Happy Tuesday!

This week - usual schedule:

<u>Thurs</u> Office Hours 12:30 - 2:30
736 Evans

<u>Fri</u> Quiz up to and including 1.9

Warmup problem What is the first vector in the span of the preceding?

$\underline{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ $\underline{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}$ $\underline{v}_4 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix}$ $\underline{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \end{bmatrix}$

Want to find first! $k = 1, 2, 3, 4,$ or 5 so that

$$\underline{v}_k = a_1 \underline{v}_1 + \cdots + a_{k-1} \underline{v}_{k-1}$$

any numbers.

Such an equation is the same as an eqn

$$a_1 \underline{v}_1 + \cdots + a_{k-1} \underline{v}_{k-1} + (-1) \cdot \underline{v}_k = \underline{0}$$

Want to solve sys $A\underline{x} = \underline{0}$

where $A = \begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_k \\ | & & | \end{bmatrix}$

$$\underline{x} = \begin{bmatrix} a_1 \\ \vdots \\ a_{k-1} \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & 2 & 2 & -2 \end{bmatrix} \rightsquigarrow \cdots$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 5 & -4 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \cdots$$

$$A_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solve $A_3 \underline{y} = \underline{0}$

$0 = 1\cdot y_1 + 1\cdot y_3$

$0 = 1\cdot y_2 + 2\cdot y_3$

$\underline{y} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Soln:

$$\underline{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} = 1 \cdot \underline{v}_1 + 2 \cdot \underline{v}_2 = 1 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

General principle A list of vectors $\underline{v}_1, \ldots, \underline{v}_k$ is lin. depend.

$$\Updownarrow$$

There exists a vector $\underline{v}_i$ in list in the span of the others.

In fact we can always find $\underline{v}_i$ in list in span of $\underline{v}_1, \dots, \underline{v}_{i-1}$.

<u>Idea of why true</u>: Suppose

$$a_1\underline{v}_1 + \dots + a_k\underline{v}_k = 0 \quad a_1, \dots, a_k \text{ not all zero}$$

Say $a_i \neq 0$

Then
$$-\frac{a_1}{a_i}\underline{v}_1 + \dots + \frac{-a_{i-1}}{a_i}\underline{v}_{i-1} + \frac{-a_{i+1}}{a_i}\underline{v}_{i+1} + \dots + \frac{-a_k}{a_i}\underline{v}_k = \underline{v}_i$$

Conversely: Suppose $\underline{v}_i = a_1\underline{v}_1 + \dots + a_{i-1}\underline{v}_{i-1} + a_{i+1}\underline{v}_{i+1} + \dots + a_k\underline{v}_k$

Then $a_1\underline{v}_1 + \dots + a_{i-1}\underline{v}_{i-1} + (-1)\underline{v}_i + a_{i+1}\underline{v}_{i+1} + \dots + a_k\underline{v}_k = \underline{0}.$

$\neq 0.$

New perspective on lin sys $A\underline{x} = \underline{b}$

$m \times n$ matrix

$\underline{x} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ $\underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

If $A = \begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_n \\ | & & | \end{bmatrix}$

then $A\underline{x} = a_1\underline{v}_1 + \cdots + a_n\underline{v}_n$

<u>Rethink question of</u>: does there exist (and how many are there) solns to eqn $A\underline{x} = b$

Think of A as a <u>map</u> / <u>mapping</u> / <u>transformation</u>

$$\mathbb{R}^n \longrightarrow \mathbb{R}^m$$

<u>domain</u> codomain

$$\underline{x} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \longmapsto A\underline{x} = a_1\underline{v}_1 + \dots + a_n\underline{v}_n$$

where $A = \begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_n \\ | & & | \end{bmatrix}$

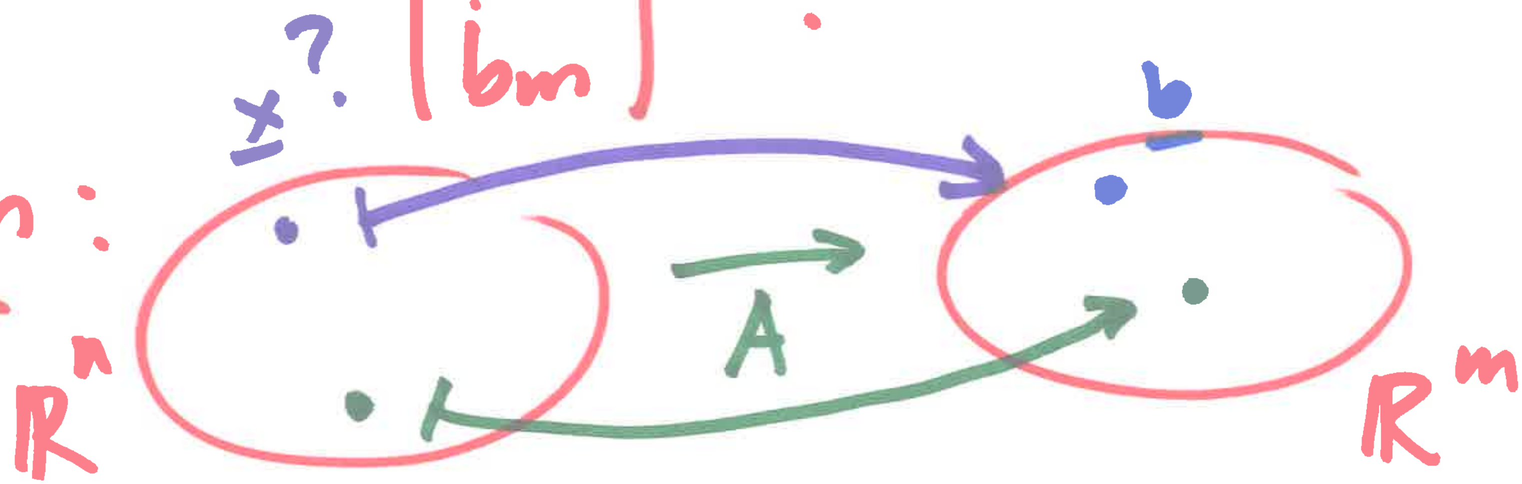
The eqn $A\underline{x} = \underline{b}$ becomes question:

Is there an $\underline{x} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ such that

the map given by A ~~hits~~ takes $\underline{x}$ to the

vector $\underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$?

Cartoon:



__Def.__ __Image__ / __Range__ of A is

$\{ \underline{b}$ in $\mathbb{R}^m$ such that there exists $\underline{x}$ in $\mathbb{R}^n$ with $A\underline{x} = \underline{b} \}$

In language of lin systs:

Image $= \{ \underline{b}$ in $\mathbb{R}^m$ such that $A\underline{x} = \underline{b}$ has soln $\}$

<u>Exer</u> What is image of map $\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$?

$$A\underline{x} = a_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\text{Image} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

<u>General principle</u> Image of A

= Span of cols of A.

<u>Exer</u> Is $\underline{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in image of map $\mathbb{R}^3 \to \mathbb{R}^2$ given by

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} ?$$

Need to solve $A\underline{x} = \underline{b}$, $\quad \underline{x} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

For example take, $\underline{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Exer Find all vectors $\underline{x} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ that are taken to $\underline{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ by map $\mathbb{R}^3 \longrightarrow \mathbb{R}^4$ given by $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Want to solve $A\underline{x} = \underline{b}$ with $\underline{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Aug matrix: $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Pivot vars: x_2, x_3

Free vars: x_1

Soln set: $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{ where } \underbrace{a_1}_{\text{any number}},\ a_2 = 1,\ a_3 = 1 \right\}$

Maps given by matrices satisfy:

1) $A(\underline{x}+\underline{y}) = A\underline{x} + A\underline{y}$

2) $A(c\cdot\underline{x}) = c\cdot(A\underline{x})$

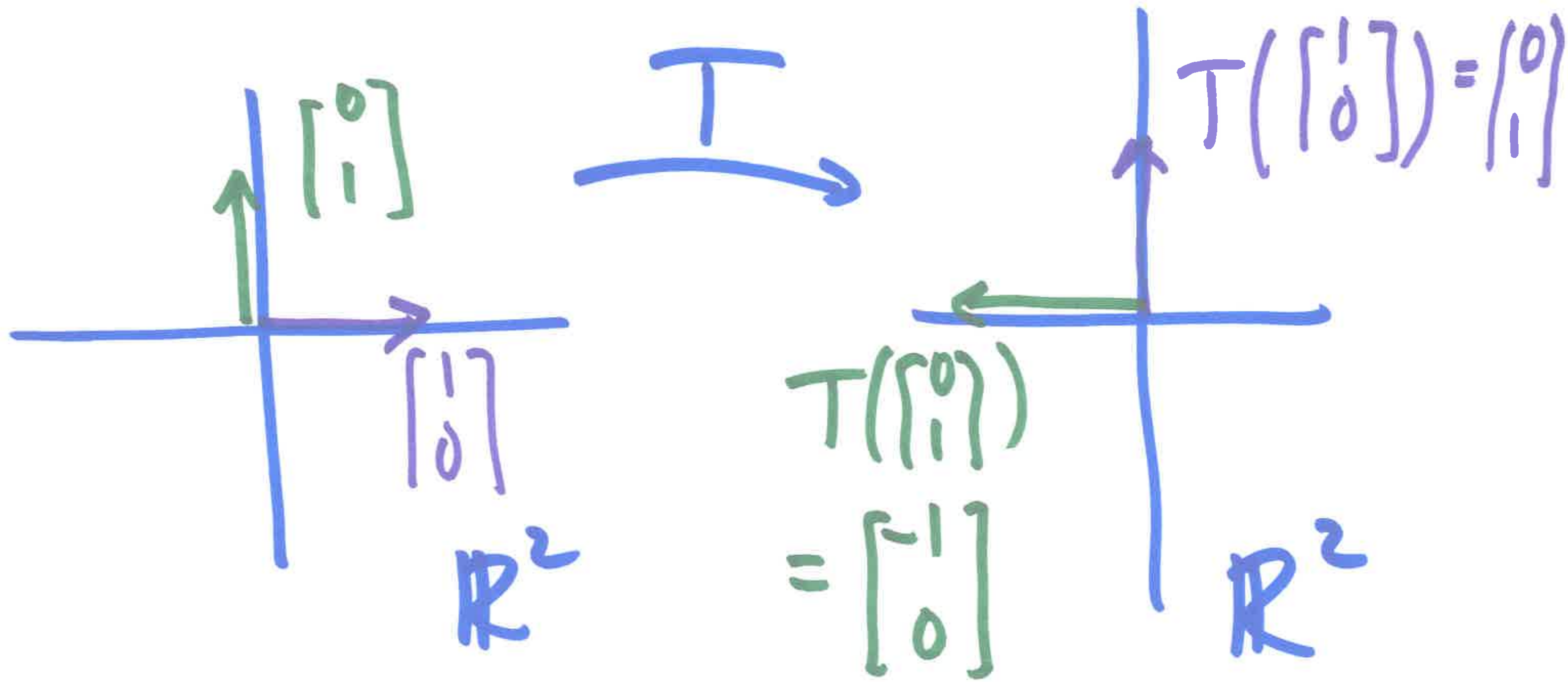
<u>Def</u>: Any map $T: \mathbb{R}^n \to \mathbb{R}^m$ satisfying above properties is called a <u>linear</u> <u>map</u>/<u>linear</u> <u>transf</u>.

__Fact__: Any linear transf $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by a $m \times n$ matrix!

$$A = \begin{bmatrix} | & & | \\ T(\underline{e}_1) & \cdots & T(\underline{e}_n) \\ | & & | \end{bmatrix}$$

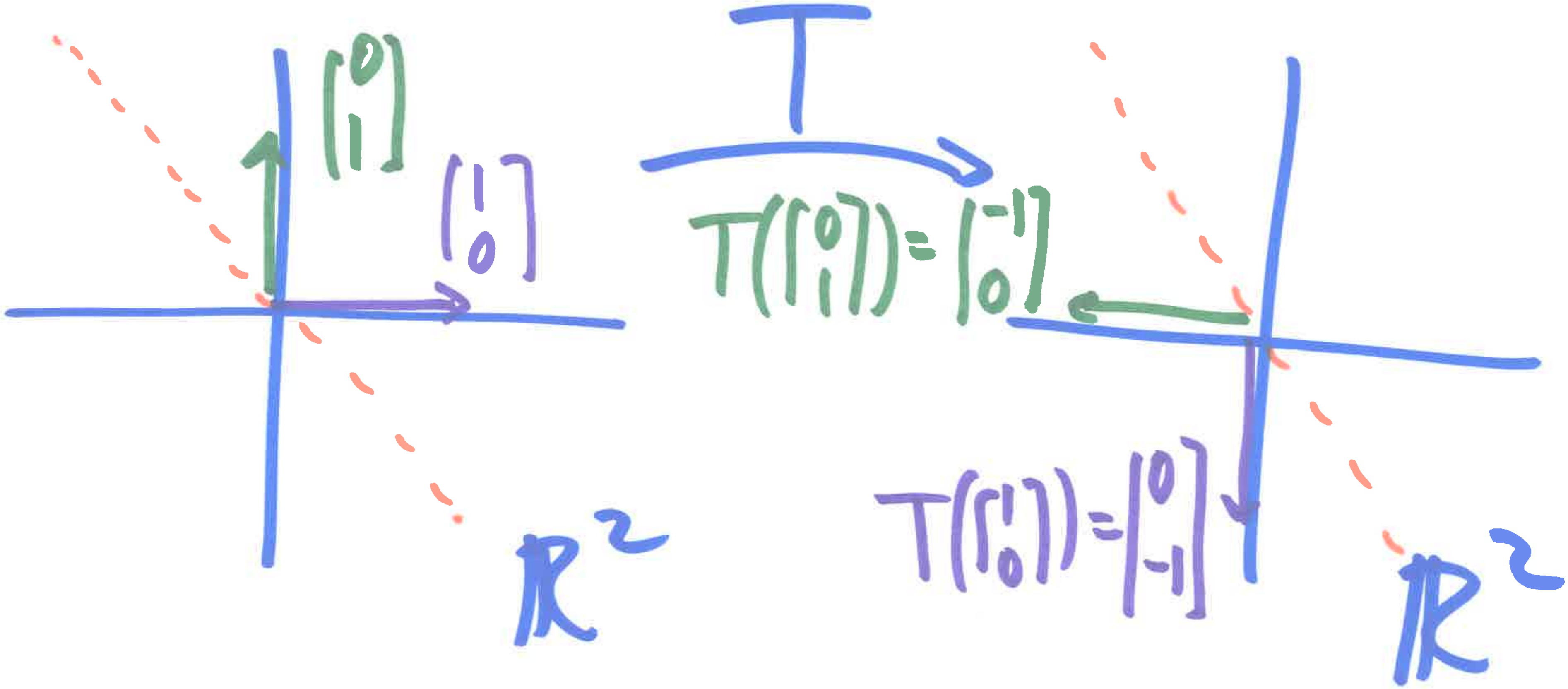
$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad \underline{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Exer Find matrix for lin transf. $T: \mathbb{R}^2 \to \mathbb{R}^2$ that rotates $90°$ counterclockwise.



$$A = \begin{bmatrix} | & | \\ T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ | & | \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

<u>Exer</u> Find matrix for lin transf $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects across line $\boxed{y = -x}$



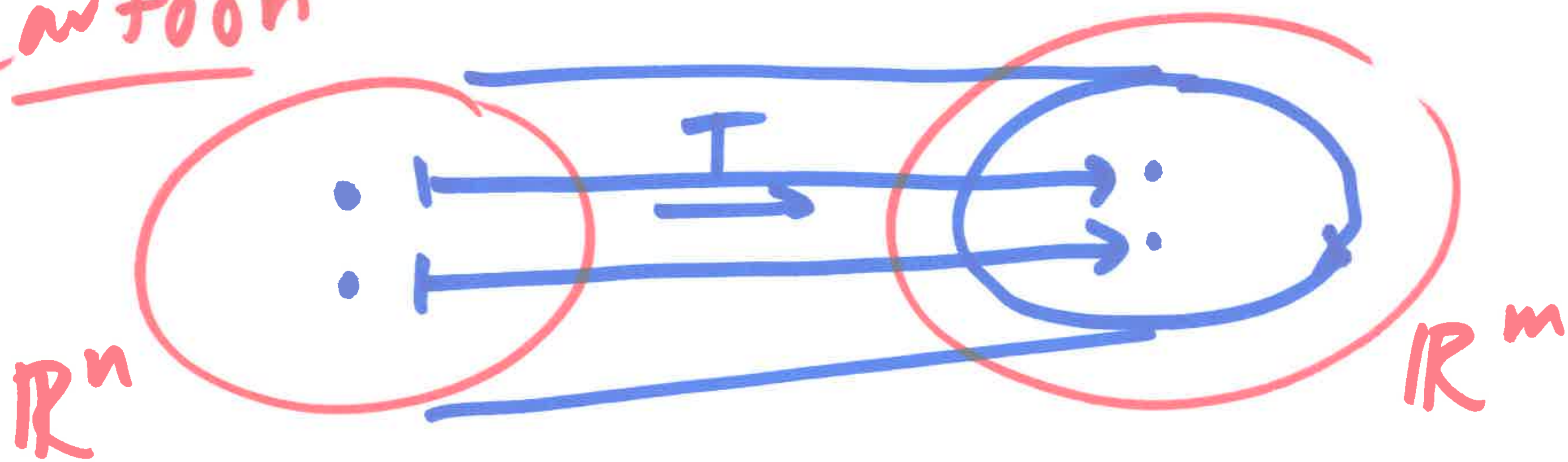
$$A = \left[\; T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \;\middle|\; T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \;\right] = \begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}$$

__New terminology:__

__Def__ 1) A __lin transf__ T is __one-to-one__ / __injective__

if whenever $T\underline{x} = T\underline{y}$, we have $\underline{x} = \underline{y}$

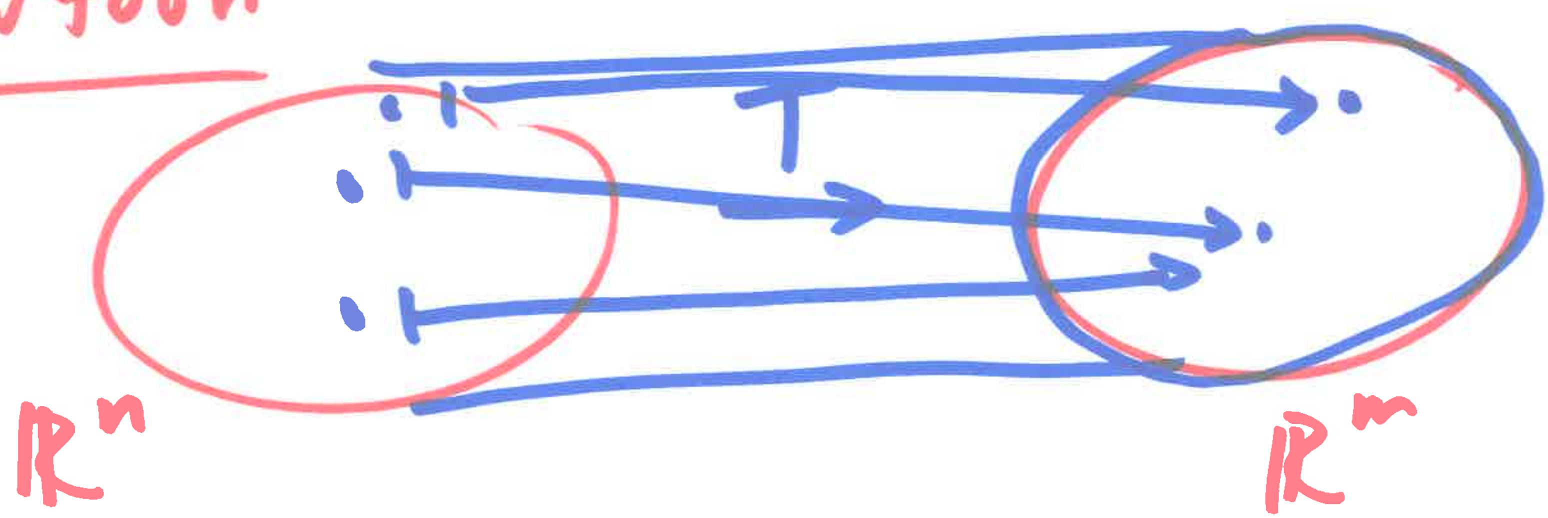
__Cartoon__



2) A lin transf T is <u>onto</u> / <u>surjective</u>

if for all $\underline{b}$, there exists $\underline{x}$ such that $T(\underline{x}) = \underline{b}$.

<u>Cartoon</u>



<u>Theorem</u> Suppose T is lin transf with matrix A

1) T injective $\iff$ cols of A lin indep

($A\underline{x} = \underline{0}$ has only triv soln $\underline{x} = \underline{0}$)

2) T surjective $\iff$ every $\underline{b}$ is in span of cols of A

($A\underline{x} = \underline{b}$ has soln for every $\underline{b}$)