\{\vec{x}, \vec{y}\}$ is linearly independent and $\vec{z} \in \text{span}\{\vec{x}, \vec{y}\}$ Then $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly independent.

   A. TRUE   B. FALSE

2. If $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ Then A^{-1}

   A. $= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ B. $= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ C. $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ D. doesn't exist

3. The augmented matrix of the system $\begin{cases} -2x_1 + x_3 = 1 \\ 2x_2 + x_1 = 0 \end{cases}$ is

   A. $\left[\begin{array}{cc|c} -2 & 1 & 1 \\ 2 & 1 & 0 \end{array}\right]$ B. $\left[\begin{array}{ccc|c} -2 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \end{array}\right]$ C. $\left[\begin{array}{ccc|c} -2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{array}\right]$

   D. $\left[\begin{array}{ccc|c} -2 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 \end{array}\right]$

4. Can 4 distinct vectors in $\mathbb{R}^3$ be linearly independent?

   A. YES   B. NO

5. Suppose $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \mathbb{R}^n$, Then The matrix $A = [\vec{v}_1 \cdots \vec{v}_n]$ is invertible.

   A. TRUE   B. FALSE

(6) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ab - cd \neq 0$, then A is invertible.

A. TRUE   (B.) FALSE

(7) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & 0 \end{bmatrix}$, Then $\det(A) =$

(A) -7 B. 0 C. 1 D. 7 E. π

(8) Suppose you have found 5 linearly independent solutions of a homogeneous system $A\vec{x} = \vec{0}$, of 21 equations and 25 unknowns. Every solution of this system is a linear combination of these 5 solutions.

Is it possible to find a vector $\vec{b}$ such that the corresponding nonhomogeneous system $A\vec{x} = \vec{b}$ has no solutions?

(A.) YES   B. NO

(9) With the same hypothesis as 8, is it possible to find a vector $\vec{b}$ such that the corresponding nonhomogeneous system $A\vec{x} = \vec{b}$ has a unique solution?

A. YES   (B.) NO

(10) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}$.

(A) Rank $(A) = 1$, Nullity $(A) = 4$

B. Rank $(A) = 0$, Nullity $(A) = 5$

C. Rank $(A) = 3$, Nullity $(A) = 2$

(11) Let $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ -4 & 2 & -6 & 0 \\ 2 & -1 & 3 & 0 \end{bmatrix}$. Find a basis for

Col(A), a basis for Null(A). Using the bases found, compute Rank(A), Nullity(A).

<u>Sol</u>:

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ -4 & 2 & -6 & 0 \\ 2 & -1 & 3 & 0 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 + 4R_1} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & -2 & 8 \\ 0 & -1 & 1 & -4 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + \frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & -2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since columns 1 & 2 have pivots, a basis for Col(A) is

$$\left\{ \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right\}$$

So Rank(A) = 2

From REF of A, we get $\begin{cases} x_1 + x_3 + 2x_4 = 0 \\ 2x_2 - 2x_3 + 8x_4 = 0 \end{cases}$

$\rightarrow$ x_3, x_4 : free $\quad x_1 = -x_3 - 2x_4, \quad x_2 = x_3 - 4x_4$

So solutions of $A\vec{x} = \vec{0}$ have the form:

$$\begin{bmatrix} -x_3 - 2x_4 \\ x_3 - 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

So a basis for Null(A) is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$

And Nullity(A) = 2.

(12) a) Let $A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Compute $\det(A)$ by row reducing A to its row echelon form.

Sol: $A_0 = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = A_1$

$$\det(A_0) = -\det(A_1)$$

A_1 is a REF of A_0 & $\det A_1 = -1$

So $\det(A_0) = 1$.

b) Find the standard matrix for the linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_3 \\ x_1 + x_2 + x_3 \end{bmatrix}$.

Sol: $A = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & T(\vec{e_3}) \end{bmatrix}$ where $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$T(\vec{e_1}) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$T(\vec{e_2}) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(\vec{e_3}) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

So $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$