The Geometric Nature of the Fundamental Lemma

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Northwestern University

Current Events Bulletin
2011 Joint Mathematics Meetings
New Orleans, LA

A long strange trip

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To understand functoriality, we must study Langlands reciprocity...

Linearization of subspaces

Let X be a measure space.

subspaces $Y \subset X \rightsquigarrow$ integral distributions $Y(\varphi) = \int_Y \varphi$

Now can add $Y_1 + Y_2$ and scale cY subspaces.

Suppose symmetry group G acts on X preserving measure.

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Now consider group G with a conjugation invariant measure.

Importance of linearization of conjugacy classes:

characters of G-representations $\rightsquigarrow$ G-invariant distributions

Given G-representation V, can form distributional character:

$$\chi_V(\varphi) = \int_G \varphi(g)\,\mathrm{Tr}_V(g)dg$$

(ignoring analytic technical difficulties).

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Example: Finite groups

Specialize to finite group G. Then

$$G\text{-invariant distributions} = \text{class functions}$$

Theorem

Characters χ_V of irreducible G-representations V form basis for class functions. Rescaled characters $\hat{\chi}_V = \chi_V / \dim V$ idempotents with respect to convolution $\hat{\chi}_V * \hat{\chi}_V = \hat{\chi}_V$.

Interpretation

Class functions = functions on space of irreducible representations. Rescaled characters $\hat{\chi}_V$ are characteristic functions of points.

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Induced representations

It is difficult to construct representations.

We have trivial representation Tr and method of induction.

Given group G, and subgroup $\Gamma \subset G$, form "unitary induction"

$$\mathrm{Ind}_\Gamma^G(\mathrm{Tr}) = L^2(G/\Gamma)$$

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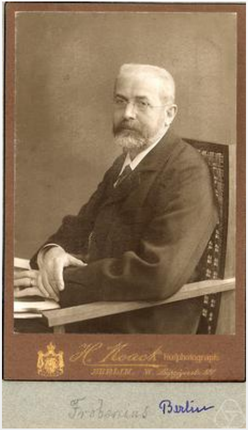
G finite group, $\Gamma \subset G$ subgroup.



Ferdinand Georg Frobenius
1849–1917

Character of induced representation $L^2(G/\Gamma)$

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Volumes of quotients of centralizers

$$a_\gamma = |\Gamma_\gamma \backslash G_\gamma|$$

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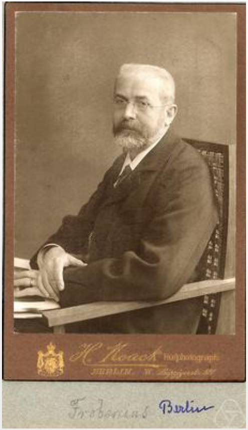

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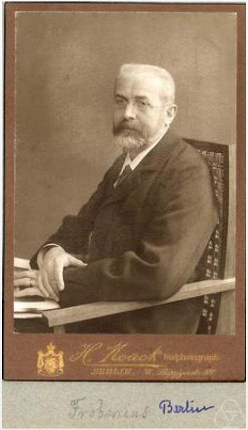

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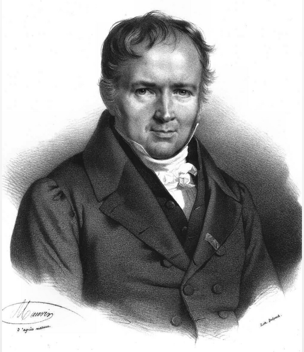
$\mathbb{R}$ additive group, $\mathbb{Z} \subset \mathbb{R}$ discrete subgroup.



Siméon Denis Poisson
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Character of induced representation $L^2(\mathbb{R}/\mathbb{Z})$

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(Fourier analysis provides isomorphism

$$L^2(\mathbb{R}/\mathbb{Z}) \simeq \widehat{\bigoplus}_{\lambda \in \mathbb{Z}} \mathbb{C}\langle e^{2\pi i \lambda} \rangle$$

Hence identification of characters

$$\sum_{n \in \mathbb{Z}} \varphi(n) = \sum_{\lambda \in \mathbb{Z}} \widehat{\varphi}(\lambda).)$$

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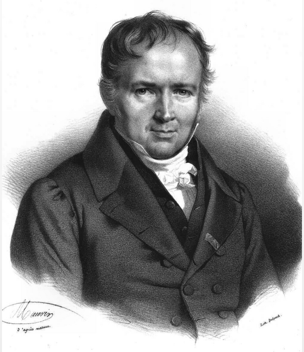

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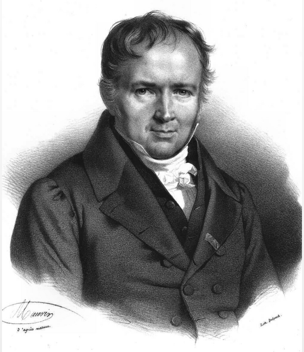

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Arthur-Selberg Trace Formula

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1917–2007

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$\mathbb{G}$ reductive algebraic group over number field F.
Think $\mathbb{G} = GL(n)$ and $F = \mathbb{Q}$.

$\mathbb{A}_F$ adèles of F of all hypothetical "Laurent series expansions" of elements in the form of p-adic and real numbers.

Then $G = \mathbb{G}(\mathbb{A}_F)$ is a locally compact group, and $\Gamma = \mathbb{G}(F) \subset \mathbb{G}(\mathbb{A}_F)$ is a discrete subgroup.

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Upshot: character involves integrals over conjugacy classes in real and p-adic Lie groups.

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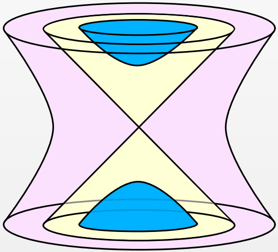
Conjugacy classes

For simplicity, let's consider the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$.



Orbits of $SL(2, \mathbb{R})$ acting on its Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \simeq \mathbb{R}^3$.

Three types of orbits under conjugation:

- hyperbolic: $\det < 0$.
- nilpotent: $\det = 0$.
- elliptic: $\det > 0$.

We will focus on the two elliptic orbits $\mathcal{O}_{A_+}, \mathcal{O}_{A_-} \subset \mathfrak{s}(2, \mathbb{R})$ through the matrices

$$A_+ = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad A_- = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

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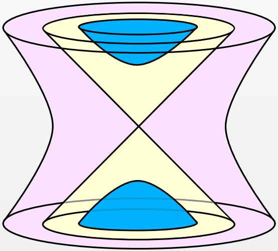

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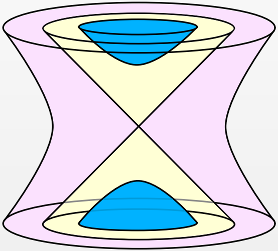

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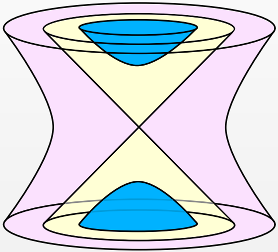

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Over the complex numbers $\mathbb{C}$, the matrices

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are both conjugate to the matrix

$$A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \in \mathfrak{sl}(2,\mathbb{C}).$$

One says that A_+ and A_- are stably conjugate.

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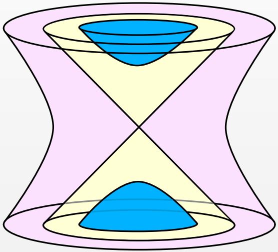

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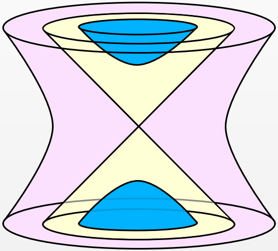

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Said another way, the two elliptic orbits

$$\mathcal{O}_{A_+}, \mathcal{O}_{A_-} \subset \mathfrak{s}(2,\mathbb{R})$$

coalesce into a single conjugacy class

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cut out by the invariant polynomial

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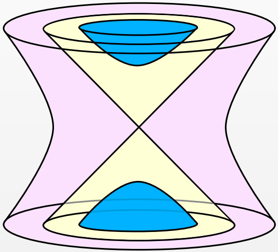

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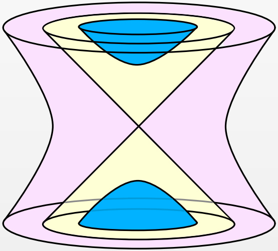
Linearization of adjoint orbits



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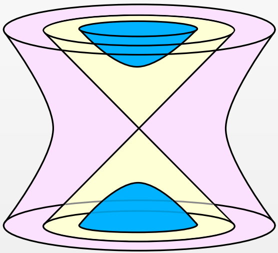
Consider the distributions given by integrating over the elliptic orbits

$$\mathcal{O}_{A_+}(\varphi) = \int_{\mathcal{O}_{A_+}} \varphi \qquad \mathcal{O}_{A_-}(\varphi) = \int_{\mathcal{O}_{A_-}} \varphi$$

with respect to an invariant measure.

Alternative basis



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The distributions $\mathcal{O}_{A_+}, \mathcal{O}_{A_-}$ span a two-dimensional complex vector space.

It admits the alternative basis

$$\mathcal{O}_{st} = \mathcal{O}_{A_+} + \mathcal{O}_{A_-}$$

$$\mathcal{O}_{tw} = \mathcal{O}_{A_+} - \mathcal{O}_{A_-}$$

Here st stands for stable and tw stands for twisted.

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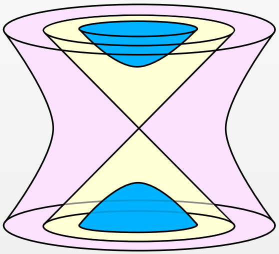

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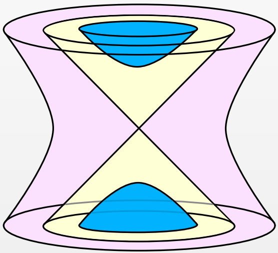

Orbits of $SL(2, \mathbb{R})$ acting on its Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \simeq \mathbb{R}^3$.

The distributions $\mathcal{O}_{A_+}, \mathcal{O}_{A_-}$ span a two-dimensional complex vector space. It admits the alternative basis

$$\mathcal{O}_{st} = \mathcal{O}_{A_+} + \mathcal{O}_{A_-}$$

$$\mathcal{O}_{tw} = \mathcal{O}_{A_+} - \mathcal{O}_{A_-}$$

Here st stands for stable and tw stands for twisted.

Stable distributions

Stable distribution

$$\mathcal{O}_{st} = \mathcal{O}_{A_+} + \mathcal{O}_{A_-}$$

is integral over union of orbits

$$\mathcal{O}_{A_+} \sqcup \mathcal{O}_{A_-}.$$

Algebraic variety defined by invariant polynomial

$$\det = 1.$$

Stable distribution is object of

algebraic geometry (finite mathematics)

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Twisted distributions

What to do with twisted distribution

$$\mathcal{O}_{tw} = \mathcal{O}_{A_+} - \mathcal{O}_{A_-}?$$

Distinguishes between

$$\mathcal{O}_{A_+} \text{ and } \mathcal{O}_{A_-}$$

though no invariant polynomial separates them.

Twisted distribution appears to be noncanonical: exchanging terms

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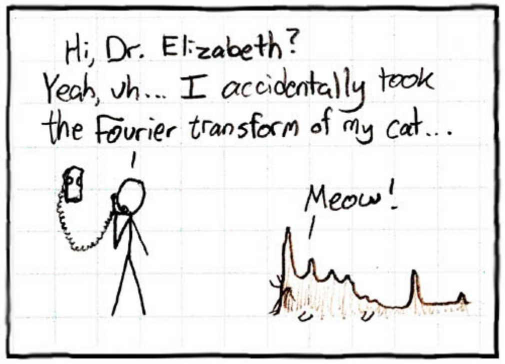
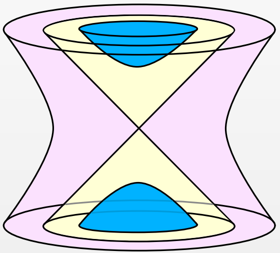


Interpretation via Fourier analysis



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results from *Fourier analysis* on set of orbits

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What is the Fundamental Lemma all about?

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Basic idea

Langlands's theory of endoscopy, and the Fundamental Lemma at its heart, confirms that one can systematically express

twisted distributions in terms of stable distributions

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nonconstant Fourier modes in terms of constant Fourier modes

Example continued

Endoscopy relates twisted distribution

$$\mathcal{O}_{tw} = \mathcal{O}_{A_+} - \mathcal{O}_{A_-}$$

to stable distribution on Lie algebra $\mathfrak{so}(2,\mathbb{R}) \simeq \mathbb{R}$ of subgroup

$$SO(2,\mathbb{R}) \subset SL(2,\mathbb{R})$$

stabilizing matrices

$$A_+ = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad A_- = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Outside of bookkeeping, this is empty of content since $SO(2,\mathbb{R})$ is abelian, and so its orbits in $\mathfrak{so}(2,\mathbb{R})$ are single points.

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Why is the Fundamental Lemma difficult?

General theory of endoscopy is deep and elaborate.

Key challenge

Extraordinary difficulty of the Fundamental Lemma, and also its mystical power, emanates from fact that sought-after stable distributions live on so-called endoscopic groups H with little apparent geometric relation to original group G.

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Endoscopic groups

To find relation between group G and endoscopic group H, one must pass to Langlands dual groups

"noncommutative Pontryagin dual group" of "geometric characters"

There one finds $H^\vee$ is naturally subgroup of $G^\vee$.

Example

Consider the symplectic group $G = Sp(2n)$.

The special orthogonal group $H = SO(2n)$ is not a subgroup.

But $H^\vee = SO(2n)$ is a subgroup of $G^\vee = SO(2n+1)$.

Endoscopy gives precise relationship

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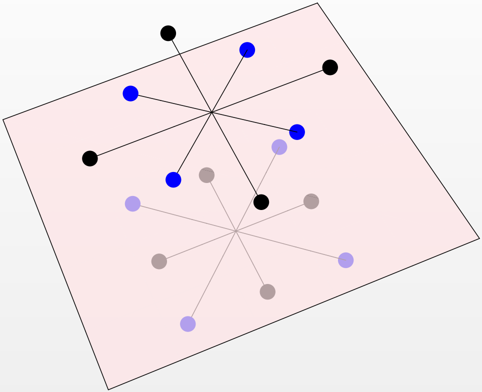
Low rank example



Foreground: roots of the group $G = Sp(4)$ with roots of the endoscopic group $H = SO(4)$ highlighted.

Background: roots of the Langlands dual group $G^\vee = SO(5)$ with roots of the subgroup $H^\vee = SO(4)$ highlighted.

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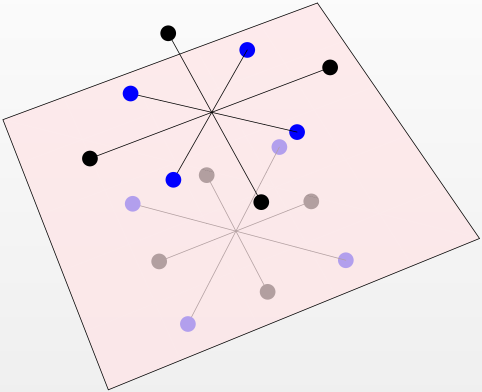

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Real versus p-adic Lie groups

Detailed conjectures organizing the intricacies of the transfer of distributions first appear in Langlands's joint work with Shelstad.

General setting needed for applications to number theory and harmonic analysis: p-adic and real Lie groups (algebraic groups over local fields).

For real Lie groups, Shelstad rapidly proved the conjectures.

What became known as the Fundamental Lemma is the most basic conjecture for p-adic groups.

Useful to have picture of real Lie groups in mind. Langlands and Shelstad: "if it were not that [transfer factors] had been proved to exist over the real field, it would have been difficult to maintain confidence in the possibility of transfer or in the usefulness of endoscopy."

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From p-adic groups to loop groups

Dictionary between arithmetic and geometry.

One-dimensional Arithmetic

- Number field F
- Rational numbers $\mathbb{Q}$
- p-adic field
- p-adic group

One-dimensional Geometry

- Smooth projective curve X
- Projective line $\mathbb{P}^1$
- Formal disk D
- Loop group LG

Theorem (Waldspurger)

To prove Fundamental Lemma, it suffices to prove its analogue in the geometric setting.

Later proof by Cluckers, Hales, and Loeser via model theory:

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Loop Grassmannians

Orbital integrals of Fundamental Lemma in geometric setting are equivalent to counting points in subvarieties of Grassmannians.

Definition

Let LG be loop group. Let $L_+G \subset LG$ be subgroup of arcs.
Loop Grassmannian Gr_G is homogenous space LG/L_+G.

Why Grassmannian? $\infty/2$-dim subspaces of ∞-dim vector space.

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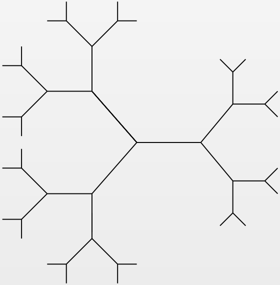

Geometric cousin of affine building.

Affine Springer fibers

Now the subvarieties...

Definition

Let ξ be element of Lie algebra of LG.
Affine Springer fiber $X_\xi \subset Gr_G$ is fixed-point locus of ξ.

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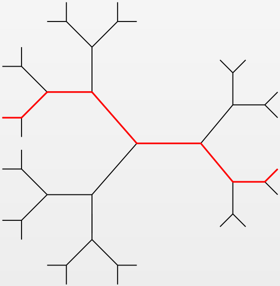

Example for ξ diagonal with distinct eigenvalues.

Basic structure of affine Springer fibers

- X_ξ is finite-dimensional increasing union of projective varieties.
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Lefschetz
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Trace formula: count points in algebraic variety by calculating traces of Galois symmetries acting on topological cohomology.

Now can stand on the shoulders of giants: Kazhdan-Lusztig, Goresky-MacPherson, Beilinson-Bernstein-Deligne-Gabber,...

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Problem: cohomology of fixed-points of vector fields on flag varieties.

Trivial case: when vector field is generic, for example sum of linearly independent commuting vector fields.

General solution: "analytically continue" solution from generic locus to all vector fields.

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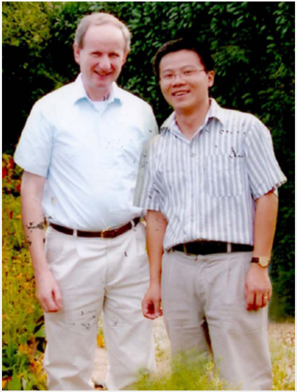

Laumon and Ngô

Beautiful insight

Affine Springer fibers modulo natural symmetries parametrize generalized line bundles on curves.

Deformations to simpler curves provide deformations to simpler affine Springer fibers.

Striking consequence

Fundamental Lemma for unitary groups!

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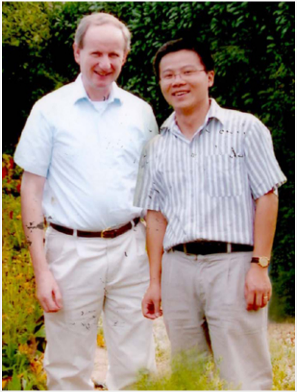

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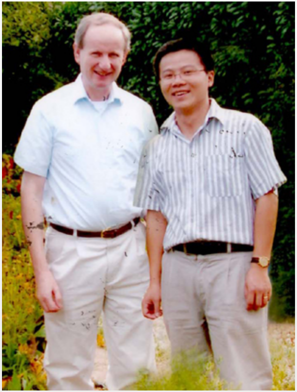

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Hitchin fibration

X smooth projective curve (Riemann surface).

Hitchin moduli $\mathcal{M}_G(X)$ parametrizes G-bundle on X together with twisted endomorphism.

Base $\mathcal{A}_G(X)$ parametrizes possible eigenvalues of twisted endomorphism (spectral curve).

Integrable system $\mathcal{M}_G(X) \to \mathcal{A}_G(X)$ assigns characteristic polynomial of endomorphism.

Fibers parametrize generalized line bundles on spectral curves.

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Meanwhile in a galaxy far, far away...

Nigel Hitchin
1946–

Hitchin fibration

X smooth projective curve (Riemann surface).

Hitchin moduli $\mathcal{M}_G(X)$ parametrizes G-bundle on X together with twisted endomorphism.

Base $\mathcal{A}_G(X)$ parametrizes possible eigenvalues of twisted endomorphism (spectral curve).

Integrable system $\mathcal{M}_G(X) \to \mathcal{A}_G(X)$ assigns characteristic polynomial of endomorphism.

Fibers parametrize generalized line bundles on spectral curves.

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Not so surprising...

Fundamental Lemma involves distributions on conjugacy classes

adjoint quotient G/G

Hitchin moduli space parametrizes twisted maps

curve $X \longrightarrow$ adjoint quotient $\mathfrak{g}/G$

Furthermore, stable conjugacy classes involve invariant polynomials

adjoint quotient $G/G \longrightarrow$ possible eigenvalues T/W

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Global curve organizes local group theory!

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Ngô's Support Theorem

Main new technical input to proof of Fundamental Lemma. Precise description of the cohomology of the fibers of an integrable system in terms of its generic fibers.

Toy model: consider a family of irreducible curves

$$f : M \to S, \text{ with } M \text{ and } S \text{ smooth.}$$

Over a Zariski open locus $S^0 \subset S$, the fibers

$$M_s = f^{-1}(s), \quad s \in S^0$$

are topologically equivalent curves, hence their cohomologies $H^*(M_s)$ form a local system of vector spaces

$$\mathcal{H} \to S^0$$

Exercise: the cohomology of any fiber can be recovered from $\mathcal{H}$.

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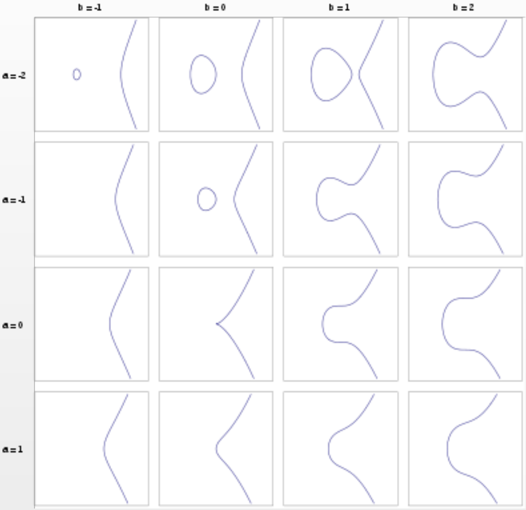
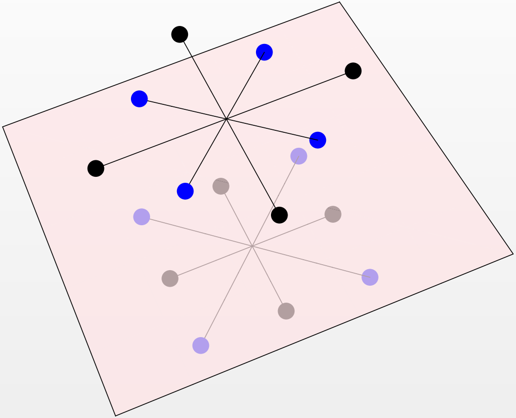
Family of plane cubics



$$y^2 = x^3 + ax + b$$

singular at $(a, b) = (0, 0)$



Thank you for listening!