Happy Determinant Day!

(Lecture 8)

This week:

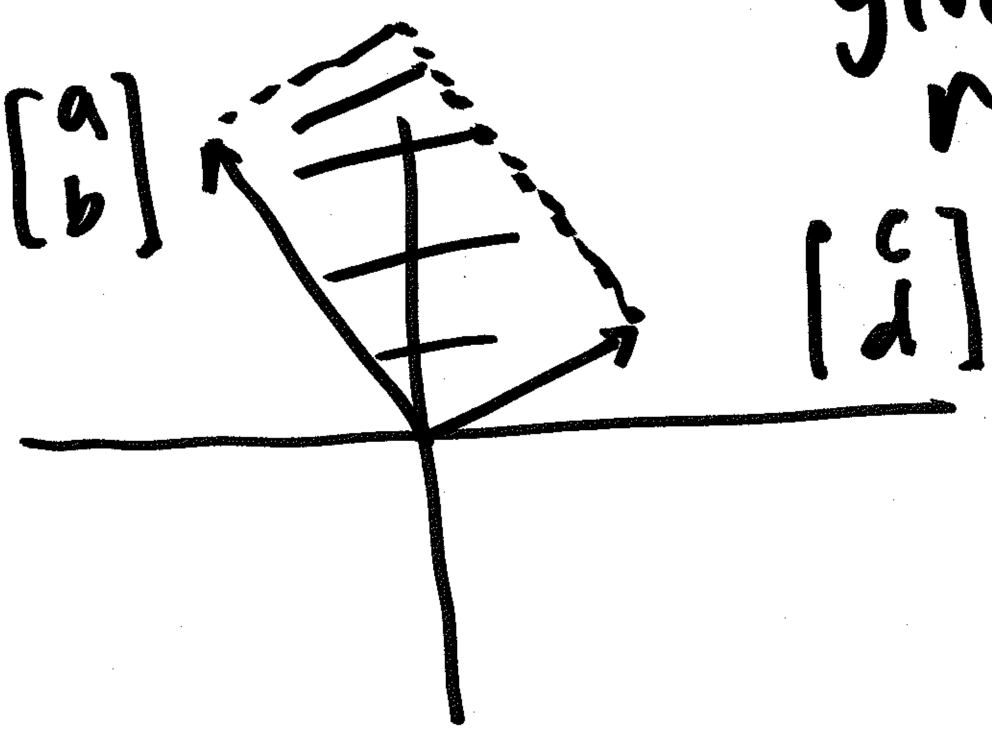
Thurs Office Hours, 736 Evans
12:30 - 2:30

Fri Quiz through 3.3

Warmup Problem $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Recall $\det A = ad - bc$

Show $|\det A| =$ Area of parallelogram given by the row vectors.



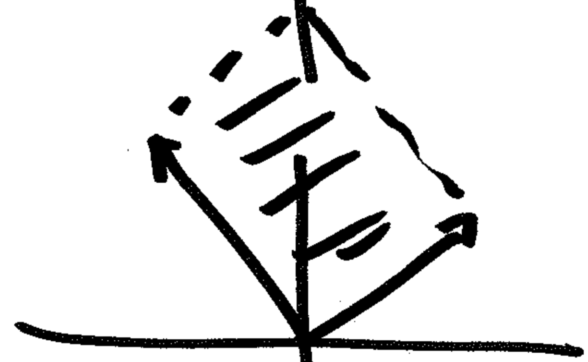
③ Favorite tool: row reduction! $A \rightsquigarrow A'$

	$\det A$	Area
R1 (add scale of row to another)	$\det A' = \det A$ (check this!)	$\text{Area}' = \text{Area}$
R2 (exchange rows)	$\det A' = -\det A$	$\text{Area}' = \text{Area}$
R3 (scales row by $c \neq 0$)	$\det A' = c \det A$	$\text{Area}' = c\,\text{Area}$

Examples:

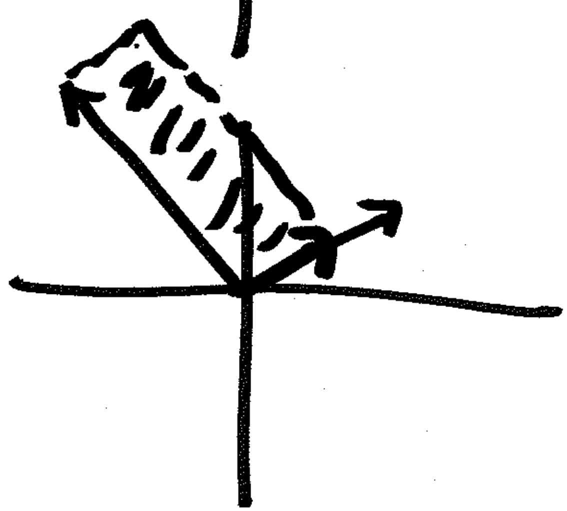
R1

R2



same parallelogram

R3



<u>Soln</u>: Use R1, R2 to put A in REF:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b' \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & b' \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}$$

$\det A'$	0	0	0	$\lvert a'd' \rvert$
	$=$	$=$	$=$	$=$
#Area'	0	0	0	$\lvert a'd' \rvert$

We will define the <u>determinant</u> of $n \times n$ matrices A so that

$$|\det A| = \text{Volume of solid given by row vectors.}$$

Also (more importantly) it will satisfy previous properties under row ops.

Inductive def of determinant

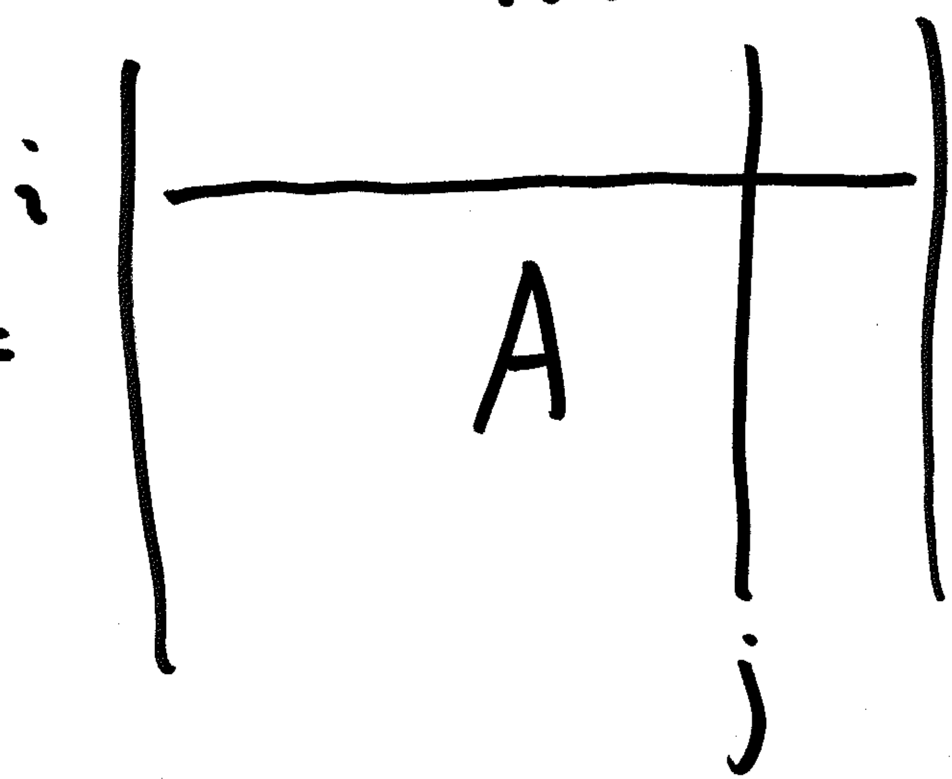
<u>$n=1$</u> $A = [a_{11}]$ $\det A = a_{11}$

Suppose you know det for $(n-1)\times(n-1)$ matrices

Take $A = [(a_{ij})]$ $n \times n$ matrix

$$= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \ddots & \\ \vdots & & \ddots \\ a_{n1} & & a_{nn} \end{bmatrix}$$

Set $A_{ij} = (n-1)\times(n-1)$ matrix given by deleting ith row & jth col from A

$A_{ij} =$ 

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

__Exer__ Calc. $\det A$ for $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix}$

$$\det A = \underline{1} \cdot \det\begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix} - 2\det\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$$

$$+ 1 \det\begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix}$$

$$= 1 \cdot (-6) - 2(1) + 1(3)$$

$$= -5$$

Also solve with row ops:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ -1 & 0 & -2 \end{pmatrix} \overset{R1}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{pmatrix}$$

$$\overset{R1}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & -\frac{5}{3} \end{pmatrix} = A'$$

$$\det A' = 1 \cdot \det \begin{pmatrix} 3 & 1 \\ 0 & -\frac{5}{3} \end{pmatrix} - 2 \cdot 0 + 1 \cdot 0$$

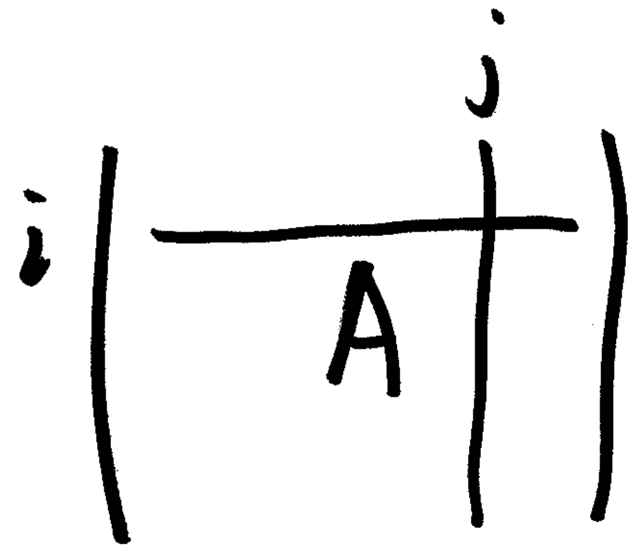
$$= 1 \cdot 3 \cdot \left(-\frac{5}{3}\right) = -5$$

Properties of det A $n \times n$ matrix

① Cofactor expansion

Define $C_{ij} = (-1)^{i+j} \det A_{ij}$

number called i,j-Cofactor



$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

cofactor expansion along ith row

(when $i=1$, this is def of $\det A$)

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

cofactor expansion along jth col

Exer Calc $\det A$ for $A = \begin{pmatrix} 0 & \boxed{0} & 1 & -1 \\ 1 & \boxed{0} & 0 & 1 \\ 1 & \boxed{1} & 0 & 0 \\ 0 & \boxed{0} & 2 & 1 \end{pmatrix}$

Cofactor exp along col $j=2$.

$$\det A = a_{32} \cdot C_{32}$$

$$C_{32} = (-1)^{3+2} \det \underbrace{\begin{pmatrix} \boxed{0} & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}}_{A_{32}\ \ A'}$$

Cof. exp along col $j=1$

$$\det A_{32} = a'_{21} C'_{21}$$

$$C_{21}' = (-1)^{2+1} \det \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$$

$$= -1 \cdot (+3) = {}^{-}3$$

So $C_{32} = (-1)^5 \cdot (-3) = +3$

So $\det A = 1 \cdot (+3) = 3$

(2) If A is upper Δ-ar (or lower Δ-ar)

$$A = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

then $\det A$ = product of diag entries

<u>Ex</u> $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 7 & 5 & 3 \end{pmatrix}$

$$\det A = 1 \cdot (-2)(3) = -6$$

④ <u>**Theorem**</u> A $n \times n$ matrix invertible

$$\Longleftrightarrow$$

$$\det A \neq 0.$$

<u>Pf</u>. Put in REF using R1, R2.

$$A \rightsquigarrow U = \underbrace{\begin{pmatrix} * & & & & & \\ & * & & & & \\ & & * & & & \\ & 0 & & * & & \\ & & & & * & \\ & & & & & * \end{pmatrix}}_{n \text{ pivots}} \quad \text{or} \quad \underbrace{\begin{pmatrix} * & * & & & & \\ & & * & & & \\ & & & * & & \\ & & & & * & * \\ & 0 & & & & 0 \\ & & & & & 0 \end{pmatrix}}_{< n \text{ pivots}}$$

U is upper Δ-ar!

We find $\det \cancel{A} U = \begin{cases} 0 & \text{if } < n \text{ pivots} \\ \text{product of pivots} & \text{if } n \text{ pivots} \end{cases}$

So $\det A \neq 0 \iff \det U \neq 0$

$\iff n$ pivots $\iff A$ invt-ble

$\blacksquare$

5 A, B $n \times n$ matrices

Then $\det(AB) = (\det A)(\det B)$

Speial case $B = A^{-1}$ if A invt-ble

Find $\underset{1}{\underset{\shortparallel}{\det(I_n)}} = \det(A \cdot A^{-1})$

$= (\det A)(\det(A^{-1}))$

So $\det(A^{-1}) = (\det A)^{-1}$.

6 <u>Recall</u> If $A = [(a_{ij})]$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \ddots & \\ \vdots & & \ddots \\ a_{n1} & & a_{nn} \end{bmatrix}$$

then $A^T = [(a_{ji})] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ \vdots & \ddots & & \\ \vdots & & \ddots & \\ a_{1n} & & & a_{nn} \end{bmatrix}$

<u>transpose</u>

Thm $\det(A^T) = \det A$

<u>Observe</u> We now know similar statements about rows of A as we have seen for cols

<u>Exer</u> Show ~~row~~ span of rows of $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ is <u>not</u> all of $\mathbb{R}^3$

This is the same as asking to see cols of $A^T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ do <u>not</u> span all of $\mathbb{R}^3$

<u>Recall</u>

Cols of A^T span $\iff$ A^T is invt-ble

$\iff \det A^T \neq 0 \iff \det A \neq 0$

But cof. exp of $\det A$ along col 2 gives $\det A = 0$ So rows of A do <u>not</u> span $\mathbb{R}^3$